CHAPTER 12 TWO-DEGREE- OF-FREEDOM-SYSTEMS

Introduction to two degree of freedom systems:

- The vibrating systems, which require two coordinates to describe its motion, are called two-degrees-of –freedom systems.
- These coordinates are called generalized coordinates when they are independent of each other and equal in number to the degrees of freedom of the system.
- Unlike single degree of freedom system, where only one co-ordinate and hence one equation of motion is required to express the vibration of the system, in two-dof systems minimum two co-ordinates and hence two equations of motion are required to represent the motion of the system. For a conservative natural system, these equations can be written by using mass and stiffness matrices.
- One may find a number of generalized co-ordinate systems to represent the motion of the same system. While using these co-ordinates the mass and stiffness matrices may be coupled or uncoupled. When the mass matrix is coupled, the system is said to be dynamically coupled and when the stiffness matrix is coupled, the system is known to be statically coupled.
- The set of co-ordinates for which both the mass and stiffness matrix are uncoupled, are known as principal co-ordinates. In this case both the system equations are independent and individually they can be solved as that of a single-dof system.
- A two-dof system differs from the single dof system in that it has two natural frequencies, and for each of the natural frequencies there corresponds a natural state of vibration with a displacement configuration known as the normal mode. Mathematical terms associated with these quantities are eigenvalues and eigenvectors.
- Normal mode vibrations are free vibrations that depend only on the mass and stiffness of the system and how they are distributed. A normal mode oscillation is defined as one in which each mass of the system undergoes harmonic motion of same frequency and passes the equilibrium position simultaneously.
- The study of two-dof- systems is important because one may extend the same concepts used in these cases to more than 2-dof- systems. Also in these cases one can easily obtain an analytical or closed-form solutions. But for more degrees of

freedom systems numerical analysis using computer is required to find natural frequencies (eigenvalues) and mode shapes (eigenvectors).

The above points will be elaborated with the help of examples in this lecture.

Few examples of two-degree-of-freedom systems

Figure 1 shows two masses m_1 and m_2 with three springs having spring stiffness k_1, k_2 and k_3 free to move on the horizontal surface. Let x_1 and x_2 be the displacement of mass m_1 and m_2 respectively.

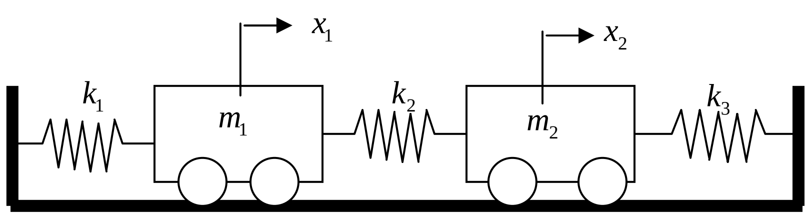


Figure 1

As described in the previous lectures one may easily derive the equation of motion by using d'Alembert principle or the energy principle (Lagrange principle or Hamilton's principle)

Using d'Alembert principle for mass m_1, from the free body diagram shown in figure 1(b)

$$m_1\ddot{x}_1 + (k_1 + k_2)x_1 - k_2x_2 = 0 \qquad (1)$$

and similarly for mass m_2

$$m_2\ddot{x}_2 - k_1x_1 + (k_2 + k_3)x_2 = 0 \qquad (2)$$

Important points to remember

- Inertia force acts opposite to the direction of acceleration, so in both the free body diagrams inertia forces are shown towards left.

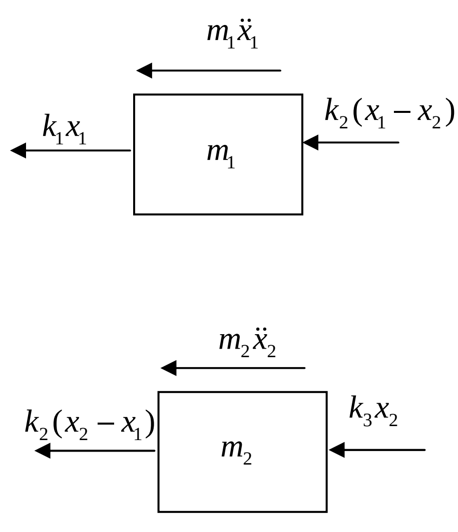


Figure 1 (b), Free body diagram

- For spring k_2, assuming $x_2 > x_1$, The spring will pull mass m_1 towards right by $k_2(x_2 - x_1)$ and it is stretched by $x_2 - x_1$ (towards right) it will exert a force of $k_2(x_2 - x_1)$ towards left on mass m_2. Similarly assuming $x_1 > x_2$, the spring get compressed by an amount $x_2 - x_1$ and exert tensile force of $k_2(x_1 - x_2)$. One may note that in both cases, free body diagram remain unchanged.

Now if one uses Lagrange principle,

The Kinetic energy = $T = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2$ and (3)

Potential energy $= U = \frac{1}{2} k_1 x_1^2 + \frac{1}{2} k_2 (x_1 - x_2)^2 + \frac{1}{2} k_3 x_2^2$ (4)

So, the Lagrangian

$$L = T - U = \left(\frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2 \right) - \left(\frac{1}{2} k_1 x_1^2 + \frac{1}{2} k_2 (x_1 - x_2)^2 + \frac{1}{2} k_3 x_2^2 \right) \tag{5}$$

The equation of motion for this free vibration case can be found from the Lagrange principle

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k} \right) - \frac{\partial L}{\partial q_k} = 0, \tag{6}$$

and noting that the generalized co-ordinate $q_1 = x_1$ and $q_2 = x_2$

which yields

$$m_1 \ddot{x}_1 + (k_1 + k_2) x_1 - k_2 x_2 = 0 \tag{7}$$

$$m_2 \ddot{x}_2 - k_1 x_1 + (k_2 + k_3) x_2 = 0 \tag{8}$$

Same as obtained before using d’Alembert principle.

Now writing the equation of motion in matrix form

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{9}$$

Here it may be noted that for the present two degree-of-freedom system, the system is dynamically uncoupled but statically coupled.

Example 2.

Consider a lathe machine, which can be modeled as a rigid bar with its center of mass not coinciding with its geometric center and supported by two springs, k_1, k_2.

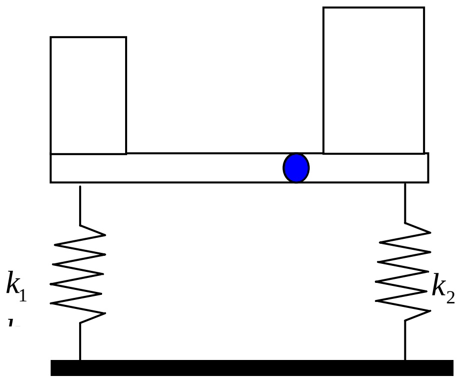


Figure 2

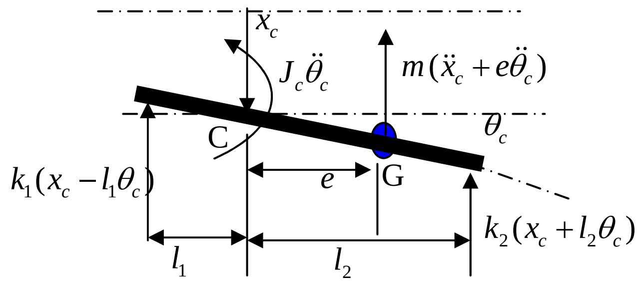


Figure 3: Free body diagram of the system

In this example, it will be shown, how the use of different coordinate systems lead to static and or dynamic coupled or uncoupled equations of motion. Clearly this is a two-degree-of freedom system and one may express the co-ordinate system in many different ways. Figure 3 shows the free body diagram of the system where point G is the center of mass. Point C represents a point on the bar at which we want to define the co-ordinates of this system. This point is at a distance l_1 from the left end and l_2 from right end. Distance between points C and G is e. Assuming x_c is the linear displacement of point C and θ_c the rotation about point C, the equation of motion of this system can be obtained by using d'Alember's principle. Now summation of all the forces, viz. the spring forces and the inertia forces must be equal to zero leads to the following equation.

$$m\ddot{x}_c + me\ddot{\theta}_c + k_1(x_c - l_1\theta_c) + k_2(x_c + l_2\theta_c) = 0 \quad (10)$$

Again taking moment of all the forces about point C

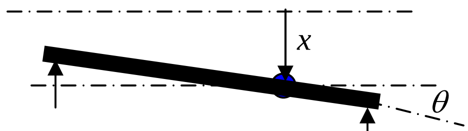
$$J_G\ddot{\theta}_c + (m\ddot{x}_c + me\ddot{\theta}_c)e - k_1(x_c - l_1\theta_c)l_1 + k_2(x_c + l_2\theta_c)l_2 = 0 \quad (11)$$

Noting $J_c = J_G + me^2$, the above two equations in matrix form can be written as

$$\begin{bmatrix} m & me \\ me & J_c \end{bmatrix}\begin{pmatrix} \ddot{x}_c \\ \ddot{\theta}_c \end{pmatrix} + \begin{bmatrix} k_1 + k_2 & k_2 l_2 - k_1 l_1 \\ k_2 l_2 - k_1 l_1 & k_1 l_1^2 + k_2 l_2^2 \end{bmatrix}\begin{pmatrix} x_c \\ \theta_c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (12)$$

Now depending on the position of point C, few cases can are studied below.

Case 1 : Considering $e = 0$, i.e., point C and G coincides, the equation of motion can be written as



$$\begin{bmatrix} m & 0 \\ 0 & J_G \end{bmatrix}\begin{pmatrix} \ddot{x} \\ \ddot{\theta} \end{pmatrix} + \begin{bmatrix} k_1 + k_2 & k_2 l_2 - k_1 l_1 \\ k_2 l_2 - k_1 l_1 & k_1 l_1^2 + k_2 l_2^2 \end{bmatrix}\begin{pmatrix} x \\ \theta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{13}$$

So in this case the system is statically coupled and if $k_1 l_1 = k_2 l_2$, this coupling disappears, and we obtained uncoupled x and θ vibrations.

Case 2 : If, $k_2 l_2 = k_1 l_1$, the equation of motion becomes

$$\begin{bmatrix} m & me \\ me & J_c \end{bmatrix}\begin{pmatrix} \ddot{x}_c \\ \ddot{\theta}_c \end{pmatrix} + \begin{bmatrix} k_1 + k_2 & 0 \\ 0 & k_1 l_1^2 + k_2 l_2^2 \end{bmatrix}\begin{pmatrix} x_c \\ \theta_c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{14}$$

Hence in this case the system is dynamically coupled but statically uncoupled.

Case 3: If we choose $l_1 = 0$, i.e. point C coincide with the left end, the equation of motion will become

$$\begin{bmatrix} m & me \\ me & J_c \end{bmatrix}\begin{pmatrix} \ddot{x}_c \\ \ddot{\theta}_c \end{pmatrix} + \begin{bmatrix} k_1 + k_2 & k_2 l_2 \\ k_2 l_2 & k_2 l_2^2 \end{bmatrix}\begin{pmatrix} x_c \\ \theta_c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{15}$$

Here the system is both statically and dynamically coupled.

Normal Mode Vibration

Again considering the problem of the spring-mass system in figure 1 with $m_1 = m$, $m_2 = 2m$, $k_1 = k_2 = k_3 = k$, the equation of motion (9) can be written as

$$\begin{aligned} m\ddot{x}_1 + k(x_1 - x_2) + kx_1 &= 0 \\ 2m\ddot{x}_2 - k(x_1 - x_2) + kx_2 &= 0 \end{aligned} \tag{16}$$

We define a normal mode oscillation as one in which each mass undergoes harmonic motion of the same frequency, passing simultaneously through the equilibrium position. For such motion, we let

$$x_1 = A_1 e^{i\omega t}, x_2 = A_2 e^{i\omega t} \tag{17}$$

Hence,

$$\begin{aligned} &(2k - m\omega^2)A_1 - kA_2 = 0 \\ &-kA_1 + (2k - 2m\omega^2)A_2 = 0 \end{aligned} \tag{18}$$

or, in matrix form

$$\begin{bmatrix} 2k - m\omega^2 & -k \\ -k & 2k - 2m\omega^2 \end{bmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{19}$$

Hence for nonzero values of A_1 and A_2 (i.e., for non-trivial response)

$$\begin{vmatrix} 2k - m\omega^2 & -k \\ -k & 2k - 2m\omega^2 \end{vmatrix} = 0\,. \tag{20}$$

Now substituting $\omega^2 = \lambda$, equation 6.1. yields

$$\lambda^2 - (3\frac{k}{m})\lambda + \frac{3}{2}(\frac{k}{m})^2 = 0 \tag{21}$$

Hence, $\lambda_1 = (\frac{3}{2} - \frac{1}{2}\sqrt{3})\frac{k}{m} = 0.634\frac{k}{m}$ and $\lambda_2 = (\frac{3}{2} + \frac{1}{2}\sqrt{3})\frac{k}{m} = 2.366\frac{k}{m}$

So, the natural frequencies of the system are $\omega_1 = \sqrt{\lambda_1} = \sqrt{0.634\frac{k}{m}}$ and $\omega_2 = \sqrt{2.366\frac{k}{m}}$

Now from equation (1)., it may be observed that for these frequencies, as both the equations are not independent, one can not get unique value of A_1 and A_2. So one should find a normalized value. One may normalize the response by finding the ratio of A_1 to A_2. From the first equation (19) the normalized value can be given by

$$\frac{A_1}{A_2} = \frac{k}{2k - m\omega^2} = \frac{k}{2k - m\lambda} \tag{22}$$

and from the second equation of (19), the normalized value can be given by

$$\frac{A_1}{A_2} = \frac{2k - 2m\omega^2}{k} = \frac{2k - 2m\lambda}{k} \tag{23}$$

Now, substituting $\omega_1^2 = \lambda_1 = 0.634\frac{k}{m}$ in equation (22) and (23) yields the same values, as both these equations are linearly dependent. Here,

$$\left(\frac{A_1}{A_2}\right)_{\lambda=\lambda_1} = \frac{0.732}{1} \tag{24}$$

and similarly for $\omega_2{}^2 = \lambda_2 = 2.366\dfrac{k}{m}$

$$\left(\frac{A_1}{A_2}\right)_{\lambda=\lambda_2} = \frac{-2.73}{1} \tag{25}$$

It may be noted

- Equation (19) gives only the ratio of the amplitudes and not their absolute values, which are arbitrary.
- If one of the amplitudes is chosen to be 1 or any number, we say that amplitudes ratio is normalized to that number.
- The normalized amplitude ratios are called the normal modes and designated by $\phi_n(x)$.

From equation (24) and (25), the two normal modes of this problem are:

$$\phi_1(x) = \begin{Bmatrix} 0.731 \\ 1.00 \end{Bmatrix} \qquad \phi_2(x) = \begin{Bmatrix} -2.73 \\ 1.00 \end{Bmatrix}$$

In the 1st normal mode, the two masses move in the same direction and are said to be in phase and in the 2nd mode the two masses move in the opposite direction and are said to be out of phase. Also in the first mode when the second mass moves unit distance, the first mass moves 0.731 units in the same direction and in the second mode, when the second mass moves unit distance; the first mass moves 2.73 units in opposite direction.

Free vibration using normal modes

When the system is disturbed from its initial position, the resulting free-vibration of the system will be a combination of the different normal modes. The participation of different modes will depend on the initial conditions of displacements and velocities. So for a system the free vibration can be given by

$$x = \phi_1 A \sin(\omega_1 t + \psi_1) + \phi_2 B \sin(\omega_2 t + \psi_2) \tag{27}$$

Here A and B are part of participation of first and second modes respectively in the resulting free vibration and ψ_1 and ψ_2 are the phase difference. They depend on the initial conditions. This is explained with the help of the following example.

Example: Let us consider the same spring-mass problem (figure 4) for which the natural frequencies and normal modes are determined. We have to determine the resulting free vibration when the system is given an initial displacement $x_1(0) = 5, x_2(0) = 1$ and initial velocity $\dot{x}_1(0) = \dot{x}_2(0) = 0$.

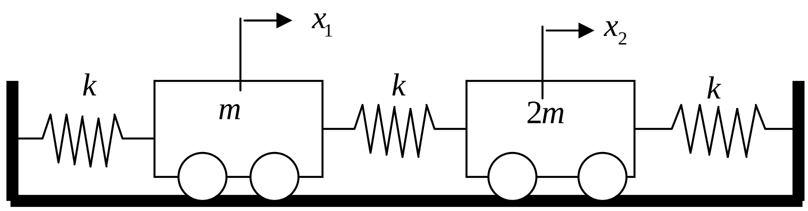


Figure 4

Solution:

Any free vibration can be considered to be the superposition of its normal modes. For each of these modes the time solution can be expressed as:

$$\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}_1 = \begin{Bmatrix} 0.731 \\ 1 \end{Bmatrix} \sin \omega_1 t$$
$$\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}_2 = \begin{Bmatrix} -2.731 \\ 1.00 \end{Bmatrix} \sin \omega_2 t$$

The general solution for the free vibration can then be written as:

$$\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = A \begin{Bmatrix} 0.731 \\ 1.00 \end{Bmatrix} \sin(\omega_1 t + \psi_1) + B \begin{Bmatrix} -2.73 \\ 1 \end{Bmatrix} \sin(\omega_2 t + \psi_2)$$

where A and B allow different amounts of each mode and ψ_1 and ψ_2 allows the two modes’ different phases or starting values.

Substituting:

$$\begin{Bmatrix} x_1(0) \\ x_2(0) \end{Bmatrix} = \begin{Bmatrix} 5 \\ 1 \end{Bmatrix} = A \begin{Bmatrix} 0.731 \\ 1 \end{Bmatrix} \sin\psi_1 + B \begin{Bmatrix} -2.731 \\ 1 \end{Bmatrix} \sin\psi_2$$

$$\begin{Bmatrix} x_1(0) \\ x_2(0) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} = \omega_1 A \begin{Bmatrix} 0.731 \\ 1 \end{Bmatrix} \cos\psi_1 + \omega_2 B \begin{Bmatrix} -2.731 \\ 1 \end{Bmatrix} \cos\psi_2$$

$\cos\psi_1 = \cos\psi_2 = 0 \Rightarrow \psi_1 = \psi_2 = 90^0$

Substituting in 1st set:

$$\begin{Bmatrix} 5 \\ 1 \end{Bmatrix} = A \begin{Bmatrix} 0.731 \\ 1 \end{Bmatrix} + B \begin{Bmatrix} -2.731 \\ 1 \end{Bmatrix}$$

$$\left.\begin{aligned} 0.731A - 2.731B &= 5 \\ A + B &= 1 \end{aligned}\right\} \quad \begin{aligned} A &= 2.233 \\ B &= -1.233 \end{aligned}$$

Hence the resulting free vibration is

$$\boxed{\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = 2.233 \begin{Bmatrix} 0.731 \\ 1.00 \end{Bmatrix} \cos\omega_1 t - 1.233 \begin{Bmatrix} -2.731 \\ 1.000 \end{Bmatrix} \cos\omega_2 t}$$

Normal modes from eigenvalues

The equation of motion for a two-degree-of freedom system can be written in matrix form as

$$M\,\ddot{x} + K\,x = 0 \tag{28}$$

where M and K are the mass and stiffness matrix respectively; x is the vector of generalized co-ordinates. Now pre-multiplying M^{-1} in both side of equation 6.2. one may get

$$I\,\ddot{x} + M^{-1}K\,x = 0 \tag{29}$$

or, $I\,\ddot{x} + A\,x = 0$ (30)

Here $A = M^{-1}K$ is known as the dynamic matrix. Now to find the normal modes,

$x_1 = X_1 e^{i\omega t}, x_2 = X_2 e^{i\omega t}$, the above equation will reduce to

$$\left[A - \lambda I\right] X = 0 \tag{31}$$

where $X = \{x_1 \quad x_2\}^T$ and $\lambda = \omega^2$.

From equation (31) it is apparent that the free vibration problem in this case is reduced to that of finding the eigenvalues and eigenvectors of the matrix *A*.

Example: Determine the normal modes of a double pendulum.

Solution

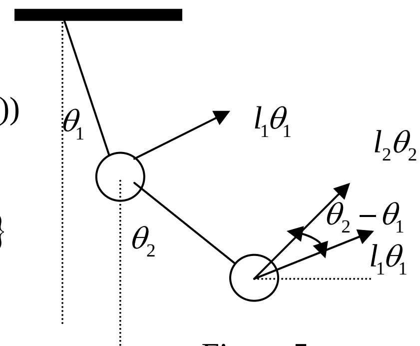


Figure 5

Kinetic energy of the system =

$$T = \frac{1}{2} m_1 l_1^2 \dot{\theta}_1^2 + \frac{1}{2} m_2 (l_1^2 \dot{\theta}_1^2 + l_2^2 \dot{\theta}_2^2 + 2 l_1 l_2 \theta_1 \theta_2 \cos(\theta_2 - \theta_1))$$

Potential energy of the system =

$$U = m_1 g (1 - \cos\theta_1) + m_2 g \{ l_1 (1 - \cos\theta_1) + l_2 (1 - \cos\theta_2) \}$$
$$= g \{ (m_1 + m_2) l_1 (1 - \cos\theta_1) + m_2 l_2 (1 - \cos\theta_2) \}$$

So Lagrangian of the system =

$$L = T - U$$
$$= \frac{1}{2} m_1 l_1^2 \dot{\theta}_1^2 + \frac{1}{2} m_2 (l_1 \dot{\theta}_1 + l_2 \dot{\theta}_2 + 2 l_1 l_2 \cos(\theta_2 - \theta_1)) - g \{ (m_1 + m_2)(1 - \cos\theta_1) + m_2 (1 - \cos\theta_2) \}$$

So using Lagrange principle, and assuming small angle of rotation, the equation of motion can be written in matrix form as

$$\begin{bmatrix} (m_1 + m_2) l_1^2 & m_2 l_1 l_2 \\ m_2 l_1 l_2 & m_2 l_2^2 \end{bmatrix} \begin{pmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{pmatrix} + \begin{bmatrix} (m_1 + m_2) l_1 g & 0 \\ 0 & m_2 l_2 g \end{bmatrix} \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Now considering a special case when $m_1 = m_2 = m$ and $l_1 = l_2 = l$, the above equation becomes

$$\begin{bmatrix} 2ml^2 & ml^2 \\ ml^2 & ml^2 \end{bmatrix} \begin{pmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{pmatrix} + \begin{bmatrix} 2mlg & 0 \\ 0 & mlg \end{bmatrix} \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

or, $ml^2 \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{pmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{pmatrix} + mlg \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

Now $A = \dfrac{1}{ml^2} \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} mlg \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} = \dfrac{g}{l} \begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix}$

To find eigenvalues of A,

$$|A-\lambda I| = 0 \Rightarrow \begin{vmatrix} 2\frac{g}{l}-\lambda & -\frac{g}{l} \\ -2\frac{g}{l} & 2\frac{g}{l}-\lambda \end{vmatrix} = 0$$

$$\text{Or, } 4\left(\frac{g}{l}\right)^2 - 4\frac{g}{l}\lambda + \lambda^2 - 2\left(\frac{g}{l}\right)^2 = 0$$

$$\text{Or, } \lambda^2 - 4\frac{g}{l}\lambda + 2\left(\frac{g}{l}\right)^2 = 0$$

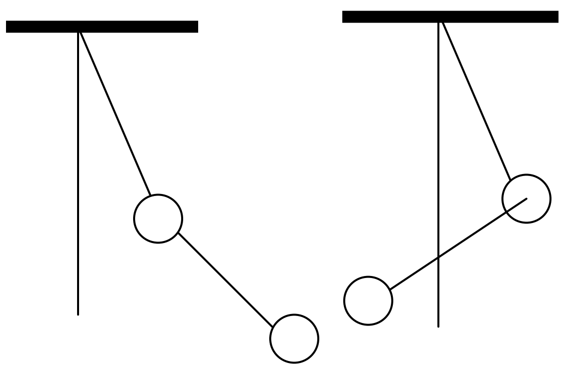
$$\text{Or, } \lambda = \frac{4\frac{g}{l} \pm \sqrt{\left(4\frac{g}{l}\right)^2 - 8\left(\frac{g}{l}\right)^2}}{2} = (2 \pm \sqrt{2})\frac{g}{l}$$

Hence natural frequencies are $\omega_1 = 0.7653\frac{g}{l}, \omega_2 = 1.8478\frac{g}{l}$

The normal modes can be determined from the eigenvalues.
The corresponding principal modes are obtained as

$$\begin{pmatrix}\theta_1 \\ \theta_2\end{pmatrix}_{\lambda=\lambda_1} = \frac{\frac{g}{l}}{(2-2+\sqrt{2})\frac{g}{l}} = \frac{1}{\sqrt{2}}$$

$$\begin{pmatrix}\theta_1 \\ \theta_2\end{pmatrix}_{\lambda=\lambda_2} = \frac{\frac{g}{l}}{(2-2-\sqrt{2})\frac{g}{l}} = -\frac{1}{\sqrt{2}}$$



It may be noted that while in the first mode
Both the pendulum moves in the same direction,

Figure 6

In the second mode they move in opposite direction
One may solve the same problem by taking x_1 and x_2 as the generalized coordinates. Here x_1 is the horizontal distance moves by mass m_1 and x_2 is the distance move by mass m_2. Figure 7 show the free body diagram of both the masses.

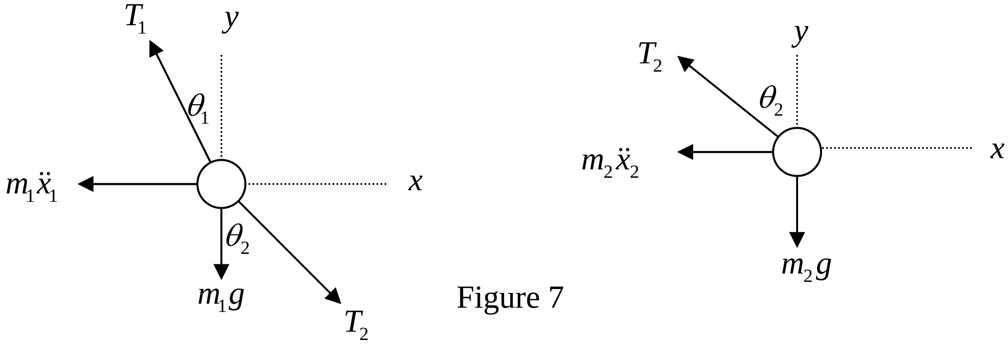


Figure 7

From the free body diagram of mass m_2,

$$T_2 \cos\theta_2 = m_2 g$$
$$T_2 \sin\theta_2 = -m_2 \ddot{x}_2$$

Also from the free body diagram of mass m_1,

$$T_1 \cos\theta_1 - T_2 \cos\theta_2 = m_1 g$$
$$T_1 \sin\theta_1 - T_2 \sin\theta_2 + m_1 \ddot{x}_2 = 0$$

Assuming θ_1 and θ_2 to be small,

$$\sin\theta_1 = \tan\theta_1 = \theta_1 = x_1 / l$$
$$\text{and} \sin\theta_2 = \tan\theta_2 = \theta_2 = (x_2 - x_1)/l$$

Hence

$T_2 = m_2 g$, and $T_1 = (m_1 + m_2) g$

$$m_1 \ddot{x}_1 + \left(\left(\frac{(m_1 + m_2) g}{l_1} \right) + \frac{m_2 g}{l_2} \right) x_1 + \frac{-m_2 g}{l_2} x_2 = 0$$

$$m_2 \ddot{x}_2 + m_2 g \left(\frac{x_2 - x_1}{l_2} \right) = 0$$

Hence in matrix form

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} + \begin{bmatrix} \left(\left(\frac{(m_1 + m_2) g}{l_1} \right) + \frac{m_2 g}{l_2} \right) & \frac{-m_2 g}{l_2} \\ \frac{-m_2 g}{l_2} & \frac{m_2 g}{l_2} \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Considering the case in which $m_1 = m_2 = m$ and $l_1 = l_2 = l$, the above equation becomes

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} + \frac{g}{l} \begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$A = \frac{g}{l}\begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix}$$

Hence and $|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} 3\frac{g}{l} - \lambda & -\frac{g}{l} \\ -\frac{g}{l} & \frac{g}{l} - \lambda \end{vmatrix} = 0$

Or, $\lambda^2 - 4\frac{g}{l}\lambda + 2\left(\frac{g}{l}\right)^2 = 0$

$or, \lambda_1 = \left(2 - \sqrt{2}\right)\frac{g}{l}\, and\; \lambda_2 = \left(2 + \sqrt{2}\right)\frac{g}{l}$

Same as those obtained by taking θ_1 and θ_2 as the generalized coordinates.

Now

$$\left(\frac{X_1}{X_2}\right)_{\lambda=\lambda_1} = \frac{\frac{g}{l}}{3\frac{g}{l} - \lambda_1} = \frac{1}{3 - 2 + \sqrt{2}} = \frac{1}{2.4142} = 0.4142$$

$$\left(\frac{X_1}{X_2}\right)_{\lambda=\lambda_2} = \frac{\frac{g}{l}}{3\frac{g}{l} - \lambda_2} = \frac{1}{3 - 2 - \sqrt{2}} = -\frac{1}{0.4142} = -2.4142$$

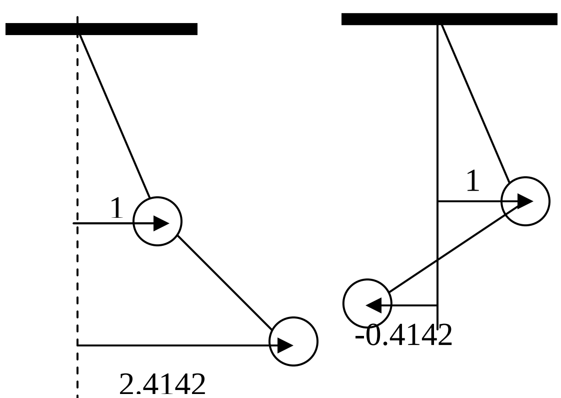


Figure 8

The different modes are as shown in the above figure.

Example Determine the equation of motion if the double pendulum is started with initial conditions $x_1(0) = x_2(0) = 0.5, \dot{x}_1(0) = \dot{x}_2(0) = 0.$

Solution:

The resulting free vibration can be considered to be the superposition of the normal modes. For each of these modes, the time solution can be written as

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}_1 = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}_1 \sin \omega_1 t \quad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}_2 = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}_2 \sin \omega_2 t$$

The general solution for the free vibration can be written as

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = A \begin{pmatrix} 0.4142 \\ 1 \end{pmatrix} \sin(\omega_1 t + \psi_1) + B \begin{pmatrix} -2.4142 \\ 1 \end{pmatrix} \sin(\omega_2 t + \psi_2)$$

where A and B are the amounts of first and second mode's participation and ψ_1 and ψ_2 are the starting values or phases of the two modes. Substituting the initial conditions in the above equation

$$\begin{pmatrix} 0.5 \\ 0.5 \end{pmatrix} = A \begin{pmatrix} 0.4142 \\ 1 \end{pmatrix} \sin \psi_1 + B \begin{pmatrix} -2.4142 \\ 1 \end{pmatrix} \sin \psi_2$$

and

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = A\omega_1 \begin{pmatrix} 0.4142 \\ 1 \end{pmatrix} \cos \psi_1 + B\omega_2 \begin{pmatrix} -2.4142 \\ 1 \end{pmatrix} \cos \psi_2$$

For the second set of equations to be satisfied, $\cos \psi_1 = \cos \psi_2 = 0$, so that $\psi_1 = \psi_2 = 90^0$. Hence $A = 0.6035$ *and* $B = -0.1036$. So the equation for free vibration can be given by

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0.6035 \begin{pmatrix} 0.4142 \\ 1 \end{pmatrix} \cos \omega_1 t - 0.1036 \begin{pmatrix} -2.4142 \\ 1 \end{pmatrix} \cos \omega_2 t$$

Damped-free vibration of two-dof systems

Consider a two degrees of freedom system with damping as shown in figure

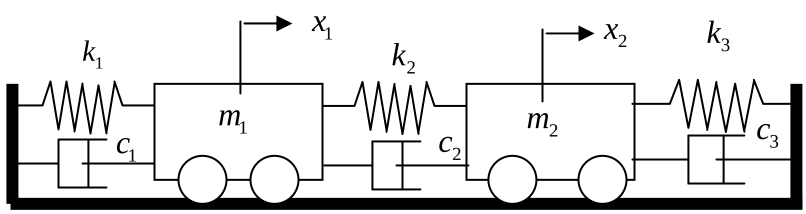


Figure 9

Now the equation of motion of this system can be given by

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{bmatrix} c_1 + c_2 & -c_2 \\ -c_2 & c_2 + c_3 \end{bmatrix} \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (32)$$

As in the previous case, here also the solution of the above equations can be written as

$$x_1 = A_1 e^{st} \text{ and } x_2 = A_2 e^{st} \tag{33}$$

where A_1, A_2 and s are constant . Substituting (33) in (32) , one may write

$$\begin{bmatrix} m_1 s^2 + (c_1 + c_2)s + k_1 + k_2 & -c_2 s - k_2 \\ -c_2 s - k_2 & m_2 s^2 + (c_2 + c_3)s + k_2 + k_3 \end{bmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{34}$$

Now for a nontrivial response i.e., for non-zero values of A_1 and A_2 , the determinant of their coefficient matrix must vanish. Hence

$$\begin{vmatrix} m_1 s^2 + (c_1 + c_2)s + k_1 + k_2 & -c_2 s - k_2 \\ -c_2 s - k_2 & m_2 s^2 + (c_2 + c_3)s + k_2 + k_3 \end{vmatrix} = 0 \tag{35}$$

or, $(m_1 s^2 + (c_1 + c_2)s + k_1 + k_2)(m_2 s^2 + (c_2 + c_3)s + k_2 + k_3) + (c_2 s + k_2)^2 = 0$ (36)

which is a fourth order equation in s and is known as the characteristic equation of the system. This equation is to be solved to get four roots. The general solution of the system can be given by

$$\begin{aligned} x_1 &= A_{11} e^{s_1 t} + A_{12} e^{s_2 t} + A_{13} e^{s_3 t} + A_{14} e^{s_4 t} \\ x_2 &= A_{21} e^{s_1 t} + A_{22} e^{s_2 t} + A_{23} e^{s_3 t} + A_{24} e^{s_4 t} \end{aligned} \tag{37}$$

Here $A_{1i}, i = 1, 2, 3, 4$ are four arbitrary constants to be determined from the initial conditions and the coefficients $A_{2i}, i = 1, 2, 3, 4$ are related to A_{2i} and can be determined from equation (34) as

$$\frac{A_{1i}}{A_{2i}} = \frac{c_2 s_i + k_2}{m_1 s_i^2 + (c_1 + c_2)s_i + k_1 + k_2} \tag{38}$$

For a physical system with damping, the motion will die out with time. For a stable system, all the four roots must be either real negative numbers or complex number with negative real parts. It may be recalled that, if the roots contain complex conjugate numbers, the motion will be oscillatory.

Example: Find the response of the system as shown in figure 9 considering $m_1 = m_2 = m$, $k_1 = k_2 = k_3 = k$ and $c_1 = c_3 = 0$ and $c_2 = c$.

Solution.

In this case the characteristics equation becomes

$$(ms^2 + cs + 2k)(ms^2 + cs + 2k) - (cs + k)^2 = 0$$

$$(ms^2+cs+2k)^2-(cs+k)^2=0$$
$$or, m^2s^4+2mcs^3+(4mk+c^2-c^2)s^2+(4kc-2kc)s+4k^2-k^2=0$$
$$or, m^2s^4+2mcs^3+4mks^2+2kcs+3k^2=0$$
$$or, ms^2(ms^2+2cs+3k)+k(ms^2+2cs+3k)=0$$
$$or, (ms^2+k)(ms^2+2cs+3k)=0$$
$$or, (ms^2+k)(ms^2+2cs+3k)=0$$

Hence the roots are

$$s_{1,2}=\pm i\sqrt{\frac{k}{m}} \text{ and } s_{3,4}=-\frac{c}{m}\pm\sqrt{\left(\frac{c}{m}\right)^2-3\frac{k}{m}}$$

So the system has a pair of complex conjugate

SEMI-DEFINITE SYSTEMS

The systems with have one of their natural frequencies equal to zero are known as semi-definite or degenerate systems. One can show that the following two systems are degenerate systems.

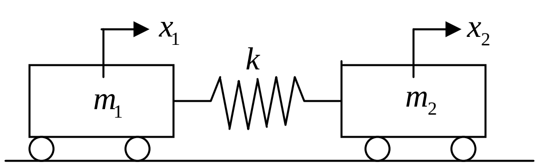


Figure 10

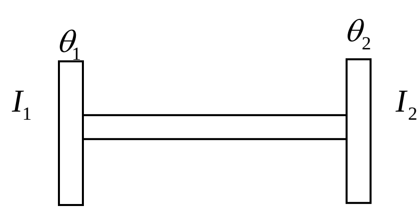


Figure 11

From figure 10 the equation of motion of the system is

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix}+\begin{bmatrix} k & -k \\ -k & k \end{bmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{39}$$

Assuming the solution $x_1 = A_1e^{i\omega t}$ and $x_2 = A_2e^{i\omega t}$ (40)

$$\begin{bmatrix} k-m_1\omega^2 & -k \\ -k & k-m_2\omega^2 \end{bmatrix}\begin{pmatrix} A_1 \\ A_2 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{41}$$

So for non-zero values of A_1, A_2,

$$\begin{vmatrix} k - m_1\omega^2 & -k \\ -k & k - m_2\omega^2 \end{vmatrix} = 0 \tag{42}$$

$$\text{or, } \left(k - m_1\omega^2\right)\left(k - m_2\omega^2\right) - k^2 = 0 \tag{43}$$

$$\text{or, } k^2 - k(m_1 + m_2)\omega^2 + m_1 m_2\omega^4 - k^2 = 0 \tag{44}$$

$$\text{or, } \omega^2(m_1 m_2\omega^2 - k(m_1 + m_2)) = 0 \tag{45}$$

$$\Rightarrow \omega_1 = 0, \text{ and, } \omega_2 = \sqrt{\frac{k(m_1 + m_2)}{m_1 m_2}} \tag{46}$$

Hence, the system is a semi-definite or degenerate system. Corresponding to the first mode frequency, i.e., $\omega_1 = 0,\ A_1 = A_2$. So the system will have a rigid-body motion. For the second mode frequency

$$\frac{A_1}{A_2} = \frac{k}{k - m_1\omega^2} = \frac{k m_1 m_2}{k m_1 m_2 - m_1 k(m_1 + m_2)} = \frac{k m_1 m_2}{-m_1 k m_1} = -\frac{m_2}{m_1} \tag{47}$$

amplitude ratio is inversely proportional to the mass ratio the system.
Similarly one may show for the two-rotor system,

$$\frac{\theta_1}{\theta_2} = -\frac{I_2}{I_1} \tag{48}$$

the ratio of angle of rotation inversely proportional to the moment of inertia of the rotors.

Forced harmonic vibration, Vibration Absorber

Consider a system excited by a harmonic force $F_1 \sin \omega t$ expressed by the matrix equation

$$\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} + \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} F \\ 0 \end{pmatrix}\sin \omega t \tag{49}$$

Since the system is undamped, the solution can be assumed as

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}\sin \omega t \tag{50}$$

Substituting equation (50) in equation (49), one obtains

$$\begin{gathered} \begin{bmatrix} k_{11} - m_{11}\omega^2 & k_{12} - m_{12}\omega^2 \\ k_{21} - m_{21}\omega^2 & k_{22} - m_{22}\omega^2 \end{bmatrix}\begin{pmatrix} X_1 \\ X_2 \end{pmatrix}\sin \omega t = \begin{pmatrix} F_1 \\ 0 \end{pmatrix}\sin \omega t \\ or, \begin{bmatrix} k_{11} - m_{11}\omega^2 & k_{12} - m_{12}\omega^2 \\ k_{21} - m_{21}\omega^2 & k_{22} - m_{22}\omega^2 \end{bmatrix}\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} F \\ 0 \end{pmatrix} \end{gathered} \tag{51}$$

$$\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{bmatrix} k_{11} - m_{11}\omega^2 & k_{12} - m_{12}\omega^2 \\ k_{21} - m_{21}\omega^2 & k_{22} - m_{22}\omega^2 \end{bmatrix}^{-1} \begin{pmatrix} F \\ 0 \end{pmatrix}$$

$$= \frac{\begin{bmatrix} k_{22} - m_{22}\omega^2 & -k_{12} + m_{12}\omega^2 \\ -k_{21} + m_{21}\omega^2 & k_{11} - m_{11}\omega^2 \end{bmatrix} \begin{pmatrix} F \\ 0 \end{pmatrix}}{\begin{vmatrix} k_{11} - m_{11}\omega^2 & k_{12} - m_{12}\omega^2 \\ k_{21} - m_{21}\omega^2 & k_{22} - m_{22}\omega^2 \end{vmatrix}} \tag{52}$$

Hence

$$X_1 = \frac{\left(k_{22} - m_{22}\omega^2\right)F}{|Z(\omega)|}, \tag{53}$$

where $[Z(\omega)] = \begin{bmatrix} k_{11} - m_{11}\omega^2 & k_{12} - m_{12}\omega^2 \\ k_{21} - m_{21}\omega^2 & k_{22} - m_{22}\omega^2 \end{bmatrix}$

$$X_2 = \frac{\left(k_{21} - m_{21}\omega^2\right)F}{|Z(\omega)|} \tag{54}$$

Example Consider the system shown in figure 12 where the mass m_1 is subjected to a force $F \sin \omega t$. Find the response of the system when $m_1 = m_2$ and $k_1 = k_2 = k_3$.

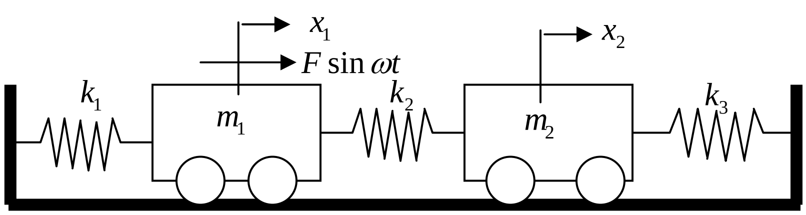


Figure 12

Solution:

The equation of motion of this system can be written as

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} F \sin \omega t \\ 0 \end{pmatrix}$$

$$\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} + \begin{bmatrix} 2k & -k \\ -k & 2k \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} F \sin \omega t \\ 0 \end{pmatrix}$$

So assuming the solution

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \sin \omega t$$ and proceeding as explained before

$$\left[Z(\omega)\right] = \begin{bmatrix} 2k - m\omega^2 & -k \\ -k & 2k - m\omega^2 \end{bmatrix}$$

$$\left|Z(\omega)\right| = \left(2k - m\omega^2\right)^2 - k^2 = m^2\omega^4 - 4mk\omega^2 + 3k^2 = m^2(\omega^4 - 4\frac{k}{m}\omega^2 + 3\frac{k^2}{m^2})$$

or, $\left|Z(\omega)\right| = m^2(\omega^2 - \frac{k}{m})(\omega^2 - 3\frac{k}{m}) = m^2(\omega^2 - \omega_1^2)(\omega^2 - \omega_2^2)$

where, $\omega_1^2 = \frac{k}{m}$ and $\omega_2^2 = 3\frac{k}{m}$ are normal mode frequencies of this system.

Hence,

$$X_1 = \frac{\left(2k - m\omega^2\right)F}{m^2(\omega^2 - \omega_1^2)(\omega^2 - \omega_2^2)}$$

$$X_2 = \frac{kF}{m^2(\omega^2 - \omega_1^2)(\omega^2 - \omega_2^2)}$$

So it may be observed that the system will have maximum vibration when $\omega = \omega_1\ or, \omega = \omega_2$. Also it may be observed that $X_1 = 0$, when $\omega^2 = 2k/m$.

Tuned Vibration Absorber

Consider a vibrating system of mass m_1, stiffness k_1, subjected to a force $F \sin \omega t$. As studied in case of forced vibration of single-degree of freedom system, the system will have a steady state response given by

$$x = \frac{F \sin \omega t}{m(\omega_n^2 - \omega^2)}, \text{ where } \omega_n = \sqrt{k_1/m_1} \qquad (55)$$

which will be maximum when $\omega = \omega_{n.}$ Now to absorb this vibration, one may add a secondary spring and mass system as shown in figure 13.

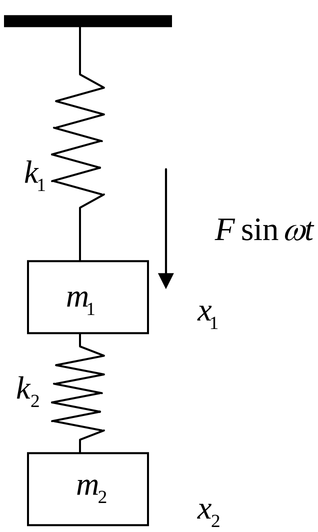


Figure 13

The equation of motion for this system can be given by

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix}+\begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}=\begin{pmatrix} F\sin\omega t \\ 0 \end{pmatrix} \qquad (56)$$

Comparing equation (49) and (56),

$m_{11}=m_1,\ m_{12}=0,\ m_{21}=0,\ m_{22}=m_{2,}\ k_{11}=k_1+k_2,\ k_{12}=-k_2,\ k_{21}=-k_2$, and $k_{22}=k$.

Hence,

$$|Z(\omega)|=\begin{vmatrix} k_1+k_2-m_1\omega^2 & -k_2 \\ -k_2 & k_2-m_2\omega^2 \end{vmatrix}=k_1k_2-m_1k_2\omega^2-k_1m_2\omega^2-k_2m_2\omega^2+m_1m_2\omega^4$$

$$=m_1m_2(\lambda_1-\omega^2)(\lambda_2-\omega^2) \qquad (57)$$

where λ_1 and λ_2 are the roots of the characteristic equation $|Z(\omega)|=0$ of the free-vibration of this system., which can be given by

$$\lambda_{1,2}=0.5\left\{\left(\frac{k_1}{m_1}+\frac{k_2}{m_2}+\frac{k_2}{m_1}\right)\pm\sqrt{\left(\frac{k_1}{m_1}+\frac{k_2}{m_2}+\frac{k_2}{m_1}\right)^2-4\frac{k_1k_2}{m_1m_2}}\right\} \qquad (58)$$

Now from equation (53) and (54)

$$X_1=\frac{\left(k_{22}-m_{22}\omega^2\right)F}{|Z(\omega)|}=\frac{\left(k_2-m_2\omega^2\right)F}{|Z(\omega)|}, \qquad (59)$$

$$X_2=\frac{-k_2F}{|Z(\omega)|} \qquad (60)$$

From equation (59), it is clear that, $X_1=0,$ when $\omega^2=\frac{k_2}{m_2}$. Hence, by suitably choosing the stiffness and mass of the secondary spring and mass system, vibration can be completely eliminated from the primary system. For $\omega^2=\frac{k_2}{m_2}$,

$$|Z(\omega)|=k_1k_2-m_1k_2\frac{k_2}{m_2}-k_1m_2\frac{k_2}{m_2}-k_2m_2\frac{k_2}{m_2}+m_1m_2\frac{k_2}{m_2}\frac{k_2}{m_2}$$

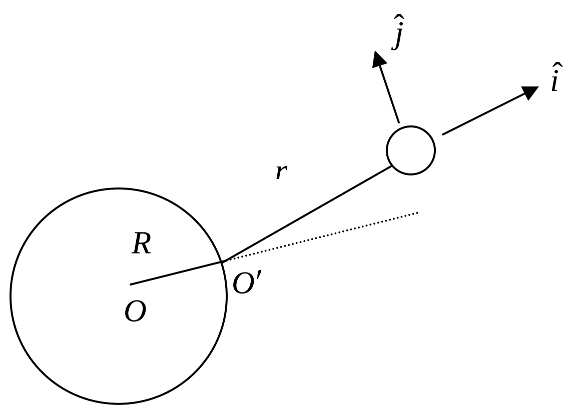
$$=k_1k_2-m_1\frac{k_2^2}{m_2}-k_1k_2-k_2^2+m_1\frac{k_2^2}{m_2}=-k_2^2 \qquad (61)$$

and $X_2=\frac{-k_2F}{-k_2^2}=\frac{F}{k_2}$ $\qquad$ (62)

Centrifugal Pendulum Vibration Absorber

The tuned vibration absorber is only effective when the frequency of external excitation equals to the natural frequency of the secondary spring and mass system. But in many cases, for example in case of an automobile engine, the exciting torques are proportional to the rotational speed 'n' which may vary over a wide range. For the absorber to be effective, its natural frequency must also be proportional to the speed. The characteristics of the centrifugal pendulum are ideally suited for this purpose.

Placing the coordinates through point O', parallel and normal to r, the line r rotates with angular velocity ($\dot{\theta}+\dot{\phi}$)



The acceleration of mass m

$$a_m = \left[-R\dot{\theta}^2\cos\phi + R\ddot{\theta}\sin\phi - r(\dot{\theta}+\dot{\phi})^2\right]\hat{i} + \left[R\dot{\theta}^2\sin\phi + R\ddot{\theta}\cos\phi + r(\ddot{\theta}+\ddot{\phi})^2\right]\hat{j} \quad (63)$$

Since the moment about O' is zero,

$$M_{O'} = m\left[R\dot{\theta}^2\sin\phi + R\ddot{\theta}\cos\phi + r(\ddot{\theta}+\ddot{\phi})^2\right]r = 0 \quad (64)$$

Assuming ϕ to be small, $\cos\phi = 1, \sin\phi = \phi$, so

$$\ddot{\phi} + \left(\frac{R}{r}\dot{\theta}^2\right)\phi = -\left(\frac{R+r}{r}\right)\ddot{\theta} \quad (65)$$

If we assume the motion of the wheel to be a steady rotation n plus a small sinusoidal oscillation of frequency ω, one may write

$$\theta = nt + \theta_0 \sin\omega t \quad (66)$$

$$\dot{\theta} = nt + \omega\theta_0 \cos\omega t \cong n \quad (67)$$

$$\ddot{\theta} = -\theta_0\omega^2 \sin\omega t \quad (68)$$

Substituting the above equations in equation (65) yields,

$$\ddot{\phi}+\left(\frac{R}{r}n^2\right)\phi=\left(\frac{R+r}{r}\right)\omega^2\theta_0\sin\omega t \tag{69}$$

Hence the natural frequency of the pendulum is

$$\omega_n=n\sqrt{\frac{R}{r}} \tag{70}$$

and its steady-state solution is

$$\phi=\frac{(R+r)/r}{-\omega^2+(Rn^2/r)}\omega^2\theta_0\sin\omega t \tag{71}$$

It may be noted that the same pendulum in a gravity field would have a natural frequency of $\sqrt{\frac{g}{r}}$. So it may be noted that for the centrifugal pendulum the gravity field is replaced by the centrifugal field Rn^2.

Torque exerted by the pendulum on the wheel

With the $\hat{j}$ component of a_m equal to zero, the pendulum force is a tension along r, given by m times the $\hat{i}$ component of a_m.

$$\begin{aligned}T&=\left(R\cos\phi\,\hat{i}+R\sin\phi\,\hat{j}\right)\times m\left[-R\dot{\theta}^2\cos\phi+R\ddot{\theta}\sin\phi-r(\dot{\theta}+\dot{\phi})^2\right]\hat{i}\\&=-mR\phi\left[-R\omega^2\theta_0\sin\omega t\sin\phi-Rn^2-rn^2-r\dot{\phi}^2-2r\dot{\theta}\dot{\phi}\right]\end{aligned} \tag{72}$$

Now assuming small angle of rotation

$$T=-m\left(R+r\right)n^2R\phi \tag{73}$$

Now substituting the (73) in (72),

$$\begin{aligned}T&=\frac{-mR(R+r)^2n^2/r}{(Rn^2/r)-\omega^2}\omega^2\theta_0\sin\omega t\\&=-\left[\frac{m(R+r)^2}{1-r\omega^2/Rn^2}\right]\ddot{\theta}=J_{eff}\ddot{\theta}\end{aligned} \tag{74}$$

Hence the effective inertia can be written as

$$J_{eff}=-\left[\frac{m(R+r)^2}{1-r\omega^2/Rn^2}\right]=-\frac{m(R+r)^2}{1-\left(\omega/\omega_n\right)^2} \tag{75}$$

which can be ∞ at its natural frequency. This possesses some difficulties in the design of the pendulum. For example to suppress a disturbing torque of frequency equal to four times the natural speed *n*, the pendulum must meet the requirement $\omega^2=(4n)^2=n^2R/r$. Hence, as the length of the pendulum $r=R/16$ becomes very small it will be difficult to design it. To avoid this one may go for Chilton bifilar design.

Exercise problems

1. In a certain refrigeration plant, a section of pipe carrying the refrigerant vibrated violently at a compressor speed of 232 rpm. To eliminate this difficulty, it was proposed to clamp a cantilever spring mass system to the pipe to act as an absorber. For a trial test, for a 905 gm. Absorber tuned to 232 cpm resulted in two natural frequencies of 198 and 272 cpm. If the absorber system is to be designed so that the natural frequencies lie outside the region 160 to 320 cpm, what must be the weight and spring stiffness?
2. Derive the normal modes of vibration of a double pendulum with same length and mass of the pendulum.
3. Develop a matlab code for determination of free-vibration of a general two-degree of freedom system.
4. Derive the equation of motion for the double pendulum shown in figure p1 in terms of θ_1 and θ_2 using Lagrange principle. Determine the natural frequencies and mode shapes of the systems. If the system is started with the following initial conditions: $x_1(0) = x_2(0) = X$, $v_1(0) = v_2(0) = 0$, (v_1 and v_2 are velocity) determine the equation of motion. If the lower mass is given an impulse $F_0\,\delta(t)$, determine the response in terms of normal modes.

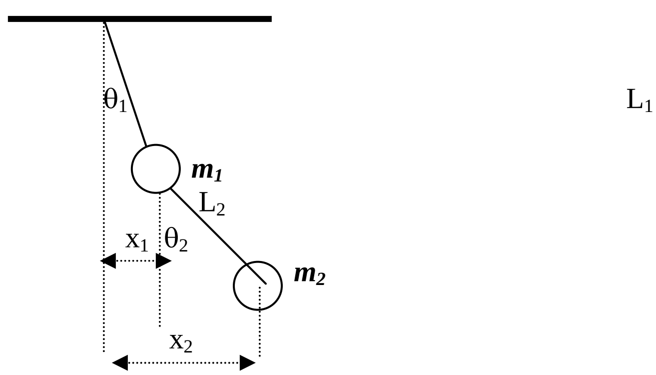


Figure P1

5. A centrifugal pump rotating at 500 rpm is driven by an electric motor at 1200 rpm through a single stage reduction gearing. The moments of inertia of the pump impeller and the motor are 1600 kg.m^2 and 500 kg.m^2 respectively. The lengths of the pump shaft and the motor shaft are 450 and 200 mm, and their diameters are 100 and 50 mm respectively. Neglecting the inertia of the gears, find the frequencies of torsional oscillations of the system. Also determine the position of the nodes.