TESTE N.º 2 – Proposta de resolução

1.

1.1. Sabemos que $A(\cos\alpha, \operatorname{sen}\alpha)$, $B(1, \operatorname{sen}\alpha)$ e $C(1, \operatorname{tg}\alpha)$.

Como A pertence ao 3.º quadrante $\cos\alpha < 0$, $\operatorname{sen}\alpha < 0$ e $\operatorname{tg}\alpha > 0$.

$$A_{[ABC]} = \frac{\overline{AB}\times\overline{BC}}{2} = \frac{(1-\cos\alpha)(-\operatorname{sen}\alpha+\operatorname{tg}\alpha)}{2} = \frac{-\operatorname{sen}\alpha+\operatorname{sen}\alpha\cos\alpha+\operatorname{tg}\alpha-\operatorname{sen}\alpha}{2} =$$
$$= \frac{-2\operatorname{sen}\alpha+\operatorname{sen}\alpha\cos\alpha+\operatorname{tg}\alpha}{2} =$$
$$= \frac{\operatorname{sen}\alpha}{2}\left(-2+\cos\alpha+\frac{\operatorname{tg}\alpha}{\operatorname{sen}\alpha}\right) =$$
$$= \frac{\operatorname{sen}\alpha}{2}\left(-2+\cos\alpha+\frac{\operatorname{sen}\alpha}{\cos\alpha\operatorname{sen}\alpha}\right) =$$
$$= \frac{\operatorname{sen}\alpha}{2}\left(-2+\cos\alpha+\frac{1}{\cos\alpha}\right) \quad \text{c.q.d.}$$

1.2. $\cos\left(-\frac{\pi}{2}-\alpha\right) = \frac{3}{5} \Leftrightarrow -\operatorname{sen}\alpha = \frac{3}{5} \Leftrightarrow \operatorname{sen}\alpha = -\frac{3}{5}$

$\operatorname{sen}^2\alpha + \cos^2\alpha = 1$

$\frac{9}{25} + \cos^2\alpha = 1 \Leftrightarrow \cos^2\alpha = \frac{16}{25} \Leftrightarrow \cos\alpha = \pm\frac{4}{5}$

Como $\alpha \in 3.^\circ\,\text{Q}$, $\cos\alpha = -\frac{4}{5}$.

$$\frac{\operatorname{sen}\alpha}{2}\times\left(-2+\cos\alpha+\frac{1}{\cos\alpha}\right) = -\frac{3}{10}\times\left(-2-\frac{4}{5}-\frac{5}{4}\right) = -\frac{3}{10}\times\left(-\frac{40}{20}-\frac{16}{20}-\frac{25}{20}\right) =$$
$$= -\frac{3}{10}\times\left(-\frac{81}{20}\right) =$$
$$= \frac{243}{200}$$

1.3. $\frac{9\pi}{8} < \alpha_1 < \frac{11\pi}{8}$

$A\left(\alpha_1 + \frac{\pi}{8}\right) = 3A\,(\alpha_1)$

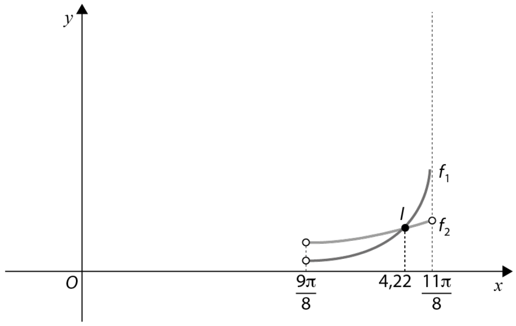
Utilizando x como variável independente:

$A\left(x + \frac{\pi}{8}\right) = 3A(x)$

Recorrendo às capacidades gráficas da calculadora:

$$f_1(x) = \frac{\operatorname{sen}\left(x+\frac{\pi}{8}\right)}{2}\left(-2+\cos\left(x+\frac{\pi}{8}\right)+\frac{1}{\cos\left(x+\frac{\pi}{8}\right)}\right)$$

$$f_2(x) = \frac{3\operatorname{sen}(x)}{2}\left(-2+\cos x+\frac{1}{\cos x}\right)$$



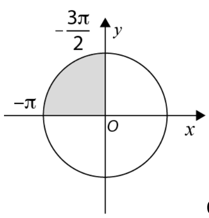
$I(a, b)$

$a \approx 4{,}22$

$b \approx 6{,}07$

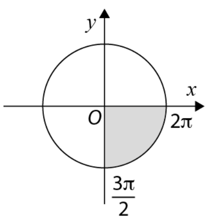
2. Opção (D)

$\alpha \in \left]-\frac{3\pi}{2}, -\pi\right[$



$\alpha \in 2.° \text{ Q}$

$\beta \in \left]\frac{3\pi}{2}, 2\pi\right[$



$\beta \in 4.° \text{ Q}$

$\text{sen}\,\alpha \times \cos\beta > 0$

$\text{tg}\,\alpha \times \text{tg}\,\beta > 0$

$\cos\alpha + \text{sen}\,\beta < 0$

$\text{sen}\,\alpha - \text{sen}\,\beta > 0$

3. Opção (D)

$\pi < 4 < \frac{3\pi}{2}$

4. $\alpha \in \left]-\pi, 0\right[$

$\text{tg}(\pi - \alpha) = -2 \Leftrightarrow -\text{tg}\,\alpha = -2$

$\Leftrightarrow \text{tg}\,\alpha = 2$

Como $\alpha \in \left]-\pi, 0\right[$ e $\text{tg}\,\alpha > 0$, então $\alpha \in 3.° \text{ Q}$.

$\cos(-\pi - \alpha) - \text{tg}(-\alpha) + \text{sen}\left(\frac{\pi}{2} - \alpha\right) + \cos(\alpha + \pi) = -\cos\alpha + \text{tg}\,\alpha + \cos\alpha - \cos\alpha =$

$= -\cos\alpha + \text{tg}\,\alpha = \frac{\sqrt{5}}{5} + 2$

Cálculo auxiliar

$1 + \text{tg}^2\,\alpha = \dfrac{1}{\cos^2\alpha}$

$1 + 4 = \dfrac{1}{\cos^2\alpha} \Leftrightarrow \dfrac{1}{\cos^2\alpha} = 5 \Leftrightarrow \cos^2\alpha = \dfrac{1}{5} \Leftrightarrow \cos\alpha = \pm\dfrac{\sqrt{5}}{5}$

Como $\alpha \in 3.^\circ\,\text{Q}$, $\cos\alpha = -\frac{\sqrt{5}}{5}$

5. Opção (A)

$D_f = \left\{x \in \mathbb{R}: 1 - \text{tg}^2(2x) \neq 0 \;\wedge\; 2x \neq \frac{\pi}{2} + k\pi, k \in \mathbb{Z}\right\}$

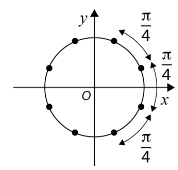
Cálculos auxiliares

- $1 - \text{tg}^2(2x) = 0 \Leftrightarrow \text{tg}^2(2x) = 1 \Leftrightarrow \text{tg}(2x) = 1 \;\vee\; \text{tg}(2x) = -1$

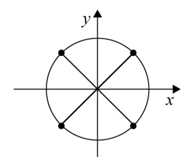
$\Leftrightarrow 2x = \frac{\pi}{4} + k\pi \;\vee\; 2x = -\frac{\pi}{4} + k\pi, k \in \mathbb{Z}$

$\Leftrightarrow x = \frac{\pi}{8} + \frac{k\pi}{2} \;\vee\; x = -\frac{\pi}{8} + \frac{k\pi}{2}, k \in \mathbb{Z}$

$\Leftrightarrow x = \frac{\pi}{8} + \frac{k\pi}{4}, k \in \mathbb{Z}$



- $2x = \frac{\pi}{2} + k\pi, k \in \mathbb{Z} \Leftrightarrow x = \frac{\pi}{4} + \frac{k\pi}{2}, k \in \mathbb{Z}$



$D_f = \mathbb{R} \setminus \left\{x : x = \dfrac{\pi}{4} + \dfrac{k\pi}{2} \vee x = \dfrac{\pi}{8} + \dfrac{k\pi}{4}, k \in \mathbb{Z}\right\}$

Averiguemos se $\frac{\pi}{2}$ é período de f:

- $\forall\, x \in D_f \Longrightarrow x + \frac{\pi}{2} \in D_f$
- $\forall\, x \in D_f \Longrightarrow f\left(x + \frac{\pi}{2}\right) = \frac{1}{1 - \text{tg}^2\left(2x + \frac{2\pi}{2}\right)} = \frac{1}{1 - \text{tg}^2(2x)} = f(x)$

$\frac{\pi}{2}$ é período da função f.

6.

6.1. Seja $x \in D_f$ qualquer:

$f(x) = (\cos x + \text{tg}\,x)^2 + (1 - \text{sen}\,x)^2 =$

$= \cos^2 x + 2\cos x\,\text{tg}\,x + \text{tg}^2\,x + 1 - 2\,\text{sen}\,x + \text{sen}^2\,x =$

$= \cos^2 x + \text{sen}^2\,x + 2\,\text{sen}\,x + \frac{1}{\cos^2 x} - 2\,\text{sen}\,x =$

$= 1 + \frac{1}{\cos^2 x}$

6.2. $\forall x \in D_f, x \in D_f \Longrightarrow -x \in D_f$

$f(-x) = 1 + \frac{1}{\cos^2(-x)} = 1 + \frac{1}{\cos^2 x} = f(x), \forall x \in D_f$

Logo, f é par.

6.3 $f(x) = 3 \Leftrightarrow 1 + \frac{1}{\cos^2 x} = 3$

$\Leftrightarrow \frac{1}{\cos^2 x} = 2$

$\Leftrightarrow \cos^2 x = \frac{1}{2}$

$\Leftrightarrow \cos x = \frac{\sqrt{2}}{2} \ \vee \ \cos x = -\frac{\sqrt{2}}{2}$

$\Leftrightarrow \cos x = \cos\left(\frac{\pi}{4}\right) \ \vee \ \cos x = \cos\left(\frac{3\pi}{4}\right)$

$\Leftrightarrow x = \frac{\pi}{4} + 2k\pi \ \vee \ x = -\frac{\pi}{4} + 2k\pi \ \vee \ x = \frac{3\pi}{4} + 2k\pi \ \vee \ x = -\frac{3\pi}{4} + 2k\pi, k \in \mathbb{Z}$

$\Leftrightarrow x = \frac{\pi}{4} + \frac{k\pi}{2}, k \in \mathbb{Z}$

7. $x^2 - 2x + y^2 + 4y = 11 \Leftrightarrow x^2 - 2x + 1 + y^2 + 4y + 4 = 11 + 1 + 4$

$\Leftrightarrow (x-1)^2 + (y+2)^2 = 16$

$C(1,-2) \qquad r = 4$

$A\hat{C}B = \alpha$

$\alpha \times r = \frac{10\pi}{3} \Leftrightarrow 4\alpha = \frac{10\pi}{3}$

$\Leftrightarrow \alpha = \frac{10\pi}{12}$

$\Leftrightarrow \alpha = \frac{5\pi}{6}$

$\overrightarrow{CA} \cdot \overrightarrow{BC} = \overrightarrow{CA} \cdot \left(-\overrightarrow{CB}\right) = -\overrightarrow{CA} \cdot \overrightarrow{CB} =$

$= -\left\|\overrightarrow{CA}\right\| \times \left\|\overrightarrow{CB}\right\| \times \cos\left(\widehat{\overrightarrow{CA}, \overrightarrow{CB}}\right) =$

$= -4 \times 4 \times \cos\left(\frac{5\pi}{6}\right) =$

$= -16 \times \left(-\frac{\sqrt{3}}{2}\right) =$

$= 8\sqrt{3}$

8. Opção (C)

$1 + \text{tg}^2\alpha = \frac{1}{\cos^2\alpha}$

$1 + \text{tg}^2\alpha = \frac{1}{\left(-\frac{1}{\sqrt{10}}\right)^2} \Leftrightarrow 1 + \text{tg}^2\alpha = 10 \Leftrightarrow \text{tg}^2\alpha = 9 \Leftrightarrow \text{tg}\,\alpha = 3 \ \vee \ \text{tg}\,\alpha = -3$

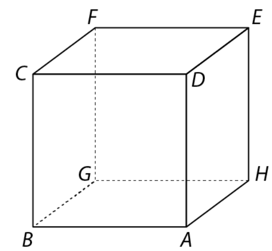
Como $\cos\alpha < 0$ e α é a inclinação da reta r, então $\alpha \in]90°, 180°[$, logo $\operatorname{tg}\alpha < 0$, ou seja, $\operatorname{tg}\alpha = -3$.

Assim, $m_r = -3$. Logo, o declive de uma reta perpendicular à reta r terá de ser igual a $\frac{1}{3}$.

9.

9.1. Opção (D)

Averiguemos qual das equações seguintes define uma reta perpendicular à reta BD e que passa no ponto F:



- $(x, y, z) = (10,5,6) + k(-4,0,1), k \in \mathbb{R}$

  O ponto de coordenadas $(10, 5, 6)$ pertence à reta

  $(-4, 0, 1) \cdot (-1, 9, 4) = 4 + 4 = 8$, logo a reta definida acima não é perpendicular à reta BD.

- $(x, y, z) = (10, 5, 6) + k(-6, 2, 3), k \in \mathbb{R}$

  O ponto de coordenadas $(10,5,6)$ pertence à reta

  $(-6, 2, 3) \cdot (-1, 9, 4) = 6 + 18 + 12 = 36$, logo a reta definida acima não é perpendicular à reta BD.

- $(x, y, z) = (7, 2, 0) + k(-1, -1, 2), k \in \mathbb{R}$

  $(10, 5, 6) = (7, 2, 0) + k(-1, -1, 2) \Leftrightarrow (10, 5, 6) = (7 - k, 2 - k, 2k)$

  $\Leftrightarrow \begin{cases} 10 = 7 - k \\ 5 = 2 - k \\ 6 = 2k \end{cases} \Leftrightarrow \begin{cases} k = -3 \\ k = -3 \\ k = 3 \end{cases}$ Condição impossível.

  O ponto de coordenadas $(10, 5, 6)$ não pertence à reta.

- $(x, y, z) = (-16, 3, 4) + k(13, 1, 1), k \in \mathbb{R}$

  $(10, 5, 6) = (-16, 3, 4) + k(13, 1, 1) \Leftrightarrow (10, 5, 6) = (-16 + 13k, 3 + k, 4 + k)$

  $\Leftrightarrow \begin{cases} 10 = -16 + 13k \\ 5 = 3 + k \\ 6 = 4 + k \end{cases} \Leftrightarrow \begin{cases} k = 2 \\ k = 2 \\ k = 2 \end{cases} \Leftrightarrow k = 2$, logo o ponto de coordenadas $(10, 5, 6)$ pertence à reta.

  $(13, 1, 1) \cdot (-1, 9, 4) = -13 + 9 + 4 = 0$, logo a reta definida acima é perpendicular à reta BD.

9.2. O ponto B é a interseção do plano BCF com a reta BD.

Determinemos, então uma equação do plano BCF.

$$\overrightarrow{FE} = E - F = (7, 11, 4) - (10, 5, 6) = (-3, 6, -2)$$

Uma equação do plano BCF é do tipo $-3x + 6y - 2z + d = 0$.

Como $F(10, 5, 6)$ pertence ao plano:

$-3 \times 10 + 6 \times 5 - 2 \times 6 + d = 0 \Leftrightarrow d = 12$

BCF: $-3x + 6y - 2z + 12 = 0$

BD: $(x, y, z) = (3, -9, -1) + k(-1, 9, 4), k \in \mathbb{R}$

Ponto genérico da reta BD: $(3 - k, -9 + 9k, -1 + 4k)$, com $k \in \mathbb{R}$

Substituindo as coordenadas do ponto genérico da reta BD na equação do plano BCF:

$$-3(3 - k) + 6(-9 + 9k) - 2(-1 + 4k) + 12 = 0 \Leftrightarrow -9 + 3k - 54 + 54k + 2 - 8k + 12 = 0$$

$$\Leftrightarrow 49k = 49$$

$$\Leftrightarrow k = 1$$

$B(3 - 1, -9 + 9, -1 + 4)$

$B(2, 0, 3)$

$\overrightarrow{BE} = E - B = (7, 11, 4) - (2, 0, 3) = (5, 11, 1)$

9.3. $O\hat{E}F = \widehat{\overrightarrow{EO}, \overrightarrow{EF}}$

$\overrightarrow{EO} = O - E = (-7, -11, -4)$

$\overrightarrow{EF} = F - E = (3, -6, 2)$

$\|\overrightarrow{EO}\| = \sqrt{49 + 121 + 16} = \sqrt{186}$

$\|\overrightarrow{EF}\| = \sqrt{9 + 36 + 4} = 7$

$$\cos(\widehat{\overrightarrow{EO}, \overrightarrow{EF}}) = \frac{\overrightarrow{EO} \,.\, \overrightarrow{EF}}{\|\overrightarrow{EO}\| \times \|\overrightarrow{EF}\|} \Leftrightarrow \cos(\widehat{\overrightarrow{EO}, \overrightarrow{EF}}) = \frac{-21 + 66 - 8}{7\sqrt{186}}$$

$$\Leftrightarrow \cos(\widehat{\overrightarrow{EO}, \overrightarrow{EF}}) = \frac{37}{7\sqrt{186}}$$

Logo, $\left(\widehat{\overrightarrow{EO}, \overrightarrow{EF}}\right) = \cos^{-1}\left(\frac{37}{7\sqrt{186}}\right)$, ou seja, $\widehat{\overrightarrow{EO}, \overrightarrow{EF}} \approx 67°$.