Nathan Newmark (1910–1981) was an American engineer and a professor of civil engineering at the University of Illinois at Champaign-Urbana. His research in earthquake resistant structures and structural dynamics is widely known. The numerical method he presented in 1959 for the dynamic response computation of linear and nonlinear systems is known as the Newmark β-method. (Courtesy of University of Illinois Urbane-Champaign).

CHAPTER 11

Numerical Integration Methods in Vibration Analysis

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When the differential equation governing the free or forced vibration of a system cannot be integrated in closed form, a numerical approach is to be used for the vibration analysis. The finite difference method, which is based on the approximation of the derivatives appearing in the equation of motion and the boundary conditions, is presented. Specifically, the central difference method is outlined for both single- and multidegree-of-freedom systems using the central difference method. The free-vibration solution of continuous systems is also considered using the finite difference method in the context of the longitudinal vibration of bars and transverse vibration of beams with different boundary conditions. The fourth-order Runge-Kutta method is presented for the solution of differential equations governing the vibration of single- and multidegree-of-freedom systems. The Houbolt, Wilson, and Newmark methods are presented for the general solution of multidegree-of-freedom systems. Finally, Matlab programs are given for the solution of multidegree-of-freedom systems with several numerical examples.

Learning Objectives

After you have finished studying this chapter, you should be able to do the following:

- Use the finite difference method for the solution of single- and multidegree-of-freedom vibration problems.
- Solve the vibration problems of continuous systems using the finite difference method.
- Solve the differential equations associated with discrete (multidegree-of-freedom) systems using the fourth-order Runge-Kutta, Houbolt, Wilson, and Newmark methods.
- Use MATLAB functions for solving discrete and continuous vibration problems.

11.1 Introduction

When the differential equation of motion of a vibrating system cannot be integrated in closed form, a numerical approach must be used. Several numerical methods are available for the solution of vibration problems [11.1–11.3].[1] Numerical integration methods have two fundamental characteristics. First, they are not intended to satisfy the governing differential equation(s) at all time t but only at discrete time intervals Δt apart. Second, a suitable type of variation of the displacement x, velocity $\dot{x}$, and acceleration $\ddot{x}$ is assumed within each time interval Δt. Different numerical integration methods can be obtained, depending on the type of variation assumed for the displacement, velocity, and acceleration, within each time interval Δt. We shall assume that the values of x and $\dot{x}$ are known to be x_0 and $\dot{x}_0$, respectively, at time $t = 0$ and that the solution of the problem is required from $t = 0$ to $t = T$. In the following, we subdivide the time duration T into n equal steps Δt so that $\Delta t = T/n$ and seek the solution at $t_0 = 0,\ t_1 = \Delta t,\ t_2 = 2\,\Delta t,\ \ldots,\ t_n = n\,\Delta t = T$. We shall derive formulas for finding the solution at $t_i = i\,\Delta t$ from the known solution at $t_{i-1} = (i-1)\,\Delta t$ according to five different numerical integration

[1]A numerical procedure using different types of interpolation functions for approximating the forcing function $F(t)$ was presented in Section 4.9.

schemes: (1) the finite difference method, (2) the Runge-Kutta method, (3) the Houbolt method, (4) the Wilson method, and (5) the Newmark method. In the finite difference and Runge-Kutta methods, the current displacement (solution) is expressed in terms of the previously determined values of displacement, velocity, and acceleration, and the resulting equations are solved to find the current displacement. These methods fall under the category of explicit integration methods. In the Houbolt, Wilson, and Newmark methods, the temporal difference equations are combined with the current equations of motion, and the resulting equations are solved to find the current displacement. These methods belong to the category of implicit integration methods.

11.2 Finite Difference Method

The main idea in the finite difference method is to use approximations to derivatives. Thus the governing differential equation of motion and the associated boundary conditions, if applicable, are replaced by the corresponding finite difference equations. Three types of formulas—forward, backward, and central difference formulas—can be used to derive the finite difference equations [11.4–11.6]. We shall consider only the central difference formulas in this chapter, since they are most accurate.

In the finite difference method, we replace the solution domain (over which the solution of the given differential equation is required) with a finite number of points, referred to as *mesh* or *grid points,* and seek to determine the values of the desired solution at these points. The grid points are usually considered to be equally spaced along each of the independent coordinates (see Fig. 11.1). By using Taylor's series expansion, x_{i+1} and x_{i-1} can be expressed about the grid point i as

$$x_{i+1} = x_i + h\dot{x}_i + \frac{h^2}{2}\ddot{x}_i + \frac{h^3}{6}\dddot{x}_i + \cdots \tag{11.1}$$

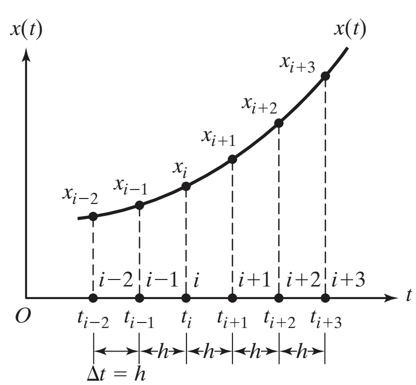


FIGURE 11.1 Grid points.

$$x_{i-1} = x_i - h\dot{x}_i + \frac{h^2}{2}\ddot{x}_i - \frac{h^3}{6}\dddot{x}_i + \cdots \tag{11.2}$$

where $x_i = x(t = t_i)$ and $h = t_{i+1} - t_i = \Delta t$. By taking two terms only and subtracting Eq. (11.2) from Eq. (11.1), we obtain the central difference approximation to the first derivative of x at $t = t_i$:

$$\dot{x}_i = \left.\frac{dx}{dt}\right|_{t_i} = \frac{1}{2h}(x_{i+1} - x_{i-1}) \tag{11.3}$$

By taking terms up to the second derivative and adding Eqs. (11.1) and (11.2), we obtain the central difference formula for the second derivative:

$$\ddot{x}_i = \left.\frac{d^2x}{dt^2}\right|_{t_i} = \frac{1}{h^2}(x_{i+1} - 2x_i + x_{i-1}) \tag{11.4}$$

11.3 Central Difference Method for Single-Degree-of-Freedom Systems

The governing equation of a viscously damped single-degree-of-freedom system is

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = F(t) \tag{11.5}$$

Let the duration over which the solution of Eq. (11.5) is required be divided into n equal parts of interval $h = \Delta t$ each. To obtain a satisfactory solution, we must select a time step Δt that is smaller than a critical time step Δt_{cri}.[2] Let the initial conditions be given by $x(t = 0) = x_0$ and $\dot{x}(t = 0) = \dot{x}_0$.

Replacing the derivatives by the central differences and writing Eq. (11.5) at grid point i gives

$$m\left\{\frac{x_{i+1} - 2x_i + x_{i-1}}{(\Delta t)^2}\right\} + c\left\{\frac{x_{i+1} - x_{i-1}}{2\,\Delta t}\right\} + kx_i = F_i \tag{11.6}$$

[2]Numerical methods that require the use of a time step (Δt) smaller than a critical time step (Δt_{cri}) are said to be *conditionally stable* [11.7]. If Δt is taken to be larger than Δt_{cri}, the method becomes unstable. This means that the truncation of higher-order terms in the derivation of Eqs. (11.3) and (11.4) (or rounding-off in the computer) causes errors that grow and make the response computations worthless in most cases. The critical time step is given by $\Delta t_{\text{cri}} = \tau_n/\pi$, where τ_n is the natural period of the system or the smallest such period in the case of a multidegree-of-freedom system [11.8]. Naturally, the accuracy of the solution always depends on the size of the time step. By using an unconditionally stable method, we can choose the time step with regard to accuracy only, not with regard to stability. This usually allows a much larger time step to be used for any given accuracy.

where $x_i = x(t_i)$ and $F_i = F(t_i)$. Solution of Eq. (11.6) for x_{i+1} yields

$$x_{i+1} = \left\{ \frac{1}{\frac{m}{(\Delta t)^2} + \frac{c}{2\,\Delta t}} \right\} \left[\left\{ \frac{2m}{(\Delta t)^2} - k \right\} x_i + \left\{ \frac{c}{2\,\Delta t} - \frac{m}{(\Delta t)^2} \right\} x_{i-1} + F_i \right] \tag{11.7}$$

This is called the *recurrence formula*. It permits us to calculate the displacement of the mass (x_{i+1}) if we know the previous history of displacements at t_i and t_{i-1}, as well as the present external force F_i. Repeated application of Eq. (11.7) yields the complete time history of the behavior of the system. Note that the solution of x_{i+1} is based on the use of the equilibrium equation at time t_i—that is, Eq. (11.6). For this reason, this integration procedure is called an *explicit integration method*. Certain care has to be exercised in applying Eq. (11.7) for $i = 0$. Since both x_0 and x_{-1} are needed in finding x_1, and the initial conditions provide only the values of x_0 and $\dot{x}_0$, we need to find the value of x_{-1}. Thus the method is not self-starting. However, we can generate the value of x_{-1} by using Eqs. (11.3) and (11.4) as follows. By substituting the known values of x_0 and $\dot{x}_0$ into Eq. (11.5), $\ddot{x}_0$ can be found:

$$\ddot{x}_0 = \frac{1}{m}[F(t = 0) - c\dot{x}_0 - kx_0] \tag{11.8}$$

Application of Eqs. (11.3) and (11.4) at $i = 0$ yields the value of x_{-1}:

$$x_{-1} = x_0 - \Delta t \dot{x}_0 + \frac{(\Delta t)^2}{2}\ddot{x}_0 \tag{11.9}$$

EXAMPLE 11.1

Response of Single-Degree-of-Freedom System

Find the response of a viscously damped single-degree-of-freedom system subjected to a force

$$F(t) = F_0\left(1 - \sin\frac{\pi t}{2t_0}\right)$$

with the following data: $F_0 = 1$, $t_0 = \pi$, $m = 1$, $c = 0.2$, and $k = 1$. Assume the values of the displacement and velocity of the mass at $t = 0$ to be zero.

Solution: The governing differential equation is

$$m\ddot{x} + c\dot{x} + kx = F(t) = F_0\left(1 - \sin\frac{\pi t}{2t_0}\right) \tag{E.1}$$

The finite difference solution of Eq. (E.1) is given by Eq. (11.7). Since the initial conditions are $x_0 = \dot{x}_0 = 0$, Eq. (11.8) yields $\ddot{x}_0 = 1$; hence Eq. (11.9) gives $x_{-1} = (\Delta t)^2/2$. Thus the solution of Eq. (E.1) can be found from the recurrence relation

$$x_{i+1} = \frac{1}{\left[\dfrac{m}{(\Delta t)^2} + \dfrac{c}{2\,\Delta t}\right]}\left[\left\{\frac{2m}{(\Delta t)^2} - k\right\}x_i + \left\{\frac{c}{2\,\Delta t} - \frac{m}{(\Delta t)^2}\right\}x_{i-1} + F_i\right], \qquad i = 0, 1, 2, \ldots \tag{E.2}$$

with $x_0 = 0$, $x_{-1} = (\Delta t)^2/2$, $x_i = x(t_i) = x(i\,\Delta t)$, and

$$F_i = F(t_i) = F_0\left(1 - \sin\frac{i\pi\,\Delta t}{2t_0}\right)$$

The undamped natural frequency and the natural period of the system are given by

$$\omega_n = \left(\frac{k}{m}\right)^{1/2} = 1 \tag{E.3}$$

and

$$\tau_n = \frac{2\pi}{\omega_n} = 2\pi \tag{E.4}$$

Thus the time step Δt must be less than $\tau_n/\pi = 2.0$. We shall find the solution of Eq. (E.1) by using the time steps $\Delta t = \tau_n/40$, $\tau_n/20$, and $\tau_n/2$. The time step $\Delta\tau = \tau_n/2 > \Delta t_{\text{cri}}$ is used to illustrate the unstable (diverging) behavior of the solution. The values of the response x_i obtained at different instants of time t_i are shown in Table 11.1.

This example can be seen to be identical to Example 4.17. The results obtained by idealization 4 (piecewise linear type interpolation) of Example 4.17 are shown in Table 11.1 up to time $t_i = \pi$ in the last column of the table. It can be observed that the finite difference method gives reasonably accurate results with time steps $\Delta t = \tau_n/40$ and $\tau_n/20$ (which are smaller than Δt_{cri}) but gives diverging results with $\Delta\tau = \tau_n/2$ (which is larger than Δt_{cri}).

■

TABLE 11.1 Comparison of Solutions of Example 11.1

	Values of $x_i = x(t_i)$ Obtained with			
Time (t_i)	$\Delta t = \frac{\tau_n}{40}$	$\Delta t = \frac{\tau_n}{20}$	$\Delta t = \frac{\tau_n}{2}$	**Value of x_i Given by Idealization 4 of Example 4.31**
0	0.00000	0.00000	0.00000	0.00000
$\pi/10$	0.04638	0.04935	—	0.04541
$2\pi/10$	0.16569	0.17169	—	0.16377
$3\pi/10$	0.32767	0.33627	—	0.32499
$4\pi/10$	0.50056	0.51089	—	0.49746
$5\pi/10$	0.65456	0.66543	—	0.65151
$6\pi/10$	0.76485	0.77491	—	0.76238
$7\pi/10$	0.81395	0.82185	—	0.81255
$8\pi/10$	0.79314	0.79771	—	0.79323
$9\pi/10$	0.70297	0.70340	—	0.70482
π	0.55275	0.54869	4.9348	0.55647
2π	0.19208	0.19898	−29.551	—
3π	2.7750	2.7679	181.90	—
4π	0.83299	0.83852	−1058.8	—
5π	−0.05926	−0.06431	6253.1	—

11.4 Runge-Kutta Method for Single-Degree-of-Freedom Systems

In the Runge-Kutta method, the approximate formula used for obtaining x_{i+1} from x_i is made to coincide with the Taylor's series expansion of x at x_{i+1} up to terms of order $(\Delta t)^n$. The Taylor's series expansion of $x(t)$ at $t + \Delta t$ is given by

$$x(t + \Delta t) = x(t) + \dot{x}\,\Delta t + \ddot{x}\frac{(\Delta t)^2}{2!} + \dddot{x}\frac{(\Delta t)^3}{3!} + \ddddot{x}\frac{(\Delta t)^4}{4!} + \cdots \tag{11.10}$$

In contrast to Eq. (11.10), which requires higher-order derivatives, the Runge-Kutta method does not require explicitly derivatives beyond the first [11.9–11.11]. For the solution of a second-order differential equation, we first reduce it to two first-order equations. For example, Eq. (11.5) can be rewritten as

$$\ddot{x} = \frac{1}{m}[F(t) - c\dot{x} - kx] = f(x, \dot{x}, t) \tag{11.11}$$

By defining $x_1 = x$ and $x_2 = \dot{x}$, Eq. (11.11) can be written as two first-order equations:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= f(x_1, x_2, t) \end{aligned} \tag{11.12}$$

By defining

$$\vec{X}(t) = \begin{Bmatrix} x_1(t) \\ x_2(t) \end{Bmatrix} \quad \text{and} \quad \vec{F}(t) = \begin{Bmatrix} x_2 \\ f(x_1, x_2, t) \end{Bmatrix}$$

the following recurrence formula is used to find the values of $\vec{X}(t)$ at different grid points t_i according to the fourth-order Runge-Kutta method

$$\vec{X}_{i+1} = \vec{X}_i + \tfrac{1}{6}[\vec{K}_1 + 2\vec{K}_2 + 2\vec{K}_3 + \vec{K}_4] \tag{11.13}$$

where

$$\vec{K}_1 = h\vec{F}(\vec{X}_i, t_i) \tag{11.14}$$

$$\vec{K}_2 = h\vec{F}(\vec{X}_i + \tfrac{1}{2}\vec{K}_1, t_i + \tfrac{1}{2}h) \tag{11.15}$$

$$\vec{K}_3 = h\vec{F}(\vec{X}_i + \tfrac{1}{2}\vec{K}_2, t_i + \tfrac{1}{2}h) \tag{11.16}$$

$$\vec{K}_4 = h\vec{F}(\vec{X}_i + \vec{K}_3, t_{i+1}) \tag{11.17}$$

The method is stable and self-starting—that is, only the function values at a single previous point are required to find the function value at the current point.

EXAMPLE 11.2 Response of Single-Degree-of-Freedom System

Find the solution of Example 11.1 using the Runge-Kutta method.

Solution: We use a step size of $\Delta t = 0.3142$ and define

$$\vec{X}(t) = \begin{Bmatrix} x_1(t) \\ x_2(t) \end{Bmatrix} = \begin{Bmatrix} x(t) \\ \dot{x}(t) \end{Bmatrix}$$

and

$$\vec{F}(t) = \begin{Bmatrix} x_2 \\ f(x_1, x_2, t) \end{Bmatrix} = \begin{Bmatrix} \dot{x}(t) \\ \dfrac{1}{m}\left[F_0\left(1 - \sin\dfrac{\pi t}{2t_0}\right) - c\dot{x}(t) - kx(t)\right] \end{Bmatrix}$$

From the known initial conditions, we have

$$\vec{X}_0 = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

The values of $\vec{X}_{i+1}$, $i = 0, 1, 2, \ldots$ obtained according to Eq. (11.13) are shown in Table 11.2.

TABLE 11.2

Step i	Time t_i	$x_1 = x$	$x_2 = \dot{x}$
1	0.3142	0.045406	0.275591
2	0.6283	0.163726	0.461502
3	0.9425	0.324850	0.547296
⋮			
19	5.9690	−0.086558	0.765737
20	6.2832	0.189886	0.985565

■

11.5 Central Difference Method for Multidegree-of-Freedom Systems

The equation of motion of a viscously damped multidegree-of-freedom system (see Eq. (6.119)) can be expressed as

$$[m]\ddot{\vec{x}} + [c]\dot{\vec{x}} + [k]\vec{x} = \vec{F} \tag{11.18}$$

where $[m]$, $[c]$, and $[k]$ are the mass, damping, and stiffness matrices, $\vec{x}$ is the displacement vector, and $\vec{F}$ is the force vector. The procedure indicated for the case of a single-degree-of-freedom system can be directly extended to this case [11.12, 11.13]. The central difference formulas for the velocity and acceleration vectors at time $t_i = i\,\Delta t$ ($\dot{\vec{x}}_i$ and $\ddot{\vec{x}}_i$) are given by

$$\dot{\vec{x}}_i = \frac{1}{2\,\Delta t}(\vec{x}_{i+1} - \vec{x}_{i-1}) \tag{11.19}$$

$$\ddot{\vec{x}}_i = \frac{1}{(\Delta t)^2}(\vec{x}_{i+1} - 2\vec{x}_i + \vec{x}_{i-1}) \tag{11.20}$$

which are similar to Eqs. (11.3) and (11.4). Thus the equation of motion, Eq. (11.18), at time t_i can be written as

$$[m]\frac{1}{(\Delta t)^2}(\vec{x}_{i+1} - 2\vec{x}_i + \vec{x}_{i-1}) + [c]\frac{1}{2\,\Delta t}(\vec{x}_{i+1} - \vec{x}_{i-1}) + [k]\vec{x}_i = \vec{F}_i \tag{11.21}$$

where $\vec{x}_{i+1} = \vec{x}(t = t_{i+1})$, $\vec{x}_i = \vec{x}(t = t_i)$, $\vec{x}_{i-1} = \vec{x}(t = t_{i-1})$, $\vec{F}_i = \vec{F}(t = t_i)$, and $t_i = i\,\Delta t$. Equation (11.21) can be rearranged to obtain

$$\left(\frac{1}{(\Delta t)^2}[m] + \frac{1}{2\,\Delta t}[c]\right)\vec{x}_{i+1} + \left(-\frac{2}{(\Delta t)^2}[m] + [k]\right)\vec{x}_i$$

$$+ \left(\frac{1}{(\Delta t)^2}[m] - \frac{1}{2\,\Delta t}[c]\right)\vec{x}_{i-1} = \vec{F}_i$$

or

$$\left(\frac{1}{(\Delta t)^2}[m] + \frac{1}{2\,\Delta t}[c]\right)\vec{x}_{i+1} = \vec{F}_i - \left([k] - \frac{2}{(\Delta t)^2}[m]\right)\vec{x}_i$$

$$-\left(\frac{1}{(\Delta t)^2}[m] - \frac{1}{2\,\Delta t}[c]\right)\vec{x}_{i-1} \tag{11.22}$$

Thus Eq. (11.22) gives the solution vector $\vec{x}_{i+1}$ once $\vec{x}_i$ and $\vec{x}_{i-1}$ are known. Since Eq. (11.22) is to be used for $i = 1, 2, \ldots, n$, the evaluation of $\vec{x}_1$ requires $\vec{x}_0$ and $\vec{x}_{-1}$. Thus a special starting procedure is needed to find $\vec{x}_{-1} = \vec{x}(t = -\Delta t)$. For this, Eqs. (11.18) to (11.20) are evaluated at $i = 0$ to obtain

$$[m]\ddot{\vec{x}}_0 + [c]\dot{\vec{x}}_0 + [k]\vec{x}_0 = \vec{F}_0 = \vec{F}(t = 0) \tag{11.23}$$

$$\dot{\vec{x}}_0 = \frac{1}{2\,\Delta t}(\vec{x}_1 - \vec{x}_{-1}) \tag{11.24}$$

$$\ddot{\vec{x}}_0 = \frac{1}{(\Delta t)^2}(\vec{x}_1 - 2\vec{x}_0 + \vec{x}_{-1}) \tag{11.25}$$

Equation (11.23) gives the initial acceleration vector as

$$\ddot{\vec{x}}_0 = [m]^{-1}(\vec{F}_0 - [c]\dot{\vec{x}}_0 - [k]\vec{x}_0) \tag{11.26}$$

and Eq. (11.24) gives the displacement vector at t_1 as

$$\vec{x}_1 = \vec{x}_{-1} + 2\,\Delta t\dot{\vec{x}}_0 \tag{11.27}$$

Substituting Eq. (11.27) for $\vec{x}_1$, Eq. (11.25) yields

$$\ddot{\vec{x}}_0 = \frac{2}{(\Delta t)^2}[\Delta t \dot{\vec{x}}_0 - \vec{x}_0 + \vec{x}_{-1}]$$

or

$$\vec{x}_{-1} = \vec{x}_0 - \Delta t \dot{\vec{x}}_0 + \frac{(\Delta t)^2}{2}\ddot{\vec{x}}_0 \tag{11.28}$$

where $\ddot{\vec{x}}_0$ is given by Eq. (11.26). Thus $\vec{x}_{-1}$ needed for applying Eq. (11.22) at $i = 1$ is given by Eq. (11.28). The computational procedure can be described by the following steps.

1. From the known initial conditions $\vec{x}(t = 0) = \vec{x}_0$ and $\dot{\vec{x}}(t = 0) = \dot{\vec{x}}_0$, compute $\ddot{\vec{x}}(t = 0) = \ddot{\vec{x}}_0$ using Eq. (11.26).
2. Select a time step Δt such that $\Delta t < \Delta t_{\text{cri}}$.
3. Compute $\vec{x}_{-1}$ using Eq. (11.28).
4. Find $\vec{x}_{i+1} = \vec{x}(t = t_{i+1})$, starting with $i = 0$, from Eq. (11.22), as

$$\vec{x}_{i+1} = \left[\frac{1}{(\Delta t)^2}[m] + \frac{1}{2\,\Delta t}[c]\right]^{-1}\left[\vec{F}_i - \left([k] - \frac{2}{(\Delta t)^2}[m]\right)\vec{x}_i - \left(\frac{1}{(\Delta t)^2}[m] - \frac{1}{2\,\Delta t}[c]\right)\vec{x}_{i-1}\right] \tag{11.29}$$

where

$$\vec{F}_i = (t = t_i) \tag{11.30}$$

If required, evaluate accelerations and velocities at t_i:

$$\ddot{\vec{x}}_i = \frac{1}{(\Delta t)^2}[\vec{x}_{i+1} - 2\vec{x}_i + \vec{x}_{i-1}] \tag{11.31}$$

and

$$\dot{\vec{x}}_i = \frac{1}{2\,\Delta t}[\vec{x}_{i+1} - \vec{x}_{i-1}] \tag{11.32}$$

Repeat Step 4 until $\vec{x}_{n+1}$ (with $i = n$) is determined. The stability of the finite difference scheme for solving matrix equations is discussed in reference [11.14].

EXAMPLE 11.3 Central Difference Method for a Two-Degree-of-Freedom System

Find the response of the two-degree-of-freedom system shown in Fig. 11.2 when the forcing functions are given by $F_1(t) = 0$ and $F_2(t) = 10$. Assume the value of c as zero and the initial conditions as $\vec{x}(t = 0) = \dot{\vec{x}}(t = 0) = \vec{0}$.

Solution:

Approach: Use $\Delta t = \tau/10$, where τ is the smallest time period in the central difference method.

The equations of motion are given by

$$[m]\ddot{\vec{x}}(t) + [c]\dot{\vec{x}}(t) + [k]\vec{x}(t) = \vec{F}(t) \tag{E.1}$$

where

$$[m] = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \tag{E.2}$$

$$[c] = \begin{bmatrix} c & -c \\ -c & c \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{E.3}$$

$$[k] = \begin{bmatrix} k_1 + k & -k \\ -k & k + k_2 \end{bmatrix} = \begin{bmatrix} 6 & -2 \\ -2 & 8 \end{bmatrix} \tag{E.4}$$

$$\vec{F}(t) = \begin{Bmatrix} F_1(t) \\ F_2(t) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 10 \end{Bmatrix} \tag{E.5}$$

and

$$\vec{x}(t) = \begin{Bmatrix} x_1(t) \\ x_2(t) \end{Bmatrix} \tag{E.6}$$

The undamped natural frequencies and the mode shapes of the system can be found by solving the eigenvalue problem

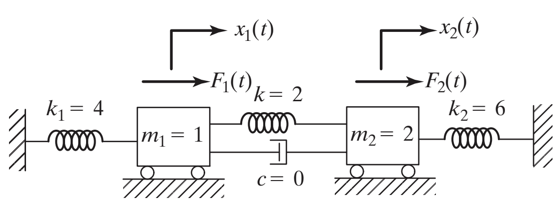


FIGURE 11.2 Two-degree-of-freedom system.

$$\left[-\omega^2\begin{bmatrix}1 & 0\\0 & 2\end{bmatrix}+\begin{bmatrix}6 & -2\\-2 & 8\end{bmatrix}\right]\begin{Bmatrix}X_1\\X_2\end{Bmatrix}=\begin{Bmatrix}0\\0\end{Bmatrix} \tag{E.7}$$

The solution of Eq. (E.7) is given by

$$\omega_1 = 1.807747, \qquad \vec{X}^{(1)} = \begin{Bmatrix}1.0000\\1.3661\end{Bmatrix} \tag{E.8}$$

$$\omega_2 = 2.594620, \qquad \vec{X}^{(2)} = \begin{Bmatrix}1.0000\\-0.3661\end{Bmatrix} \tag{E.9}$$

Thus the natural periods of the system are

$$\tau_1 = \frac{2\pi}{\omega_1} = 3.4757 \qquad \text{and} \qquad \tau_2 = \frac{2\pi}{\omega_2} = 2.4216$$

We shall select the time step (Δt) as $\tau_2/10 = 0.24216$. The initial value of $\ddot{\vec{x}}$ can be found as follows:

$$\ddot{\vec{x}}_0 = [m]^{-1}\{\vec{F} - [k]\vec{x}_0\} = \begin{bmatrix}1 & 0\\0 & 2\end{bmatrix}^{-1}\begin{Bmatrix}0\\10\end{Bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix}2 & 0\\0 & 1\end{bmatrix}\begin{Bmatrix}0\\10\end{Bmatrix} = \begin{Bmatrix}0\\5\end{Bmatrix} \tag{E.10}$$

and the value of $\vec{x}_{-1}$ as follows:

$$\vec{x}_{-1} = \vec{x}_0 - \Delta t\,\dot{\vec{x}}_0 + \frac{(\Delta t)^2}{2}\ddot{\vec{x}}_0 = \begin{Bmatrix}0\\0.1466\end{Bmatrix} \tag{E.11}$$

Now Eq. (11.29) can be applied recursively to obtain $\vec{x}_1, \vec{x}_2, \ldots$. The results are shown in Table 11.3.

■

11.6 Finite Difference Method for Continuous Systems

11.6.1 Longitudinal Vibration of Bars

Equation of Motion. The equation of motion governing the free longitudinal vibration of a uniform bar (see Eqs. (8.49) and (8.20)) can be expressed as

$$\frac{d^2U}{dx^2} + \alpha^2 U = 0 \tag{11.33}$$

TABLE 11.3

Time ($t_i = i\,\Delta t$)	$\vec{x}_i = \vec{x}(t = t_i)$
t_1	$\begin{Bmatrix} 0 \\ 0.1466 \end{Bmatrix}$
t_2	$\begin{Bmatrix} 0.0172 \\ 0.5520 \end{Bmatrix}$
t_3	$\begin{Bmatrix} 0.0931 \\ 1.1222 \end{Bmatrix}$
t_4	$\begin{Bmatrix} 0.2678 \\ 1.7278 \end{Bmatrix}$
t_5	$\begin{Bmatrix} 0.5510 \\ 2.2370 \end{Bmatrix}$
t_6	$\begin{Bmatrix} 0.9027 \\ 2.5470 \end{Bmatrix}$
t_7	$\begin{Bmatrix} 1.2354 \\ 2.6057 \end{Bmatrix}$
t_8	$\begin{Bmatrix} 1.4391 \\ 2.4189 \end{Bmatrix}$
t_9	$\begin{Bmatrix} 1.4202 \\ 2.0422 \end{Bmatrix}$
t_{10}	$\begin{Bmatrix} 1.1410 \\ 1.5630 \end{Bmatrix}$
t_{11}	$\begin{Bmatrix} 0.6437 \\ 1.0773 \end{Bmatrix}$
t_{12}	$\begin{Bmatrix} 0.0463 \\ 0.6698 \end{Bmatrix}$

where

$$\alpha^2 = \frac{\omega^2}{c^2} = \frac{\rho\omega^2}{E} \tag{11.34}$$

To obtain the finite difference approximation of Eq. (11.33), we first divide the bar of length l into $n - 1$ equal parts each of length $h = l/(n - 1)$ and denote the mesh points as 1, 2, 3, $\ldots$, i, $\ldots$, n, as shown in Fig. 11.3. Then, by denoting the value of U at mesh point i as U_i and using a formula for the second derivative similar to Eq. (11.4), Eq. (11.33) for mesh point i can be written as

$$\frac{1}{h^2}(U_{i+1} - 2U_i + U_{i-1}) + \alpha^2 U_i = 0$$

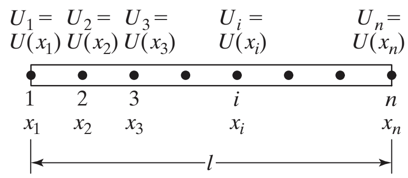


FIGURE 11.3 Division of a bar for finite difference approximation.

or

$$U_{i+1} - (2 - \lambda)U_i + U_{i-1} = 0 \tag{11.35}$$

where $\lambda = h^2\alpha^2$. The application of Eq. (11.35) at mesh points $i = 2, 3, \ldots, n - 1$ leads to the equations

$$\begin{aligned} U_3 - (2 - \lambda)U_2 + U_1 &= 0 \\ U_4 - (2 - \lambda)U_3 + U_2 &= 0 \\ &\vdots \\ U_n - (2 - \lambda)U_{n-1} + U_{n-2} &= 0 \end{aligned} \tag{11.36}$$

which can be stated in matrix form as

$$\begin{bmatrix} -1 & (2-\lambda) & -1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -1 & (2-\lambda) & -1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & -1 & (2-\lambda) & -1 & \cdots & 0 & 0 & 0 \\ . & . & . & . & . & & . & . & . \\ . & . & . & . & . & \cdots & . & . & . \\ . & . & . & . & . & & . & . & . \\ 0 & 0 & 0 & 0 & 0 & \cdots & -1 & (2-\lambda) & -1 \end{bmatrix}$$

$$\begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ . \\ . \\ . \\ U_n \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ . \\ . \\ . \\ 0 \end{Bmatrix} \tag{11.37}$$

Boundary Conditions

Fixed End. The deflection is zero at a fixed end. Assuming that the bar is fixed at $x = 0$ and $x = l$, we set $U_1 = U_n = 0$ in Eq. (11.37) and obtain the equation

$$[[A] - \lambda[I]]\vec{U} = \vec{0} \tag{11.38}$$

where

$$[A] = \begin{bmatrix} 2 & -1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & \cdots & 0 & 0 & 0 \\ . & . & . & . & \cdots & . & . & . \\ . & . & . & . & \cdots & . & . & . \\ . & . & . & . & \cdots & . & . & . \\ 0 & 0 & 0 & 0 & \cdots & 0 & -1 & 2 \end{bmatrix} \tag{11.39}$$

$$\vec{U} = \begin{Bmatrix} U_2 \\ U_3 \\ . \\ . \\ . \\ U_{n-1} \end{Bmatrix} \tag{11.40}$$

and $[I]$ = identity matrix of order $n - 2$.

Note that the eigenvalue problem of Eq. (11.38) can be solved easily, since the matrix $[A]$ is a tridiagonal matrix [11.15–11.17].

Free End. The stress is zero at a free end, so $(dU)/(dx) = 0$. We can use a formula for the first derivative similar to Eq. (11.3). To illustrate the procedure, let the bar be free at $x = 0$ and fixed at $x = l$. The boundary conditions can then be stated as

$$\left.\frac{dU}{dx}\right|_1 \simeq \frac{U_2 - U_{-1}}{2h} = 0 \quad \text{or} \quad U_{-1} = U_2 \tag{11.41}$$

$$U_n = 0 \tag{11.42}$$

In order to apply Eq. (11.41), we need to imagine the function $U(x)$ to be continuous beyond the length of the bar and create a fictitious mesh point -1 so that U_{-1} becomes the fictitious displacement of the point x_{-1}. The application of Eq. (11.35) at mesh point $i = 1$ yields

$$U_2 - (2 - \lambda)U_1 + U_{-1} = 0 \tag{11.43}$$

By incorporating the condition $U_{-1} = U_2$ (Eq. 11.41), Eq. (11.43) can be written as

$$(2 - \lambda)U_1 - 2U_2 = 0 \tag{11.44}$$

By adding Eqs. (11.44) and (11.37), we obtain the final equations:

$$[[A] - \lambda[I]]\vec{U} = \vec{0} \tag{11.45}$$

where

$$[A] = \begin{bmatrix} 2 & -2 & 0 & 0 & \cdots & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & \cdots & 0 & 0 & 0 \\ \cdot & & & & & & & \\ \cdot & & & & & & & \\ \cdot & & & & & & & \\ 0 & 0 & 0 & 0 & \cdots & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & -1 & 2 \end{bmatrix} \tag{11.46}$$

and

$$\vec{U} = \begin{Bmatrix} U_1 \\ U_2 \\ \cdot \\ \cdot \\ \cdot \\ U_{n-1} \end{Bmatrix} \tag{11.47}$$

11.6.2 Transverse Vibration of Beams

Equation of Motion. The governing differential equation for the transverse vibration of a uniform beam is given by Eq. (8.83):

$$\frac{d^4W}{dx^4} - \beta^4 W = 0 \tag{11.48}$$

where

$$\beta^4 = \frac{\rho A \omega^2}{EI} \tag{11.49}$$

By using the central difference formula for the fourth derivative,[3] Eq. (11.48) can be written at any mesh point i as

$$W_{i+2} - 4W_{i+1} + (6 - \lambda)W_i - 4W_{i-1} + W_{i-2} = 0 \tag{11.50}$$

where

$$\lambda = h^4\beta^4 \tag{11.51}$$

Let the beam be divided into $n - 1$ equal parts with n mesh points and $h = l/(n - 1)$. The application of Eq. (11.50) at the mesh points $i = 3, 4, \ldots, n - 2$ leads to the equations

$$\begin{bmatrix} 1 & -4 & (6-\lambda) & -4 & 1 & 0 & 0\cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -4 & (6-\lambda) & -4 & 1 & 0\cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -4 & (6-\lambda) & -4 & 1\cdots & 0 & 0 & 0 & 0 & 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 & 0 & 0 & 0 & 0 & 0 & 0\cdots & 1 & -4 & (6-\lambda) & -4 & 1 \end{bmatrix}$$

$$\begin{Bmatrix} W_1 \\ W_2 \\ W_3 \\ \cdot \\ \cdot \\ \cdot \\ W_n \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{Bmatrix} \tag{11.52}$$

Boundary Conditions

Fixed End. The deflection W and the slope $(dW)/(dx)$ are zero at a fixed end. If the end $x = 0$ is fixed, we introduce a fictitious node -1 on the left-hand side of the beam, as shown in Fig. 11.4, and state the boundary conditions, using the central difference formula for $(dW)/(dx)$, as

$$W_1 = 0$$

[3]The central difference formula for the fourth derivative (see Problem 11.3) is given by

$$\left.\frac{d^4 f}{dx^4}\right|_i \simeq \frac{1}{h^4}(f_{i+2} - 4f_{i+1} + 6f_i - 4f_{i-1} + f_{i-2})$$

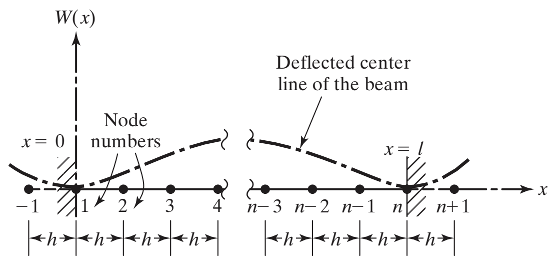


FIGURE 11.4 Beam with fixed ends.

$$\left.\frac{dW}{dx}\right|_1 = \frac{1}{2h}(W_2 - W_{-1}) = 0 \qquad \text{or} \qquad W_{-1} = W_2 \tag{11.53}$$

where W_i denotes the value of W at node i. If the end $x = l$ is fixed, we introduce the fictitious node $n + 1$ on the right side of the beam, as shown in Fig. 11.4, and state the boundary conditions as

$$W_n = 0$$

$$\left.\frac{dW}{dx}\right|_n = \frac{1}{2h}(W_{n+1} - W_{n-1}) = 0 \qquad \text{or} \qquad W_{n+1} = W_{n-1} \tag{11.54}$$

Simply Supported End. If the end $x = 0$ is simply supported (see Fig. 11.5), we have

$$W_1 = 0$$

$$\left.\frac{d^2W}{dx^2}\right|_1 = \frac{1}{h^2}(W_2 - 2W_1 + W_{-1}) = 0 \qquad \text{or} \qquad W_{-1} = -W_2 \tag{11.55}$$

Similar equations can be written if the end $x = l$ is simply supported.

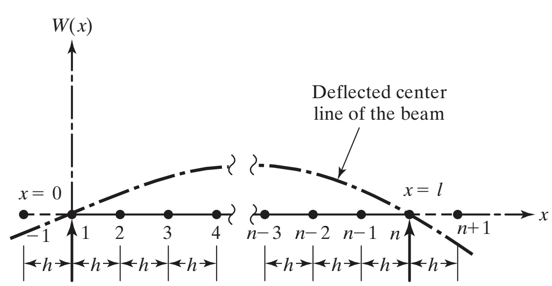


FIGURE 11.5 Beam with simply supported ends.

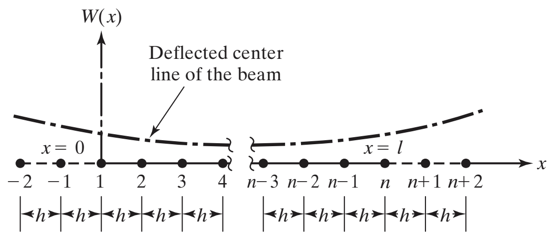


FIGURE 11.6 Beam with free ends.

Free End. Since bending moment and shear force are zero at a free end, we introduce two fictitious nodes outside the beam, as shown in Fig. 11.6, and use central difference formulas for approximating the second and the third derivatives of the deflection W. For example, if the end $x = 0$ is free, we have

$$\left.\frac{d^2W}{dx^2}\right|_1 = \frac{1}{h^2}(W_2 - 2W_1 + W_{-1}) = 0$$

$$\left.\frac{d^3W}{dx^3}\right|_1 = \frac{1}{2h^3}(W_3 - 2W_2 + 2W_{-1} - W_{-2}) = 0 \tag{11.56}$$

EXAMPLE 11.4 Pinned-Fixed Beam

Find the natural frequencies of the simply supported-fixed beam shown in Fig. 11.7. Assume that the cross section of the beam is constant along its length.

Solution: We shall divide the beam into four segments and express the governing equation

$$\frac{d^4W}{dx^4} - \beta^4 W = 0 \tag{E.1}$$

in finite difference form at each of the interior mesh points. This yields the equations

$$W_0 - 4W_1 + (6 - \lambda)W_2 - 4W_3 + W_4 = 0 \tag{E.2}$$

$$W_1 - 4W_2 + (6 - \lambda)W_3 - 4W_4 + W_5 = 0 \tag{E.3}$$

$$W_2 - 4W_3 + (6 - \lambda)W_4 - 4W_5 + W_6 = 0 \tag{E.4}$$

where W_0 and W_6 denote the values of W at the fictitious nodes 0 and 6, respectively, and

$$\lambda = h^4\beta^4 = \frac{h^4 \rho A \omega^2}{EI} \tag{E.5}$$

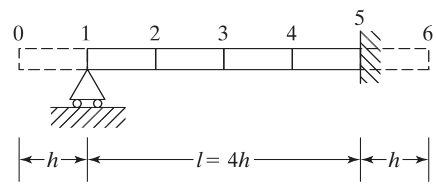


FIGURE 11.7 Simply supported-fixed beam.

The boundary conditions at the simply supported end (mesh point 1) are

$$W_1 = 0$$
$$W_0 = -W_2 \tag{E.6}$$

At the fixed end (mesh point 5) the boundary conditions are

$$W_5 = 0$$
$$W_6 = W_4 \tag{E.7}$$

With the help of Eqs. (E.6) and (E.7), Eqs. (E.2) to (E.4) can be reduced to

$$(5 - \lambda)W_2 - 4W_3 + W_4 = 0 \tag{E.8}$$

$$-4W_2 + (6 - \lambda)W_3 - 4W_4 = 0 \tag{E.9}$$

$$W_2 - 4W_3 + (7 - \lambda)W_4 = 0 \tag{E.10}$$

Equations (E.8) to (E.10) can be written in matrix form as

$$\begin{bmatrix} (5-\lambda) & -4 & 1 \\ -4 & (6-\lambda) & -4 \\ 1 & -4 & (7-\lambda) \end{bmatrix} \begin{Bmatrix} W_2 \\ W_3 \\ W_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \tag{E.11}$$

The solution of the eigenvalue problem (Eq. (E.11)) gives the following results:

$$\lambda_1 = 0.7135, \quad \omega_1 = \frac{0.8447}{h^2}\sqrt{\frac{EI}{\rho A}}, \quad \begin{Bmatrix} W_2 \\ W_3 \\ W_4 \end{Bmatrix}^{(1)} = \begin{Bmatrix} 0.5880 \\ 0.7215 \\ 0.3656 \end{Bmatrix} \tag{E.12}$$

$$\lambda_2 = 5.0322, \quad \omega_2 = \frac{2.2433}{h^2}\sqrt{\frac{EI}{\rho A}}, \quad \begin{Bmatrix} W_2 \\ W_3 \\ W_4 \end{Bmatrix}^{(2)} = \begin{Bmatrix} 0.6723 \\ -0.1846 \\ -0.7169 \end{Bmatrix} \tag{E.13}$$

$$\lambda_3 = 12.2543, \quad \omega_3 = \frac{3.5006}{h^2}\sqrt{\frac{EI}{\rho A}}, \quad \begin{Bmatrix} W_2 \\ W_3 \\ W_4 \end{Bmatrix}^{(3)} = \begin{Bmatrix} 0.4498 \\ -0.6673 \\ 0.5936 \end{Bmatrix} \tag{E.14}$$

■

11.7 Runge-Kutta Method for Multidegree-of-Freedom Systems

In the Runge-Kutta method, the matrix equations of motion, Eq. (11.18), are used to express the acceleration vector as

$$\ddot{\vec{x}}(t) = [m]^{-1}(\vec{F}(t) - [c]\dot{\vec{x}}(t) - [k]\vec{x}(t)) \tag{11.57}$$

By treating the displacements as well as velocities as unknowns, a new vector, $\vec{X}(t)$, is defined as $\vec{X}(t) = \begin{Bmatrix} \vec{x}(t) \\ \dot{\vec{x}}(t) \end{Bmatrix}$ so that

$$\dot{\vec{X}} = \begin{Bmatrix} \dot{\vec{x}} \\ \ddot{\vec{x}} \end{Bmatrix} = \begin{Bmatrix} \dot{\vec{x}} \\ [m]^{-1}(\vec{F} - [c]\dot{\vec{x}} - [k]\vec{x}) \end{Bmatrix} \tag{11.58}$$

Equation (11.58) can be rearranged to obtain

$$\dot{\vec{X}}(t) = \begin{bmatrix} [0] & [I] \\ -[m]^{-1}[k] & -[m]^{-1}[c] \end{bmatrix} \begin{Bmatrix} \vec{x}(t) \\ \dot{\vec{x}}(t) \end{Bmatrix} + \begin{Bmatrix} 0 \\ [m]^{-1}\vec{F}(t) \end{Bmatrix}$$

that is,

$$\dot{\vec{X}}(t) = \vec{f}(\vec{X}, t) \tag{11.59}$$

where

$$\vec{f}(\vec{X}, t) = [A]\vec{X}(t) + \vec{\underset{\sim}{F}}(t) \tag{11.60}$$

$$[A] = \begin{bmatrix} [0] & [I] \\ -[m]^{-1}[k] & -[m]^{-1}[c] \end{bmatrix} \tag{11.61}$$

and

$$\vec{\underset{\sim}{F}}(t) = \begin{Bmatrix} \vec{0} \\ [m]^{-1}\vec{F}(t) \end{Bmatrix} \tag{11.62}$$

With this, the recurrence formula to evaluate $\vec{X}(t)$ at different grid points t_i according to the fourth order Runge-Kutta method becomes [11.10]

$$\vec{X}_{i+1} = \vec{X}_i + \tfrac{1}{6}[\vec{K}_1 + 2\vec{K}_2 + 2\vec{K}_3 + \vec{K}_4] \tag{11.63}$$

where

$$\vec{K}_1 = h\vec{f}(\vec{X}_i, t_i) \tag{11.64}$$

$$\vec{K}_2 = h\vec{f}(\vec{X}_i + \tfrac{1}{2}\vec{K}_1, t_i + \tfrac{1}{2}h) \tag{11.65}$$

$$\vec{K}_3 = h\vec{f}(\vec{X}_i + \tfrac{1}{2}\vec{K}_2, t_i + \tfrac{1}{2}h) \tag{11.66}$$

$$\vec{K}_4 = h\vec{f}(\vec{X}_i + \vec{K}_3, t_{i+1}) \tag{11.67}$$

EXAMPLE 11.5 Runge-Kutta Method for a Two-Degree-of-Freedom System

Find the response of the two-degree-of-freedom system considered in Example 11.3 using the fourth-order Runge-Kutta method.

Solution:

Approach: Use the Runge-Kutta method with $\Delta t = 0.24216$.

Using the initial conditions $\vec{x}(t = 0) = \dot{\vec{x}}(t = 0) = \vec{0}$, Eq. (11.63) is sequentially applied with $\Delta t = 0.24216$ to obtain the results shown in Table 11.4.

TABLE 11.4

Time $t_i = i\,\Delta t$	$\vec{x}_i = \vec{x}(t = t_i)$
t_1	$\begin{Bmatrix} 0.0014 \\ 0.1437 \end{Bmatrix}$
t_2	$\begin{Bmatrix} 0.0215 \\ 0.5418 \end{Bmatrix}$
t_3	$\begin{Bmatrix} 0.0978 \\ 1.1041 \end{Bmatrix}$
t_4	$\begin{Bmatrix} 0.2668 \\ 1.7059 \end{Bmatrix}$
t_5	$\begin{Bmatrix} 0.5379 \\ 2.2187 \end{Bmatrix}$
t_6	$\begin{Bmatrix} 0.8756 \\ 2.5401 \end{Bmatrix}$
t_7	$\begin{Bmatrix} 1.2008 \\ 2.6153 \end{Bmatrix}$
t_8	$\begin{Bmatrix} 1.4109 \\ 2.4452 \end{Bmatrix}$
t_9	$\begin{Bmatrix} 1.4156 \\ 2.0805 \end{Bmatrix}$
t_{10}	$\begin{Bmatrix} 1.1727 \\ 1.6050 \end{Bmatrix}$
t_{11}	$\begin{Bmatrix} 0.7123 \\ 1.1141 \end{Bmatrix}$
t_{12}	$\begin{Bmatrix} 0.1365 \\ 0.6948 \end{Bmatrix}$

■

11.8 Houbolt Method

We shall consider the Houbolt method with reference to a multidegree-of-freedom system. In this method, the following finite difference expansions are employed:

$$\dot{\vec{x}}_{i+1} = \frac{1}{6\,\Delta t}(11\vec{x}_{i+1} - 18\vec{x}_i + 9\vec{x}_{i-1} - 2\vec{x}_{i-2}) \tag{11.68}$$

$$\ddot{\vec{x}}_{i+1} = \frac{1}{(\Delta t)^2}(2\vec{x}_{i+1} - 5\vec{x}_i + 4\vec{x}_{i-1} - \vec{x}_{i-2}) \tag{11.69}$$

To derive Eqs. (11.68) and (11.69), consider the function $x(t)$. Let the values of x at the equally spaced grid points $t_{i-2} = t_i - 2\,\Delta t$, $t_{i-1} = t_i - \Delta t$, t_i, and $t_{i+1} = t_i + \Delta t$ be given by x_{i-2}, x_{i-1}, x_i, and x_{i+1}, respectively, as shown in Fig. 11.8 [11.18]. The Taylor's series expansion, with backward step, gives several possibilities.

- With Step Size $= \Delta t$:

$$x(t) = x(t + \Delta t) - \Delta t\,\dot{x}(t + \Delta t) + \frac{(\Delta t)^2}{2!}\ddot{x}(t + \Delta t) - \frac{(\Delta t)^3}{3!}\dddot{x}(t + \Delta t)$$

or

$$x_i = x_{i+1} - \Delta t\,\dot{x}_{i+1} + \frac{(\Delta t)^2}{2}\ddot{x}_{i+1} - \frac{(\Delta t)^3}{6}\dddot{x}_{i+1} + \cdots \tag{11.70}$$

- With Step Size $= 2\,\Delta t$:

$$x(t - \Delta t) = x(t + \Delta t) - (2\,\Delta t)\dot{x}(t + \Delta t) + \frac{(2\,\Delta t)^2}{2!}\ddot{x}(t + \Delta t) - \frac{(2\,\Delta t)^3}{3!}\dddot{x}(t + \Delta t) + \cdots$$

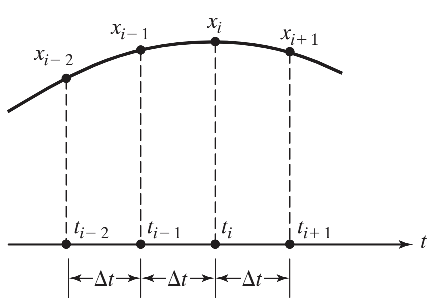


FIGURE 11.8 Equally spaced grid points.

or

$$x_{i-1} = x_{i+1} - 2\,\Delta t\,\dot{x}_{i+1} + 2(\Delta t)^2\ddot{x}_{i+1} - \tfrac{4}{3}(\Delta t)^3\,\dddot{x}_{i+1} + \cdots \qquad (11.71)$$

- With Step Size $= 3\,\Delta t$:

$$x(t - 2\,\Delta t) = x(t + \Delta t) - (3\,\Delta t)\dot{x}(t + \Delta t) + \frac{(3\,\Delta t)^2}{2!}\ddot{x}(t + \Delta t) - \frac{(3\,\Delta t)^3}{3!}\dddot{x}(t + \Delta t) + \cdots$$

or

$$x_{i-2} = x_{i+1} - 3\,\Delta t\,\dot{x}_{i+1} + \tfrac{9}{2}(\Delta t)^2\ddot{x}_{i+1} - \tfrac{9}{2}(\Delta t)^3\,\dddot{x}_{i+1} + \cdots \qquad (11.72)$$

By considering terms up to $(\Delta t)^3$ only, Eqs. (11.70) to (11.72) can be solved to express $\dot{x}_{i+1}$, $\ddot{x}_{i+1}$, and $\dddot{x}_{i+1}$ in terms of x_{i-2}, x_{i-1}, x_i, and x_{i+1}. This gives $\dot{x}_{i+1}$ and $\ddot{x}_{i+1}$ as in [11.18]:

$$\dot{x}_{i+1} = \frac{1}{6(\Delta t)}(11x_{i+1} - 18x_i + 9x_{i-1} - 2x_{i-2}) \qquad (11.73)$$

$$\ddot{x}_{i+1} = \frac{1}{(\Delta t)^2}(2x_{i+1} - 5x_i + 4x_{i-1} - x_{i-2}) \qquad (11.74)$$

Equations (11.68) and (11.69) represent the vector form of these equations.

To find the solution at step $i + 1(\vec{x}_{i+1})$, we consider Eq. (11.18) at t_{i+1}, so that

$$[m]\ddot{\vec{x}}_{i+1} + [c]\dot{\vec{x}}_{i+1} + [k]\vec{x}_{i+1} = \vec{F}_{i+1} \equiv \vec{F}(t = t_{i+1}) \qquad (11.75)$$

By substituting Eqs. (11.68) and (11.69) into Eq. (11.75), we obtain

$$\left(\frac{2}{(\Delta t)^2}[m] + \frac{11}{6\,\Delta t}[c] + [k]\right)\vec{x}_{i+1} = \vec{F}_{i+1} + \left(\frac{5}{(\Delta t)^2}[m] + \frac{3}{\Delta t}[c]\right)\vec{x}_i - \left(\frac{4}{(\Delta t)^2}[m] + \frac{3[c]}{2\,\Delta t}\right)\vec{x}_{i-1} + \left(\frac{1}{(\Delta t)^2}[m] + \frac{[c]}{3\,\Delta t}\right)\vec{x}_{i-2} \qquad (11.76)$$

Note that the equilibrium equation at time t_{i+1}, Eq. (11.75), is used in finding the solution $\vec{X}_{i+1}$ through Eq. (11.76). This is also true of the Wilson and Newmark methods. For this reason, these methods are called *implicit integration methods*.

It can be seen from Eq. (11.76) that a knowledge of $\vec{x}_i$, $\vec{x}_{i-1}$, and $\vec{x}_{i-2}$ is required to find the solution $\vec{x}_{i+1}$. Thus the values of $\vec{x}_{-1}$ and $\vec{x}_{-2}$ are to be found before attempting to find the vector $\vec{x}_1$ using Eq. (11.76). Since there is no direct method to find $\vec{x}_{-1}$ and $\vec{x}_{-2}$, we cannot use Eq. (11.76) to find $\vec{x}_1$ and $\vec{x}_2$. This makes the method non–self-starting. To start the method, we can use the central difference method described in Section 11.5 to find $\vec{x}_1$ and $\vec{x}_2$. Once $\vec{x}_0$ is known from the given initial conditions of the problem and $\vec{x}_1$ and $\vec{x}_2$ are known from the central difference method, the subsequent solutions $\vec{x}_3$, $\vec{x}_4$, ... can be found by using Eq. (11.76).

The step-by-step procedure to be used in the Houbolt method is as follows:

1. From the known initial conditions $\vec{x}(t = 0) = \vec{x}_0$ and $\dot{\vec{x}}(t = 0) = \dot{\vec{x}}_0$, find $\ddot{\vec{x}}_0 = \ddot{\vec{x}}(t = 0)$ using Eq. (11.26).
2. Select a suitable time step Δt.
3. Determine $\vec{x}_{-1}$ using Eq. (11.28).
4. Find $\vec{x}_1$ and $\vec{x}_2$ using the central difference equation (11.29).
5. Compute $\vec{x}_{i+1}$, starting with $i = 2$ and using Eq. (11.76):

$$\begin{aligned} \vec{x}_{i+1} = & \left[\frac{2}{(\Delta t)^2}[m] + \frac{11}{6\,\Delta t}[c] + [k]\right]^{-1} \\ & \times \left\{\vec{F}_{i+1} + \left(\frac{5}{(\Delta t)^2}[m] + \frac{3}{\Delta t}[c]\right)\vec{x}_i \right. \\ & - \left(\frac{4}{(\Delta t)^2}[m] + \frac{3}{2\,\Delta t}[c]\right)\vec{x}_{i-1} \\ & \left. + \left(\frac{1}{(\Delta t)^2}[m] + \frac{1}{3\,\Delta t}[c]\right)\vec{x}_{i-2}\right\} \end{aligned} \tag{11.77}$$

If required, evaluate the velocity and acceleration vectors $\dot{\vec{x}}_{i+1}$ and $\ddot{\vec{x}}_{i+1}$ using Eqs. (11.68) and (11.69).

EXAMPLE 11.6 Houbolt Method for a Two-Degree-of-Freedom System

Find the response of the two-degree-of-freedom system considered in Example 11.3 using the Houbolt method.

Solution

Approach: Use the Houbolt method with $\Delta t = 0.24216$.

The value of $\ddot{\vec{x}}_0$ can be found using Eq. (11.26):

$$\ddot{\vec{x}}_0 = \begin{Bmatrix} 0 \\ 5 \end{Bmatrix}$$

TABLE 11.5

Time $t_i = i\,\Delta t$	$\vec{x}_i = \vec{x}(t = t_i)$
t_1	$\begin{Bmatrix} 0.0000 \\ 0.1466 \end{Bmatrix}$
t_2	$\begin{Bmatrix} 0.0172 \\ 0.5520 \end{Bmatrix}$
t_3	$\begin{Bmatrix} 0.0917 \\ 1.1064 \end{Bmatrix}$
t_4	$\begin{Bmatrix} 0.2501 \\ 1.6909 \end{Bmatrix}$
t_5	$\begin{Bmatrix} 0.4924 \\ 2.1941 \end{Bmatrix}$
t_6	$\begin{Bmatrix} 0.7867 \\ 2.5297 \end{Bmatrix}$
t_7	$\begin{Bmatrix} 1.0734 \\ 2.6489 \end{Bmatrix}$
t_8	$\begin{Bmatrix} 1.2803 \\ 2.5454 \end{Bmatrix}$
t_9	$\begin{Bmatrix} 1.3432 \\ 2.2525 \end{Bmatrix}$
t_{10}	$\begin{Bmatrix} 1.2258 \\ 1.8325 \end{Bmatrix}$
t_{11}	$\begin{Bmatrix} 0.9340 \\ 1.3630 \end{Bmatrix}$
t_{12}	$\begin{Bmatrix} 0.5178 \\ 0.9224 \end{Bmatrix}$

By using a value of $\Delta t = 0.24216$, Eq. (11.29) can be used to find $\vec{x}_1$ and $\vec{x}_2$, and then Eq. (11.77) can be used recursively to obtain $\vec{x}_3, \vec{x}_4, \ldots$, as shown in Table 11.5.

■

11.9 Wilson Method

The Wilson method assumes that the acceleration of the system varies linearly between two instants of time. In particular, the two instants of time are taken as indicated in Fig. 11.9. Thus the acceleration is assumed to be linear from time $t_i = i\,\Delta t$ to time $t_{i+\theta} = t_i + \theta\,\Delta t$,

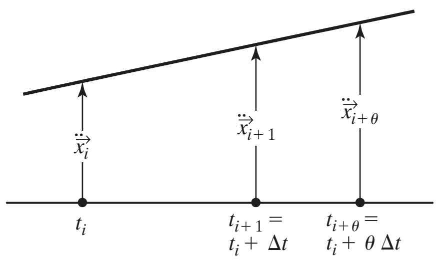


FIGURE 11.9 Linear acceleration assumption of the Wilson method.

where $\theta \geq 1.0$ [11.19]. For this reason, this method is also called the *Wilson θ method.* If $\theta = 1.0$, this method reduces to the linear acceleration scheme [11.20].

A stability analysis of the Wilson method shows that it is unconditionally stable provided that $\theta \geq 1.37$. In this section, we shall consider the Wilson method for a multidegree-of-freedom system.

Since $\ddot{\vec{x}}(t)$ is assumed to vary linearly between t_i and $t_{i+\theta}$, we can predict the value of $\ddot{\vec{x}}$ at any time $t_i + \tau$, $0 \leq \tau \leq \theta\,\Delta t$:

$$\ddot{\vec{x}}(t_i + \tau) = \ddot{\vec{x}}_i + \frac{\tau}{\theta\,\Delta t}(\ddot{\vec{x}}_{i+\theta} - \ddot{\vec{x}}_i) \tag{11.78}$$

By integrating Eq. (11.78), we obtain[4]

$$\dot{\vec{x}}(t_i + \tau) = \dot{\vec{x}}_i + \ddot{\vec{x}}_i\tau + \frac{\tau^2}{2\theta\,\Delta t}(\ddot{\vec{x}}_{i+\theta} - \ddot{\vec{x}}_i) \tag{11.79}$$

and

$$\vec{x}(t_i + \tau) = \vec{x}_i + \dot{\vec{x}}_i\tau + \frac{1}{2}\ddot{\vec{x}}_i\tau^2 + \frac{\tau^3}{6\theta\,\Delta t}(\ddot{\vec{x}}_{i+\theta} - \ddot{\vec{x}}_i) \tag{11.80}$$

By substituting $\tau = \theta\,\Delta t$ into Eqs. (11.79) and (11.80), we obtain

$$\dot{\vec{x}}_{i+\theta} = \dot{\vec{x}}(t_i + \theta\,\Delta t) = \dot{\vec{x}}_i + \frac{\theta\,\Delta t}{2}(\ddot{\vec{x}}_{i+\theta} + \ddot{\vec{x}}_i) \tag{11.81}$$

$$\vec{x}_{i+\theta} = \vec{x}(t_i + \theta\,\Delta t) = \vec{x}_i + \theta\,\Delta t\,\dot{\vec{x}}_i + \frac{\theta^2(\Delta t)^2}{6}(\ddot{\vec{x}}_{i+\theta} + 2\ddot{\vec{x}}_i) \tag{11.82}$$

[4] $\dot{\vec{x}}_i$ and $\vec{x}_i$ have been substituted in place of the integration constants in Eqs. (11.79) and (11.80), respectively.

Equation (11.82) can be solved to obtain

$$\ddot{\vec{x}}_{i+\theta} = \frac{6}{\theta^2(\Delta t)^2}(\vec{x}_{i+\theta} - \vec{x}_i) - \frac{6}{\theta\,\Delta t}\dot{\vec{x}}_i - 2\ddot{\vec{x}}_i \tag{11.83}$$

By substituting Eq. (11.83) into Eq. (11.81), we obtain

$$\dot{\vec{x}}_{i+\theta} = \frac{3}{\theta\,\Delta t}(\vec{x}_{i+\theta} - \vec{x}_i) - 2\dot{\vec{x}}_i - \frac{\theta\,\Delta t}{2}\ddot{\vec{x}}_i \tag{11.84}$$

To obtain the value of $x_{i+\theta}$, we consider the equilibrium equation (11.18) at time $t_{i+\theta} = t_i + \theta\,\Delta t$ and write

$$[m]\ddot{\vec{x}}_{i+\theta} + [c]\dot{\vec{x}}_{i+\theta} + [k]\vec{x}_{i+\theta} = \underset{\sim}{F}_{i+\theta} \tag{11.85}$$

where the force vector $\underset{\sim}{F}_{i+\theta}$ is also obtained by using the linear assumption:

$$\underset{\sim}{F}_{i+\theta} = \vec{F}_i + \theta(\vec{F}_{i+1} - \vec{F}_i) \tag{11.86}$$

Substituting Eqs. (11.83), (11.84), and (11.86) for $\ddot{\vec{x}}_{i+\theta}$, $\dot{\vec{x}}_{i+\theta}$, and $\underset{\sim}{F}_{i+\theta}$, Eq. (11.85) gives

$$\begin{aligned}\left\{\frac{6}{\theta^2(\Delta t)^2}[m] + \frac{3}{\theta\,\Delta t}[c] + [k]\right\}\vec{x}_{i+1} \\ = \vec{F}_i + \theta(\vec{F}_{i+1} - \vec{F}_i) + \left\{\frac{6}{\theta^2(\Delta t)^2}[m] + \frac{3}{\theta\,\Delta t}[c]\right\}\vec{x}_i \\ + \left\{\frac{6}{\theta\,\Delta t}[m] + 2[c]\right\}\dot{\vec{x}}_i + \left\{2[m] + \frac{\theta\,\Delta t}{2}[c]\right\}\ddot{\vec{x}}_i\end{aligned} \tag{11.87}$$

which can be solved for $\vec{x}_{i+1}$.

The Wilson method can be described by the following steps:

1. From the known initial conditions $\vec{x}_0$ and $\dot{\vec{x}}_0$, find $\ddot{\vec{x}}_0$ using Eq. (11.26).
2. Select a suitable time step Δt and a suitable value of θ (θ is usually taken as 1.4).
3. Compute the effective load vector $\underset{\approx}{F}_{i+\theta}$, starting with $i = 0$:

$$\begin{aligned}\underset{\approx}{\vec{F}}_{i+\theta} = \vec{F}_i + \theta(\vec{F}_{i+1} - \vec{F}_i) + [m]\left(\frac{6}{\theta^2(\Delta t)^2}\vec{x}_i + \frac{6}{\theta\,\Delta t}\dot{\vec{x}}_i + 2\ddot{\vec{x}}_i\right) \\ + [c]\left(\frac{3}{\theta\,\Delta t}\vec{x}_i + 2\dot{\vec{x}}_i + \frac{\theta\,\Delta t}{2}\ddot{\vec{x}}_i\right)\end{aligned} \tag{11.88}$$

4. Find the displacement vector at time $t_{i+\theta}$:

$$\vec{x}_{i+\theta} = \left[\frac{6}{\theta^2(\Delta t)^2}[m] + \frac{3}{\theta\,\Delta t}[c] + [k]\right]^{-1} \underset{\approx}{F}_{i+\theta} \tag{11.89}$$

5. Calculate the acceleration, velocity, and displacement vectors at time t_{i+1}:

$$\ddot{\vec{x}}_{i+1} = \frac{6}{\theta^3(\Delta t)^2}(\vec{x}_{i+\theta} - \vec{x}_i) - \frac{6}{\theta^2\Delta t}\dot{\vec{x}}_i + \left(1 - \frac{3}{\theta}\right)\ddot{\vec{x}}_i \tag{11.90}$$

$$\dot{\vec{x}}_{i+1} = \dot{\vec{x}}_i + \frac{\Delta t}{2}(\ddot{\vec{x}}_{i+1} + \ddot{\vec{x}}_i) \tag{11.91}$$

$$\vec{x}_{i+1} = \vec{x}_i + \Delta t\,\dot{\vec{x}}_i + \frac{(\Delta t)^2}{6}(\ddot{\vec{x}}_{i+1} + 2\ddot{\vec{x}}_i) \tag{11.92}$$

EXAMPLE 11.7 Wilson Method for a Two-Degree-of-Freedom System

Find the response of the system considered in Example 11.3, using the Wilson θ method with $\theta = 1.4$.

Solution:
Approach: Use Wilson method with $\Delta t = 0.24216$.

The value of $\ddot{\vec{x}}_0$ can be obtained as in the case of Example 11.3:

$$\ddot{\vec{x}}_0 = \begin{Bmatrix} 0 \\ 5 \end{Bmatrix}$$

Then, by using Eqs. (11.90) to (11.92) with a time step of $\Delta t = 0.24216$, we obtain the results indicated in Table 11.6.

■

11.10 Newmark Method

The Newmark integration method is also based on the assumption that the acceleration varies linearly between two instants of time. The resulting expressions for the velocity and displacement vectors $\dot{\vec{x}}_{i+1}$ and $\vec{x}_{i+1}$, for a multidegree-of-freedom system [11.21], are written as in Eqs. (11.79) and (11.80):

$$\dot{\vec{x}}_{i+1} = \dot{\vec{x}}_i + [(1 - \beta)\ddot{\vec{x}}_i + \beta\ddot{\vec{x}}_{i+1}]\,\Delta t \tag{11.93}$$

$$\vec{x}_{i+1} = \vec{x}_i + \Delta t\,\dot{\vec{x}}_i + [(\tfrac{1}{2} - \alpha)\ddot{\vec{x}}_i + \alpha\ddot{\vec{x}}_{i+1}](\Delta t)^2 \tag{11.94}$$

TABLE 11.6

Time $t_i = i\,\Delta t$	$\vec{x}_i = \vec{x}(t = t_i)$
t_1	$\begin{Bmatrix} 0.0033 \\ 0.1392 \end{Bmatrix}$
t_2	$\begin{Bmatrix} 0.0289 \\ 0.5201 \end{Bmatrix}$
t_3	$\begin{Bmatrix} 0.1072 \\ 1.0579 \end{Bmatrix}$
t_4	$\begin{Bmatrix} 0.2649 \\ 1.6408 \end{Bmatrix}$
t_5	$\begin{Bmatrix} 0.5076 \\ 2.1529 \end{Bmatrix}$
t_6	$\begin{Bmatrix} 0.8074 \\ 2.4981 \end{Bmatrix}$
t_7	$\begin{Bmatrix} 1.1035 \\ 2.6191 \end{Bmatrix}$
t_8	$\begin{Bmatrix} 1.3158 \\ 2.5056 \end{Bmatrix}$
t_9	$\begin{Bmatrix} 1.3688 \\ 2.1929 \end{Bmatrix}$
t_{10}	$\begin{Bmatrix} 1.2183 \\ 1.7503 \end{Bmatrix}$
t_{11}	$\begin{Bmatrix} 0.8710 \\ 1.2542 \end{Bmatrix}$
t_{12}	$\begin{Bmatrix} 0.3897 \\ 0.8208 \end{Bmatrix}$

where the parameters α and β indicate how much the acceleration at the end of the interval enters into the velocity and displacement equations at the end of the interval Δt. In fact, α and β can be chosen to obtain the desired accuracy and stability characteristics [11.22]. When $\beta = \frac{1}{2}$ and $\alpha = \frac{1}{6}$, Eqs. (11.93) and (11.94) correspond to the linear acceleration method (which can also be obtained using $\theta = 1$ in the Wilson method). When $\beta = \frac{1}{2}$ and $\alpha = \frac{1}{4}$, Eqs. (11.93) and (11.94) correspond to the assumption of constant acceleration between t_i and t_{i+1}. To find the value of $\vec{\ddot{x}}_{i+1}$, the equilibrium equation (11.18) is considered at $t = t_{i+1}$, so that

$$[m]\vec{\ddot{x}}_{i+1} + [c]\vec{\dot{x}}_{i+1} + [k]\vec{x}_{i+1} = \vec{F}_{i+1} \tag{11.95}$$

Equation (11.94) can be used to express $\ddot{\vec{x}}_{i+1}$ in terms of $\vec{x}_{i+1}$, and the resulting expression can be substituted into Eq. (11.93) to express $\dot{\vec{x}}_{i+1}$ in terms of $\vec{x}_{i+1}$. By substituting these expressions for $\dot{\vec{x}}_{i+1}$ and $\ddot{\vec{x}}_{i+1}$ into Eq. (11.95), we can obtain a relation for finding $\vec{x}_{i+1}$:

$$\begin{aligned}\vec{x}_{i+1} = &\left[\frac{1}{\alpha(\Delta t)^2}[m] + \frac{\beta}{\alpha\,\Delta t}[c] + [k]\right]^{-1} \\ &\times \left\{\vec{F}_{i+1} + [m]\left(\frac{1}{\alpha(\Delta t)^2}\vec{x}_i + \frac{1}{\alpha\,\Delta t}\dot{\vec{x}}_i + \left(\frac{1}{2\alpha} - 1\right)\ddot{\vec{x}}_i\right)\right. \\ &\quad + [c]\left(\frac{\beta}{\alpha\,\Delta t}\vec{x}_i + \left(\frac{\beta}{\alpha} - 1\right)\dot{\vec{x}}_i \right. \\ &\quad \left.\left. + \left(\frac{\beta}{\alpha} - 2\right)\frac{\Delta t}{2}\ddot{\vec{x}}_i\right)\right\}\end{aligned} \tag{11.96}$$

The Newmark method can be summarized in the following steps:

1. From the known values of $\vec{x}_0$ and $\dot{\vec{x}}_0$, find $\ddot{\vec{x}}_0$ using Eq. (11.26).
2. Select suitable values of Δt, α, and β.
3. Calculate the displacement vector $\vec{x}_{i+1}$, starting with $i = 0$ and using Eq. (11.96).
4. Find the acceleration and velocity vectors at time t_{i+1}:

$$\ddot{\vec{x}}_{i+1} = \frac{1}{\alpha(\Delta t)^2}(\vec{x}_{i+1} - \vec{x}_i) - \frac{1}{\alpha\,\Delta t}\dot{\vec{x}}_i - \left(\frac{1}{2\alpha} - 1\right)\ddot{\vec{x}}_i \tag{11.97}$$

$$\dot{\vec{x}}_{i+1} = \dot{\vec{x}}_i + (1 - \beta)\,\Delta t\,\ddot{\vec{x}}_i + \beta\,\Delta t\,\ddot{\vec{x}}_{i+1} \tag{11.98}$$

It is important to note that unless β is taken as $\frac{1}{2}$, there is a spurious damping introduced, proportional to $(\beta - \frac{1}{2})$. If β is taken as zero, a negative damping results; this involves a self-excited vibration arising solely from the numerical procedure. Similarly, if β is greater than $\frac{1}{2}$, a positive damping is introduced. This reduces the magnitude of response even without real damping in the problem [11.21]. The method is unconditionally stable for $\alpha \geq \frac{1}{4}(\beta + \frac{1}{2})^2$ and $\beta \geq \frac{1}{2}$.

EXAMPLE 11.8 Newmark Method for a Two-Degree-of-Freedom System

Find the response of the system considered in Example 11.3, using the Newmark method with $\alpha = \frac{1}{6}$ and $\beta = \frac{1}{2}$.

Solution

Approach: Use the Newmark method with $\Delta t = 0.24216$.

TABLE 11.7

Time $t_i = i\,\Delta t$	$\vec{x}_i = \vec{x}(t = t_i)$
t_1	$\begin{Bmatrix} 0.0026 \\ 0.1411 \end{Bmatrix}$
t_2	$\begin{Bmatrix} 0.0246 \\ 0.5329 \end{Bmatrix}$
t_3	$\begin{Bmatrix} 0.1005 \\ 1.0884 \end{Bmatrix}$
t_4	$\begin{Bmatrix} 0.2644 \\ 1.6870 \end{Bmatrix}$
t_5	$\begin{Bmatrix} 0.5257 \\ 2.2027 \end{Bmatrix}$
t_6	$\begin{Bmatrix} 0.8530 \\ 2.5336 \end{Bmatrix}$
t_7	$\begin{Bmatrix} 1.1730 \\ 2.6229 \end{Bmatrix}$
t_8	$\begin{Bmatrix} 1.3892 \\ 2.4674 \end{Bmatrix}$
t_9	$\begin{Bmatrix} 1.4134 \\ 2.1137 \end{Bmatrix}$
t_{10}	$\begin{Bmatrix} 1.1998 \\ 1.6426 \end{Bmatrix}$
t_{11}	$\begin{Bmatrix} 0.7690 \\ 1.1485 \end{Bmatrix}$
t_{12}	$\begin{Bmatrix} 0.2111 \\ 0.7195 \end{Bmatrix}$

The value of $\ddot{\vec{x}}_0$ can be found using Eq. (11.26):

$$\ddot{\vec{x}}_0 = \begin{Bmatrix} 0 \\ 5 \end{Bmatrix}$$

With the values of $\alpha = \frac{1}{6}$, $\beta = 0.5$, and $\Delta t = 0.24216$, Eq. (11.96) gives the values of $\vec{x}_i = \vec{x}(t = t_i)$, as shown in Table 11.7.

■

11.11 Examples Using MATLAB

EXAMPLE 11.9 MATLAB Solution of a Single-Degree-of-Freedom System

Using the MATLAB function **ode23**, solve Example 11.1.

Solution: Defining $x_1 = x$ and $x_2 = \dot{x}$, Eq. (E.1) of Example 11.1 can be expressed as a set of two first-order differential equations:

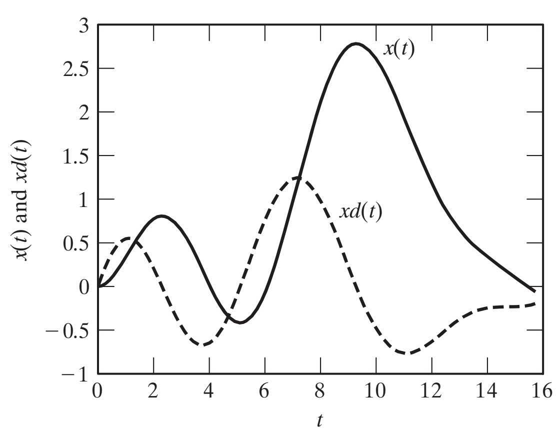
$$\dot{x}_1 = x_2 \tag{E.1}$$

$$\dot{x}_2 = \frac{1}{m}\left[F_0\left(1 - \sin\frac{\pi t}{2t_0}\right) - cx_2 - kx_1\right] \tag{E.2}$$

with initial conditions $x_1(0) = x_2(0) = 0$. The MATLAB program to solve Eqs. (E.1) and (E.2) is given below.

```
% Ex11_9.m
tspan = [0: 0.1: 5*pi];
x0 = [0; 0];
[t,x] = ode23 ('dfunc11_9', tspan, x0);
plot (t,x(:,1));
xlabel ('t');
ylabel ('x(t) and xd(t)');
gtext ('x(t)');
hold on;
plot (t,x(:,2), '--');
gtext ('xd(t)')

%dfunc11_9.m
function f = dfunc11_9(t,x)
m = 1;
k = 1;
c = 0.2;
t0 = pi;
F0 = 1;
f = zeros (2,1);
f(1) = x(2);
f(2) = (F0* (1 - sin(pi*t/(2*t0))) - c*x(2) - k*x(1) )/m;
```



■

EXAMPLE 11.10 MATLAB Solution of Multidegree-of-Freedom System

Using the MATLAB function **ode23**, solve Example 11.3.

Solution: The equations of motion of the two-degree-of-freedom system in Eq. (E.1) of Example 11.3 can be expressed as a system of four first-order differential equations in terms of

$$y_1 = x_1, \qquad y_2 = \dot{x}_1, \qquad y_3 = x_2, \qquad y_4 = \dot{x}_2$$

as

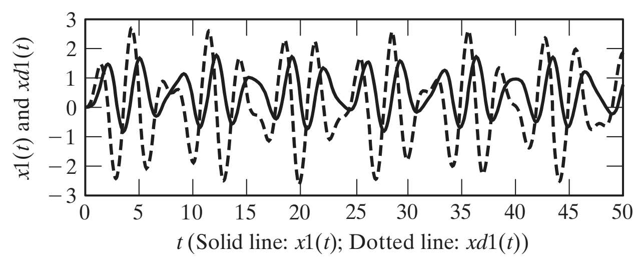
$$\dot{y}_1 = y_2 \tag{E.1}$$

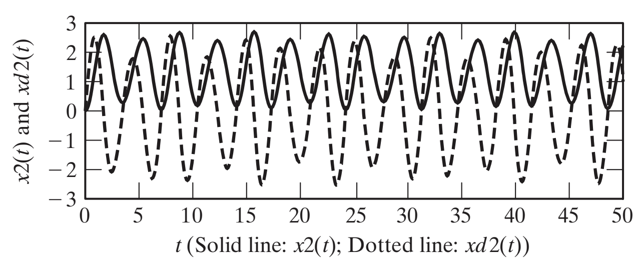
$$\dot{y}_2 = \frac{1}{m_1}\{F_1(t) - cy_2 + cy_4 - (k_1 + k)y_1 + ky_3\} = -6y_1 + y_3 \tag{E.2}$$

$$\dot{y}_3 = y_4 \tag{E.3}$$

$$\dot{y}_4 = \frac{1}{m_2}\{F_2(t) + cy_2 - cy_4 + ky_1 - (k + k_2)y_3\}$$

$$= \frac{1}{2}\{10 + 2\,y_1 - 8\,y_3\} = 5 + y_1 - 4\,y_3 \tag{E.4}$$





with initial conditions $y_i(0) = 0$, $i = 1, 2, 3, 4$. The MATLAB program to solve Eqs. (E.1) to (E.4) is given below.

```
% Ex11_10.m
tspan = [0: 0.05: 50];
y0 = [0; 0; 0; 0];
[t,y] = ode23 ('dfunc11_10', tspan, y0);
```

```
subplot (211);
plot (t,y(:,1));
xlabel ('t ( Solid line: x1 (t) Dotted line: xd1 (t) ) ');
ylabel ('x1 (t) amd xd1 (t)');
hold on;
plot (t,y(:, 2), '--');
subplot (212);
plot (t,y(:, 3));
xlabel ('t ( Solid line: x2 (t) Dotted line: xd2 (t) )');
ylabel ('x2 (t) amd xd2 (t) ');
hold on;
plot (t,y (:,4), '--');
%dfunc11_10.m
function f = dfunc11_10 (t,y)
m1 = 1;
m2 = 2;
k1 = 4;
k2 = 6;
k = 2;
c = 0;
F1 = 0;
F2 = 10;
f = zeros (4,1);
f(1) = y(2);
f(2) = ( F1 2 c*y(2) + c*y(4) - (k1+k) *y(1) + k*y(3) )/m1;
f(3) = y(4);
f(4) = ( F2 + c*y(2) - c*y(4) + k*y(1) - (k + k2) *y(3) )/m2;
```

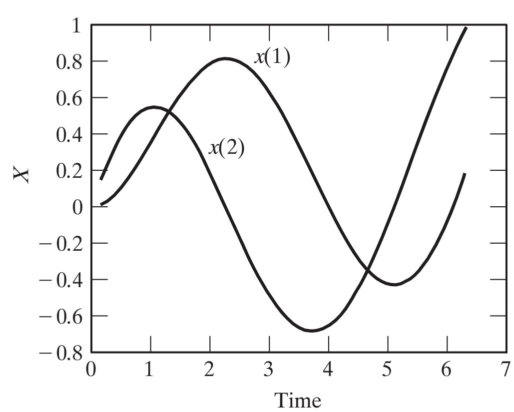
■

EXAMPLE 11.11 Program to Implement Fourth-Order Runge-Kutta Method

Develop a general MATLAB program called **Program14.m** for solving a set of first-order differential equations using the fourth-order Runge-Kutta method. Use the program to solve Example 11.2.

Solution: **Program14.m** is developed to accept the following input data:

n = number of first-order differential equations
xx = initial values $x_i(0)$, a vector of size n
dt = time increment



The program requires a subprogram to define the functions $f_i(\vec{x}, t)$, $i = 1, 2, \ldots, n$. The program gives the values of $x_i(t)$, $i = 1, 2, \ldots, n$ at different values of time t.

```
I        Time(I)          x(1)            x(2)

1     1.570800e-001    1.186315e-002   1.479138e-001
2     3.141600e-001    4.540642e-002   2.755911e-001
3     4.712400e-001    9.725706e-002   3.806748e-001
4     6.283200e-001    1.637262e-001   4.615022e-001
5     7.854000e-001    2.409198e-001   5.171225e-001
.
.
.
36    5.654880e+000   -2.868460e-001   5.040887e-001
37    5.811960e+000   -1.969950e-001   6.388500e-001
38    5.969040e+000   -8.655813e-002   7.657373e-001
39    6.126120e+000    4.301693e-002   8.821039e-001
40    6.283200e+000    1.898865e-001   9.855658e-001
```

■

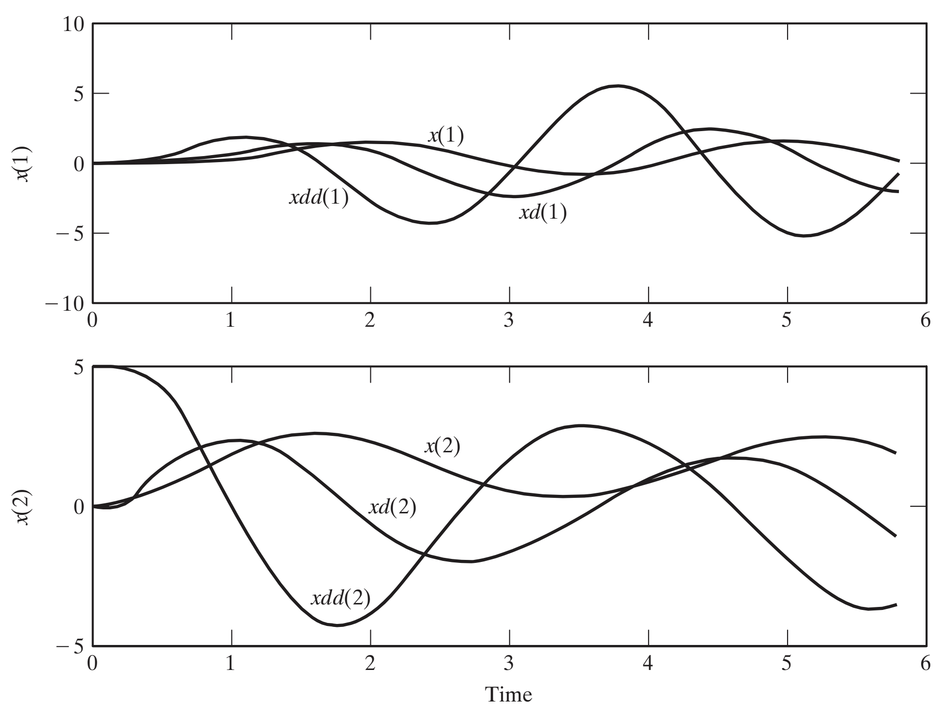
EXAMPLE 11.12 Program for Central Difference Method

Using the central difference method, develop a general MATLAB program called **`Program15.m`** to find the dynamic response of a multidegree-of-freedom system. Use the program to find the solution of Example 11.3.

Solution: **`Program15.m`** is developed to accept the following input data:

n = degree of freedom of the system

m = mass matrix, of size $n \times n$



c = damping matrix, of size $n \times n$
k = stiffness matrix, of size $n \times n$
xi = initial values of x_i, a vector of size n
xdi = initial values of $\dot{x}^i$, a vector of size n
nstep (nstp) = number of time steps at which solution is to be found
delt = increment between time steps

The program requires a subprogram to define the forcing functions $f_i(t)$, $i = 1, 2, \ldots, n$ at any time t. It gives the values of the response at different time steps i as $x_j(i)$, $\dot{x}_j(i)$, and $\ddot{x}_j(i)$, $j = 1, 2, \ldots, n$.

```
Solution by central difference method
Given data:
n= 2 nstp= 24 delt=2.421627e-001
Solution:
step    time      x(i,1)        xd(i,1)       xdd(i,1)      x(i,2)        xd(i,2)       xdd(i,2)
   1   0.0000   0.0000e+000   0.0000e+000   0.0000e+000   0.0000e+000   0.0000e+000   5.0000e+000
   2   0.2422   0.0000e+000   0.0000e+000   0.0000e+000   1.4661e-001   0.0000e+000   5.0000e+000
   3   0.4843   1.7195e-002   3.5503e-002   2.9321e-001   5.5204e-001   1.1398e+000   4.4136e+000
   4   0.7265   9.3086e-002   1.9220e-001   1.0009e+000   1.1222e+000   2.0143e+000   2.8090e+000
   5   0.9687   2.6784e-001   5.1752e-001   1.6859e+000   1.7278e+000   2.4276e+000   6.0429e-001
   .
   .
   .
  21   4.8433   1.6034e+000   1.7764e+000  -4.0959e+000   2.2077e+000   1.6763e+000  -1.0350e+000
  22   5.0854   1.6083e+000   6.5025e-001  -5.2053e+000   2.4526e+000   1.2813e+000  -2.2272e+000
  23   5.3276   1.3349e+000  -5.5447e-001  -4.7444e+000   2.5098e+000   6.2384e-001  -3.2023e+000
  24   5.5697   8.8618e-001  -1.4909e+000  -2.9897e+000   2.3498e+000  -2.1242e-001  -3.7043e+000
  25   5.8119   4.0126e-001  -1.9277e+000  -6.1759e-001   1.9837e+000  -1.0863e+000  -3.5128e+000
```

■

EXAMPLE 11.13 Program for Houbolt Method

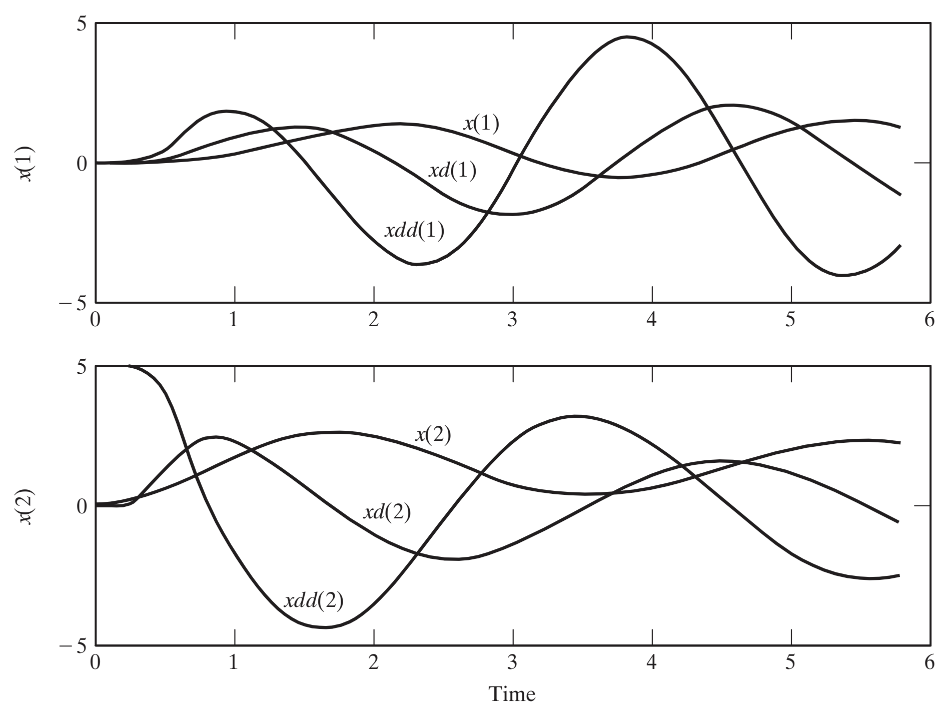
Using the Houbolt method, develop a general MATLAB program called **`Program16.m`** to find the dynamic response of a multidegree-of-freedom system. Use the program to find the solution of Example 11.6.

Solution: **`Program16.m`** is developed to accept the following input data:

n = degree of freedom of the system
m = mass matrix, of size $n \times n$
c = damping matrix, of size $n \times n$
k = stiffness matrix, of size $n \times n$
xi = initial values of x_i, a vector of size n
xdi = initial values of $\dot{x}_i$, a vector of size n
nstep (nstp) = number of time steps at which solution is to be found
delt = increment between time steps

The program requires a subprogram to define the forcing functions $f_i(t)$, $i = 1, 2, \ldots, n$ at any time t. It gives the values of the response at different time stations i as $x_j(i)$, $\dot{x}_j(i)$, and $\ddot{x}_j(i)$, $j = 1, 2, \ldots, n$.

```
Solution by Houbolt method
Given data:
n= 2 nstp= 24 delt=2.421627e-001
Solution:
step    time        x(i,1)        xd(i,1)       xdd(i,1)       x(i,2)        xd(i,2)       xdd(i,2)
   1   0.0000   0.0000e+000   0.0000e+000   0.0000e+000   0.0000e+000   0.0000e+000   5.0000e+000
   2   0.2422   0.0000e+000   0.0000e+000   0.0000e+000   1.4661e-001   0.0000e+000   5.0000e+000
   3   0.4843   1.7195e-002   3.5503e-002   2.9321e-001   5.5204e-001   1.1398e+000   4.4136e+000
   4   0.7265   9.1732e-002   4.8146e-001   1.6624e+000   1.1064e+000   2.4455e+000   6.6609e-001
   5   0.9687   2.5010e-001   8.6351e-001   1.8812e+000   1.6909e+000   2.3121e+000  -1.5134e+000
   .
   .
   .
  21   4.8433   8.7373e-001   1.7900e+000  -1.7158e+000   1.7633e+000   1.3850e+000  -1.1795e+000
  22   5.0854   1.2428e+000   1.1873e+000  -3.3403e+000   2.0584e+000   1.0125e+000  -1.9907e+000
  23   5.3276   1.4412e+000   3.6619e-001  -4.1553e+000   2.2460e+000   4.9549e-001  -2.5428e+000
  24   5.5697   1.4363e+000  -4.8458e-001  -4.0200e+000   2.2990e+000  -9.6748e-002  -2.7595e+000
  25   5.8119   1.2410e+000  -1.1822e+000  -3.0289e+000   2.2085e+000  -6.8133e-001  -2.5932e+000
```



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CHAPTER SUMMARY

Numerical methods are to be used in situations where the differential equations governing free and forced vibration cannot be solved to find closed-form solutions. We presented the finite difference method for the solution of the governing equations of discrete and continuous systems. We outlined the use of the fourth-order Runge-Kutta, Houbolt, Wilson, and Newmark methods for the solution of vibration problems related to multidegree-of-freedom systems. Finally, we presented the use of MATLAB for the numerical solution of vibration problems.

Now that you have finished this chapter, you should be able to answer the review questions and solve the problems given below.

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REVIEW QUESTIONS

11.1 Give brief answers to the following:

1. Describe the procedure of the finite difference method.
2. Using Taylor's series expansion, derive the central difference formulas for the first and the second derivatives of a function,
3. What is a conditionally stable method?
4. What is the main difference between the central difference method and the Runge-Kutta method?
5. Why is it necessary to introduce fictitious mesh points in the finite difference method of solution?
6. Define a tridiagonal matrix.
7. What is the basic assumption of the Wilson method?
8. What is a linear acceleration method?
9. What is the difference between explicit and implicit integration methods?
10. Can we use the numerical integration methods discussed in this chapter to solve nonlinear vibration problems?

11.2 Indicate whether each of the following statements is true or false:

1. The grid points in the finite difference methods are required to be uniformly spaced.
2. The Runge-Kutta method is stable.
3. The Runge-Kutta method is self-starting.
4. The finite difference method is an implicit integration method.
5. The Newmark method is an implicit integration method.

6. For a beam with grid points $-1, 1, 2, 3, \ldots$, the central difference equivalence of the condition $\left.\frac{dW}{dx}\right|_1 = 0$ is $W_{-1} = W_2$.
7. For a beam with grid points $-1, 1, 2, 3, \ldots$, the central difference approximation of a simply supported end condition at grid point 1 is given by $W_{-1} = W_2$.
8. For a beam with grid points $-1, 1, 2, 3, \ldots$, the central difference approximation of $\left.\frac{d^2W}{dx^2}\right|_1 = 0$ yields $W_2 - 2W_1 + W_{-1} = 0$.

11.3 Fill in each of the following blanks with the appropriate word:

1. Numerical methods are to be used when the equations of motion cannot be solved in _____ form.
2. In finite difference methods, approximations are used for _____.
3. Finite difference equations can be derived using _____ different approaches.
4. In finite difference methods, the solution domain is to be replaced by _____ points.
5. The finite difference approximations are based on _____ series expansion.
6. Numerical methods that require the use of a time step (Δt) smaller than a critical value (Δt_{cri}) are said to be _____ stable.
7. In a conditionally stable method, the use of Δt larger than Δt_{cri} makes the method _____.
8. A _____ formula permits the computation of x_i from known values of x_{i-1}.

11.4 Select the most appropriate answer out of the choices given:

1. The central difference approximation of dx/dt at t_i is given by
   a. $\frac{1}{2h}(x_{i+1} - x_i)$
   b. $\frac{1}{2h}(x_i - x_{i-1})$
   c. $\frac{1}{2h}(x_{i+1} - x_{i-1})$
2. The central difference approximation of d^2x/dt^2 at t_i is given by
   a. $\frac{1}{h^2}(x_{i+1} - 2x_i + x_{i-1})$
   b. $\frac{1}{h^2}(x_{i+1} - x_{i-1})$
   c. $\frac{1}{h^2}(x_i - x_{i-1})$
3. An integration method in which the computation of x_{i+1} is based on the equilibrium equation at t_i is known as
   a. explicit method
   b. implicit method
   c. regular method
4. In a non–self-starting method, we need to generate the value of the following quantity using the finite difference approximations of $\dot{x}_i$ and $\ddot{x}_i$:
   a. $\dot{x}_{-1}$ **b.** $\ddot{x}_{-1}$ **c.** x_{-1}
5. Runge-Kutta methods find the approximate solutions of
   a. algebraic equations **b.** differential equations **c.** matrix equations

6. The finite difference approximation of $d^2U/dx^2 + \alpha^2U = 0$ at x_i is given by
a. $U_{i+1} - (2 - h^2\alpha^2)U_i + U_{i-1} = 0$
b. $U_{i+1} - 2U_i + U_{i+1} = 0$
c. $U_{i+1} - (2 - \alpha^2)U_i + U_{i-1} = 0$

7. The finite difference method requires the use of finite difference approximations in
a. governing differential equation only
b. boundary conditions only
c. governing differential equation as well as boundary conditions

8. If a bar under longitudinal vibration is fixed at node 1, the forward difference formula gives
a. $U_1 = 0$ **b.** $U_1 = U_2$ **c.** $U_1 = U_{-1}$

9. If a bar under longitudinal vibration is free at node 1, the forward difference formula gives
a. $U_1 = 0$ **b.** $U_1 = U_2$ **c.** $U_1 = U_{-1}$

10. The central difference approximation of $d^4W/dx^4 - \beta^4W = 0$ at grid point i with step size h is
a. $W_{i+2} - 4W_{i+1} + (6 - h^4\beta^4)W_i - 4W_{i-1} + W_{i-2} = 0$
b. $W_{i+2} - 6W_{i+1} + (6 - h^4\beta^4)W_i - 6W_{i-1} + W_{i-2} = 0$
c. $W_{i+3} - 4W_{i+1} + (6 - h^4\beta^4)W_i - 4W_{i-1} + W_{i-3} = 0$

11.5 Match the items in the two columns below:

1. Houbolt method	**a.** Assumes that acceleration varies linearly between t_i and $t_i + \theta\,\Delta t$; $\theta \geq 1$
2. Wilson method	**b.** Assumes that acceleration varies linearly between t_i and t_{i+1}; can lead to negative damping
3. Newmark method	**c.** Based on the solution of equivalent system of first-order equations
4. Runge-Kutta method	**d.** Same as Wilson method with $\theta = 1$
5. Finite difference method	**e.** Uses finite difference expressions for $\dot{x}_{i+1}$ and $\ddot{x}_{i+1}$ in terms of x_{i-2}, x_{i-1}, x_i, and x_{i+1}
6. Linear acceleration method	**f.** Conditionally stable

PROBLEMS

Section 11.2 Finite Difference Approach

11.1 The forward difference formulas make use of the values of the function to the right of the base grid point. Thus the first derivative at point $i(t = t_i)$ is defined as

$$\frac{dx}{dt} = \frac{x(t + \Delta t) - x(t)}{\Delta t} = \frac{x_{i+1} - x_i}{\Delta t}$$

Derive the forward difference formulas for $(d^2x)/(dt^2)$, $(d^3x)/(dt^3)$, and $(d^4x)/(dt^4)$ at t_i.

11.2 The backward difference formulas make use of the values of the function to the left of the base grid point. Accordingly, the first derivative at point $i(t = t_i)$ is defined as

$$\frac{dx}{dt} = \frac{x(t) - x(t - \Delta t)}{\Delta t} = \frac{x_i - x_{i-1}}{\Delta t}$$

Derive the backward difference formulas for $(d^2x)/(dt^2)$, $(d^3x)/(dt^3)$, and $(d^4x)/(dt^4)$ at t_i.

11.3 Derive the formula for the fourth derivative, $(d^4x)/(dt^4)$, according to the central difference method.

Section 11.3 Central Difference Method for Single-Degree-Of-Freedom Systems

11.4 Find the free vibratory response of an undamped single-degree-of-freedom system with $m = 1$ and $k = 1$, using the central difference method. Assume $x_0 = 0$ and $\dot{x}_0 = 1$. Compare the results obtained with $\Delta t = 1$ and $\Delta t = 0.5$ with the exact solution $x(t) = \sin t$.

11.5 Integrate the differential equation

$$-\frac{d^2x}{dt^2} + 0.1x = 0 \qquad \text{for} \qquad 0 \le t \le 10$$

using the backward difference formula with $\Delta t = 1$. Assume the initial conditions as $x_0 = 1$ and $\dot{x}_0 = 0$.

11.6 Find the free-vibration response of a viscously damped single-degree-of-freedom system with $m = k = c = 1$, using the central difference method. Assume that $x_0 = 0$, $\dot{x}_0 = 1$, and $\Delta t = 0.5$.

11.7 Solve Problem 11.6 by changing c to 2.

11.8 Solve Problem 11.6 by taking the value of c as 4.

11.9 Find the solution of the equation $4\ddot{x} + 2\dot{x} + 3000x = F(t)$, where $F(t)$ is as shown in Fig. 11.10 for the duration $0 \le t \le 1$. Assume that $x_0 = \dot{x}_0 = 0$ and $\Delta t = 0.05$.

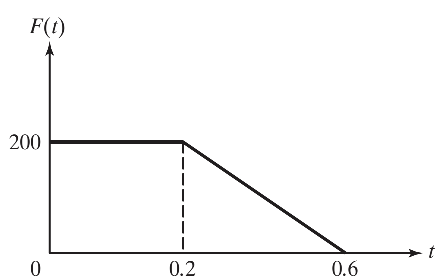

FIGURE 11.10

11.10 Find the solution of a spring-mass-damper system governed by the equation $m\ddot{x} + c\dot{x} + kx = F(t) = \delta F.\, t$ with $m = c = k = 1$ and $\delta F = 1$. Assume the initial values of x and $\dot{x}$ to be zero and $\Delta t = 0.5$. Compare the central difference solution with the exact solution given in Example 4.9.

Section 11.4 Runge-Kutta Method for Single-Degree-Of-Freedom Systems

11.11 Express the following nth-order differential equation as a system of n first-order differential equations:

$$a_n \frac{d^n x}{dt^n} + a_{n-1} \frac{d^{n-1} x}{dt^{n-1}} + \cdots + a_1 \frac{dx}{dt} = g(x, t)$$

11.12 Find the solution of the following equations by using the fourth-order Runge-Kutta method with $\Delta t = 0.1$:

(a) $\dot{x} = x - 1.5e^{-0.5t}$; $x_0 = 1$
(b) $\dot{x} = -tx^2$; $x_0 = 1$.

11.13 The second-order Runge-Kutta formula is given by

$$\vec{X}_{i+1} = \vec{X}_i + \tfrac{1}{2}(\vec{K}_1 + \vec{K}_2)$$

where

$$\vec{K}_1 = h\vec{F}(\vec{X}_i, t_i) \qquad \text{and} \qquad \vec{K}_2 = h\vec{F}(\vec{X}_i + \vec{K}_1, t_i + h)$$

Using this formula, solve the problem considered in Example 11.2.

11.14 The third-order Runge-Kutta formula is given by

$$\vec{X}_{i+1} = \vec{X}_i + \tfrac{1}{6}(\vec{K}_1 + 4\vec{K}_2 + \vec{K}_3)$$

where

$$\vec{K}_1 = h\vec{F}(\vec{X}_i, t_i)$$
$$\vec{K}_2 = h\vec{F}(\vec{X}_i + \tfrac{1}{2}\vec{K}_1, t_i + \tfrac{1}{2}h)$$

and

$$\vec{K}_3 = h\vec{F}(\vec{X}_i - \vec{K}_1 + 2\vec{K}_2, t_i + h)$$

Using this formula, solve the problem considered in Example 11.2.

11.15 Using the second-order Runge-Kutta method, solve the differential equation $\ddot{x} + 1000x = 0$ with the initial conditions $x_0 = 5$ and $\dot{x}_0 = 0$. Use $\Delta t = 0.01$.

11.16 Using the third-order Runge-Kutta method, solve Problem 11.15.

11.17 Using the fourth-order Runge-Kutta method, solve Problem 11.15.

Section 11.5 Central Difference Method for Multidegree-Of-Freedom Systems

11.18 Using the central difference method, find the response of the two-degree-of-freedom system shown in Fig. 11.2 when $c = 2$, $F_1(t) = 0$, $F_2(t) = 10$.

11.19 Using the central difference method, find the response of the system shown in Fig. 11.2 when $F_1(t) = 10 \sin 5t$ and $F_2(t) = 0$.

11.20 The equations of motion of a two-degree-of-freedom system are given by $2\ddot{x}_1 + 6x_1 - 2x_2 = 5$ and $\ddot{x}_2 - 2x_1 + 4x_2 = 20 \sin 5t$. Assuming the initial conditions as $x_1(0) = \dot{x}_1(0) = x_2(0) = \dot{x}_2(0) = 0$, find the response of the system, using the central difference method with $\Delta t = 0.25$.

Section 11.6 Central Difference Method for Continuous Systems

11.21 The ends of a beam are elastically restrained by linear and torsional springs, as shown in Fig. 11.11. Using the finite difference method, express the boundary conditions.

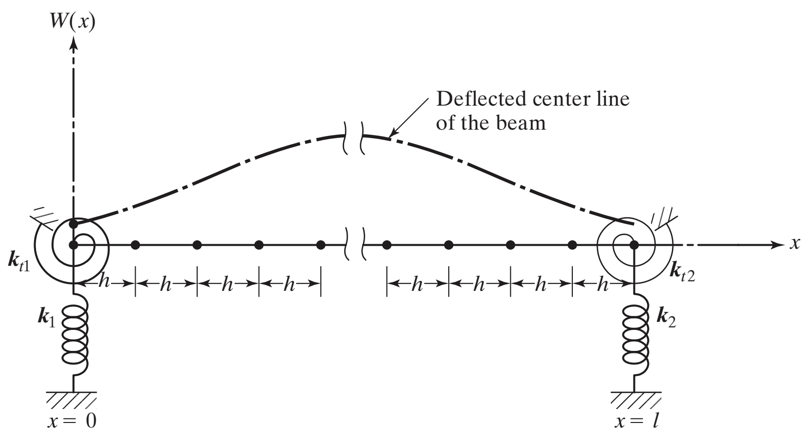

FIGURE 11.11

11.22 Using the fourth-order Runge-Kutta method, solve Problem 11.20.

11.23 Find the natural frequencies of a fixed-fixed bar undergoing longitudinal vibration, using three mesh points in the range $0 < x < l$.

11.24 Derive the finite difference equations governing the forced longitudinal vibration of a fixed-free uniform bar, using a total of n mesh points. Find the natural frequencies of the bar, using $n = 4$.

11.25 Derive the finite difference equations for the forced vibration of a fixed-fixed uniform shaft under torsion, using a total of n mesh points.

11.26 Find the first three natural frequencies of a uniform fixed-fixed beam.

11.27 Derive the finite difference equations for the forced vibration of a cantilever beam subjected to a transverse force $f(x, t) = f_0 \cos \omega t$ at the free end.

11.28 Derive the finite difference equations for the forced-vibration analysis of a rectangular membrane, using m and n mesh points in the x and y directions, respectively. Assume the membrane to be fixed along all the edges. Use the central difference formula.

Sections 11.7, 11.11 Runge-Kutta Method for Multidegree-Of-Freedom Systems and MATLAB Problems

11.29 Using `Program14.m` (fourth-order Runge-Kutta method), solve Problem 11.18 with $c = 1$.

11.30 Using `Program14.m` (fourth-order Runge-Kutta method), solve Problem 11.19.

11.31 Using `Program15.m` (central difference method), solve Problem 11.20.

Section 11.8, 11.11 Houbolt Method

11.32 Using `Program15.m` (central difference method), solve Problem 11.18 with $c = 1$.

11.33 Using `Program16.m` (Houbolt method), solve Problem 11.19.

11.34 Using `Program16.m` (Houbolt method), solve Problem 11.20.

Section 11.9 Wilson Method

11.35 Using the Wilson method with $\theta = 1.4$, solve Problem 11.18.

11.36 Using the Wilson method with $\theta = 1.4$, solve Problem 11.19.

11.37 Using the Wilson method with $\theta = 1.4$, solve Problem 11.20.

Section 11.10 Newmark Method

11.38 Using the Newmark method with $\alpha = \frac{1}{6}$ and $\beta = \frac{1}{2}$, solve Problem 11.18.

11.39 Using the Newmark method with $\alpha = \frac{1}{6}$ and $\beta = \frac{1}{2}$, solve Problem 11.19.

11.40 Using the Newmark method with $\alpha = \frac{1}{6}$ and $\beta = \frac{1}{2}$, solve Problem 11.20.

Section 11.11 MATLAB Problems

11.41 Using MATLAB function `ode23`, solve the differential equation $5\ddot{x} + 4\dot{x} + 3x = 6 \sin t$ with $x(0) = \dot{x}(0) = 0$.

11.42 The equations of motion of a two-degree-of-freedom system are given by

$$\begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + 5 \begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} F_1(t) \\ 0 \end{Bmatrix}$$

where $F_1(t)$ denotes a rectangular pulse of magnitude 5 acting over $0 \le t \le 2$. Find the solution of the equations using MATLAB.

11.43 Find the response of a simple pendulum numerically by solving the linearized equation:

$$\ddot{\theta} + \frac{g}{l}\theta = 0$$

with $\frac{g}{l} = 0.01$ and plot the response, $\theta(t)$, for $0 \le t \le 150$. Assume the initial conditions as $\theta(t = 0) = \theta_0 = 1$ rad and $\dot{\theta}(t = 0) = \dot{\theta}_0 = 1.5$ rad/s. Use the MATLAB function `ode23` for numerical solution.

11.44 Find the response of a simple pendulum numerically by solving the exact equation:

$$\ddot{\theta} + \frac{g}{l}\sin\theta = 0$$

with $\frac{g}{l} = 0.01$ and plot the response, $\theta(t)$, for $0 \le t \le 150$. Assume the initial conditions as $\theta(t = 0) = \theta_0 = 1$ rad and $\dot{\theta}(t = 0) = \dot{\theta}_0 = 1.5$ rad/s. Use the MATLAB function `ode23` for numerical solution.

11.45 Find the response of a simple pendulum numerically by solving the nonlinear equation:

$$\ddot{\theta} + \frac{g}{l}\left(\theta - \frac{\theta^3}{6}\right) = 0$$

with $\frac{g}{l} = 0.01$ and plot the response, $\theta(t)$, for $0 \leq t \leq 150$. Assume the initial conditions as $\theta(t = 0) = \theta_0 = 1$ rad and $\dot{\theta}(t = 0) = \dot{\theta}_0 = 1.5$ rad/s. Use the MATLAB function **`ode23`** for numerical solution.

11.46 Write a subroutine **`WILSON`** for implementing the Wilson method. Use this program to find the solution of Example 11.7.

11.47 Write a subroutine **`NUMARK`** for implementing the Newmark method. Use this subroutine to find the solution of Example 11.8.