TESTE N.º 1 – Proposta de resolução

1. Opção (C)

$\frac{\overline{BC}}{\overline{DC}} = \sqrt{5} \Leftrightarrow \overline{BC} = \sqrt{5} \times \overline{DC}$

$\overline{BC} = \overline{DP}$

$\overline{AP} = \overline{DC}$

$\overline{AD}^2 = \overline{AP}^2 + \overline{DP}^2$

$\overline{AD}^2 = \overline{AP}^2 + \left(\sqrt{5} \times \overline{AP}\right)^2 \Leftrightarrow \overline{AD}^2 = \overline{AP}^2 + \left(\sqrt{5} \times \overline{AP}\right)^2 \Leftrightarrow \overline{AD}^2 = 6\overline{AP}^2$

$\overline{AD} = \sqrt{6} \times \overline{AP}$

$\text{sen}(\alpha) = \frac{\overline{DP}}{\overline{AD}} = \frac{\sqrt{5}\times\overline{AP}}{\sqrt{6}\times\overline{AP}} = \frac{\sqrt{5}}{\sqrt{6}} = \frac{\sqrt{30}}{6}$

$\text{tg}(\alpha) = \frac{\overline{DP}}{\overline{AP}} = \frac{\sqrt{5}\times\overline{AP}}{\overline{AP}} = \sqrt{5}$

$\text{sen}(\beta) = \frac{\overline{AP}}{\overline{AD}} = \frac{\overline{AP}}{\sqrt{6}\times\overline{AP}} = \frac{1}{\sqrt{6}} = \frac{\sqrt{6}}{6}$

$\text{tg}(\beta) = \frac{\overline{AP}}{\overline{DP}} = \frac{\overline{AP}}{\sqrt{5}\times\overline{AP}} = \frac{1}{\sqrt{5}} = \frac{\sqrt{5}}{5}$

2. Opção (D)

$\text{sen}(90° - \alpha) = \cos(\alpha)$

Para $\alpha \in]-30°, 60°]$, $\frac{1}{2} \leq \cos(\alpha) \leq 1$

$\frac{1}{2} \leq \frac{1-\sqrt{3}k}{2} \leq 1$

$\Leftrightarrow 1 \leq 1 - \sqrt{3}k \leq 2$

$\Leftrightarrow 0 \leq -\sqrt{3}k \leq 1$

$\Leftrightarrow -1 \leq \sqrt{3}k \leq 0$

$\Leftrightarrow \frac{-1}{\sqrt{3}} \leq k \leq 0$

$\Leftrightarrow \frac{-\sqrt{3}}{3} \leq k \leq 0$

$\text{C.S.} = \left[-\frac{\sqrt{3}}{3}, 0\right]$

3. $A_{[ABCD]} = \frac{\overline{BC}+\overline{AD}}{2} \times \overline{AB}$

$\overline{AB} = 1 - \cos(\alpha)$

$\overline{AD} = \text{sen}(\alpha)$

$\overline{BC} = \text{tg}(\alpha)$

$$\text{sen}^2\alpha + \cos^2\alpha = 1 \Leftrightarrow \left(\frac{2}{3}\right)^2 + \cos^2\alpha = 1$$

$$\Leftrightarrow \frac{4}{9} + \cos^2\alpha = 1$$

$$\Leftrightarrow \cos^2\alpha = 1 - \frac{4}{9}$$

$$\Leftrightarrow \cos^2\alpha = \frac{5}{9}$$

$$\Leftrightarrow \cos\alpha = \pm\frac{\sqrt{5}}{3}$$

$0 < \alpha < \frac{\pi}{2}$, pelo que $\cos\alpha > 0$, logo $\cos\alpha = \frac{\sqrt{5}}{3}$

$$\text{tg}(\alpha) = \frac{\text{sen}(\alpha)}{\cos(\alpha)} \Leftrightarrow \text{tg}(\alpha) = \frac{\frac{2}{3}}{\frac{\sqrt{5}}{3}}$$

$$\Leftrightarrow \text{tg}(\alpha) = \frac{2}{\sqrt{5}}$$

$$\Leftrightarrow \text{tg}(\alpha) = \frac{2\sqrt{5}}{5}$$

Assim:

$$A_{[ABCD]} = \frac{\frac{2\sqrt{5}}{5} + \frac{2}{3}}{2} \times \left(1 - \frac{\sqrt{5}}{3}\right) =$$

$$= \left(\frac{\sqrt{5}}{5} + \frac{1}{3}\right) \times \left(1 - \frac{\sqrt{5}}{3}\right) =$$

$$= \frac{\sqrt{5}}{5} - \frac{1}{3} + \frac{1}{3} - \frac{\sqrt{5}}{9} =$$

$$= \frac{9\sqrt{5}}{45} - \frac{5\sqrt{5}}{45} =$$

$$= \frac{4\sqrt{5}}{45}$$

4. O argumento da função seno toma valores de um intervalo com amplitude superior a 2π.

$$-1 \leq \text{sen}\left(2x - \frac{\pi}{6}\right) \leq 1$$

$$\Leftrightarrow -b \leq b\text{sen}\left(2x - \frac{\pi}{6}\right) \leq b$$

$$\Leftrightarrow -3 - b \leq -3 + b\text{sen}\left(2x - \frac{\pi}{6}\right) \leq -3 + b$$

$$\Leftrightarrow a - 3 - b \leq a - 3 + b\text{sen}\left(2x - \frac{\pi}{6}\right) \leq a - 3 + b$$

Como $D'_f = [-4, 2]$,

$$\begin{cases} a - 3 - b = -4 \\ a - 3 + b = 2 \end{cases} \Leftrightarrow \begin{cases} a - b = -1 \\ a + b = 5 \end{cases} \Leftrightarrow \begin{cases} a = b - 1 \\ b - 1 + b = 5 \end{cases} \Leftrightarrow \begin{cases} a = b - 1 \\ 2b = 6 \end{cases} \Leftrightarrow \begin{cases} a = 3 - 1 \\ b = 3 \end{cases} \Leftrightarrow \begin{cases} a = 2 \\ b = 3 \end{cases}$$

5. Opção (D)

(I) $D_f = \left\{x \in \mathbb{R}: \cos^2(x) - 1 \neq 0 \ \wedge \ x \neq \frac{\pi}{2} + k\pi \,, k \in \mathbb{Z}\right\} = \left\{x \in \mathbb{R}: x \neq k\pi \ \wedge \ x \neq \frac{\pi}{2} + k\pi \,, k \in \mathbb{Z}\right\}$

$D_f = \mathbb{R} \setminus \left\{x: x = \dfrac{\pi}{2} + k\dfrac{\pi}{2} \,, k \in \mathbb{Z}\right\}$

C.A.:

$\cos^2(x) - 1 = 0 \Leftrightarrow \cos^2(x) = 1 \Leftrightarrow \cos x = -1 \vee \cos x = 1 \Leftrightarrow x = k\pi \,, k \in \mathbb{Z}$

(II) $f(x + \pi) = \frac{\text{tg}(x+\pi)}{\cos^2(x+\pi)-1} = \frac{\text{tg}(x)}{\cos^2(x)-1} = f(x), \forall x \in \ D_f$

De onde se conclui que π é período da funçãof, pelo que apenas a proposição (II) é verdadeira.

6. $\frac{\sqrt{2}}{2}\text{sen}\left(\frac{\pi}{2} - \alpha\right) \times \cos\left(-\frac{\pi}{2} - \alpha\right) - \cos^2\left(\frac{3\pi}{2} - \alpha\right) - \text{tg}(-\alpha) =$

$= \frac{\sqrt{2}}{2}\cos(\alpha) \times \left(-\text{sen}(\alpha)\right) - \text{sen}^2(\alpha) + \text{tg}(\alpha) =$

$= -\frac{\sqrt{2}}{2}\cos(\alpha)\text{sen}(\alpha) - \text{sen}^2(\alpha) + \text{tg}(\alpha) =$

$= -\frac{\sqrt{2}}{2} \times \left(-\frac{\sqrt{3}}{3}\right) \times \frac{\sqrt{6}}{3} - \left(\frac{\sqrt{6}}{3}\right)^2 + \left(-\sqrt{2}\right) =$

$= \frac{1}{3} - \frac{2}{3} - \sqrt{2} =$

$= -\frac{1}{3} - \sqrt{2}$

C.A.:

$\text{tg}(\alpha) = -\sqrt{2}$ e $\text{tg}^2(\alpha) + 1 = \frac{1}{\cos^2(\alpha)}$

$\Leftrightarrow \left(-\sqrt{2}\right)^2 + 1 = \dfrac{1}{\cos^2(\alpha)}$

$\Leftrightarrow \cos^2(\alpha) = \dfrac{1}{3}$

$\Leftrightarrow \cos(\alpha) = \pm\dfrac{\sqrt{3}}{3}$

$\alpha \in \left]\frac{\pi}{2}, \pi\right[$, pelo que $\cos(\alpha) = -\frac{\sqrt{3}}{3}$

$\text{tg}(\alpha) = \dfrac{\text{sen}(\alpha)}{\cos(\alpha)} \Leftrightarrow \text{sen}(\alpha) = \text{tg}(\alpha) \times \cos(\alpha)$

$\text{sen}(\alpha) = -\sqrt{2} \times \left(-\frac{\sqrt{3}}{3}\right) = \frac{\sqrt{6}}{3}$

7. $\frac{\left(\text{tg}^2x+1\right)\left(1-2\cos^2x+\cos^4x\right)}{\text{tg}^2x} =$

$= \frac{\frac{1}{\cos^2x}\left(1-2\text{co}\ {}^2x+\cos^4x\right)}{\text{tg}^2x} =$

$= \frac{\frac{1}{\cos^2x}\left(1-\cos^2x\right)^2}{\text{tg}^2x} =$

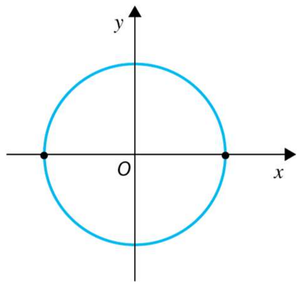
$= \frac{\frac{1}{\cos^2x}\left(\text{sen}^2x\right)^2}{\text{tg}^2x} =$

$= \frac{\frac{\text{sen}^2x \times \text{sen}^2x}{\cos^2x}}{\text{tg}^2x} =$

$= \frac{\text{tg}^2x \times \text{sen}^2x}{\text{tg}^2x} =$

$= \text{sen}^2x$

8. Opção (B)



$\cos^2 x = 1 \Leftrightarrow \cos x = -1 \vee \cos x = 1 \Leftrightarrow x = k\pi\,, k \in \mathbb{Z}$

Em $[0, 2\pi[$ a equação tem 2 soluções.

Em $[0, 2022\pi[$ a equação tem 2022 soluções.

Em $[0, 2023\pi[$ a equação tem 2023 soluções.

9.

9.1 Opção (A)

$$g\left(x+\frac{\pi}{2}\right) = -2\cos\left(2\left(x+\frac{\pi}{2}\right)\right)+1 = -2\cos(2x+\pi)+1 = 2\cos(2x)+1$$

$$f\left(x-\frac{\pi}{6}\right) = 2\cos\left(-\left(x-\frac{\pi}{6}\right)+\frac{\pi}{3}\right)+1 = 2\cos\left(-x+\frac{\pi}{6}+\frac{\pi}{3}\right)+1 = 2\cos\left(-x+\frac{\pi}{2}\right)+1 = 2\text{sen}x+1$$

$$g\left(x+\frac{\pi}{2}\right)-f\left(x-\frac{\pi}{6}\right) = 2\cos(2x)+1-(2\text{sen}x+1) = 2\cos(2x)+1-2\text{sen}x-1 = 2\cos(2x)-2\text{sen}x$$

9.2 $f(x) = g(x)$

$$2\cos\left(-x+\frac{\pi}{3}\right)+1 = -2\cos(2x)+1$$

$$\Leftrightarrow 2\cos\left(-x+\frac{\pi}{3}\right) = -2\cos(2x)$$

$$\Leftrightarrow \cos\left(-x+\frac{\pi}{3}\right) = -\cos(2x)$$

$$\Leftrightarrow \cos\left(-x+\frac{\pi}{3}\right) = \cos(\pi-2x)$$

$$\Leftrightarrow -x+\frac{\pi}{3} = \pi-2x+2k\pi \ \vee\ -x+\frac{\pi}{3} = -(\pi-2x)+2k\pi\,, k \in \mathbb{Z}$$

$$\Leftrightarrow x = \frac{2\pi}{3}+2k\pi \vee -x+\frac{\pi}{3} = -\pi+2x+2k\pi\,, k \in \mathbb{Z}$$

$$\Leftrightarrow x = \frac{2\pi}{3}+2k\pi \ \vee\ -3x = -\frac{4\pi}{3}+2k\pi\,, k \in \mathbb{Z}$$

$$\Leftrightarrow x = \frac{2\pi}{3}+2k\pi \ \vee\ x = \frac{4\pi}{9}+\frac{2k\pi}{3}\,, k \in \mathbb{Z}$$

$x \in [-\pi, \pi]$:

$k = -2$: $x = \frac{2\pi}{3}-4\pi \ \vee\ x = \frac{4\pi}{9}-\frac{4\pi}{3} \Leftrightarrow x = -\frac{10\pi}{3} \vee x = -\frac{8\pi}{9}$

$k = -1$: $x = \frac{2\pi}{3}-2\pi \ \vee\ x = \frac{4\pi}{9}-\frac{2\pi}{3} \Leftrightarrow x = -\frac{4\pi}{3} \vee x = -\frac{2\pi}{9}$

$k = 0$: $x = \frac{2\pi}{3} \ \vee\ x = \frac{4\pi}{9}$

C.S. $= \left\{-\frac{8\pi}{9}, -\frac{2\pi}{9}, \frac{2\pi}{3}, \frac{4\pi}{9}\right\}$

9.3 $f(x) = g(x) + 2$

$$2\cos\left(-x + \frac{\pi}{3}\right) + 1 = -2\cos(2x) + 1 + 2$$

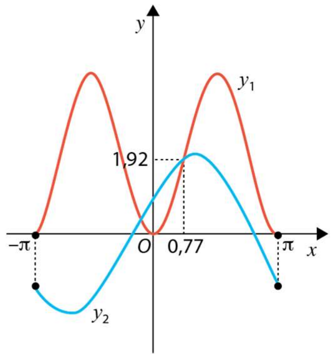
$$\Leftrightarrow 2\cos\left(-x + \frac{\pi}{3}\right) = -2\cos(2x) + 2$$

Recorrendo à calculadora gráfica:

$$y_1 = 2\cos\left(-x + \frac{\pi}{3}\right)$$

$$y_2 = -2\cos(2x) + 2$$

Assim, $x \approx 0{,}8$.



10. O arco de circunferência AB tem comprimento 2π, logo $\alpha \times 3 = 2\pi \Leftrightarrow \alpha = \frac{2\pi}{3}$

Assim:

$$A_{sc} = \frac{\frac{2\pi}{3} \times 3^2}{2} = 3\pi$$

11. Uma vez que o lado $[BC]$ é tangente à circunferência no ponto T, o triângulo $[OTC]$ é retângulo em T.

Como $C\hat{O}T = \frac{\pi}{2} - \alpha$, então $O\hat{C}T = \alpha$.

Assim, $\text{sen}\alpha = \frac{1}{\overline{OC}} \Leftrightarrow \overline{OC} = \frac{1}{\text{se}}$

Seja P, o ponto de interseção de $[AB]$ com o eixo Oy.

$$\overline{CP} = \frac{1}{\text{sen}\alpha} + 1$$

$$\text{tg}\alpha = \frac{\overline{PB}}{\overline{CP}} \Leftrightarrow \overline{PB} = \text{tg}\alpha \times \overline{CP} \Leftrightarrow \overline{PB} = \text{tg}\alpha \times \left(1 + \frac{1}{\text{sen}\alpha}\right)$$

$$\overline{AB} = 2 \times \overline{PB} = 2 \times \text{tg}\alpha \times \left(1 + \frac{1}{\text{sen}\alpha}\right)$$

$$A_{[ABC]} = \frac{2 \times \text{tg}\alpha \times \left(1 + \frac{1}{\text{sen}\alpha}\right) \times \left(1 + \frac{1}{\text{sen}\alpha}\right)}{2} =$$

$$= \text{tg}\alpha \times \left(1 + \frac{1}{\text{sen}\alpha}\right)^2 =$$

$$= \text{tg}\alpha \times \left(\frac{\text{sen}\alpha}{\text{sen}\alpha} + \frac{1}{\text{sen}\alpha}\right)^2 =$$

$$= \frac{\text{sen}\alpha}{\cos\alpha} \times \left(\frac{\text{sen}\alpha + 1}{\text{sen}\alpha}\right)^2 =$$

$$= \frac{\text{sen}\alpha \times (\text{sen}\,\alpha + 1)^2}{\text{co} \quad \times \text{sen}^2\alpha} =$$

$$= \frac{(\text{sen}\,\alpha + 1)^2}{\text{sen}\,\alpha \times \cos\alpha}$$