

## CHAPTER 1

### Laplace Transform Methods

Laplace transform is a method frequently employed by engineers. By applying the Laplace transform, one can change an ordinary differential equation into an algebraic equation, as algebraic equation is generally easier to deal with. Another advantage of Laplace transform is in dealing the external force is either impulsive, (the force lasts a very short time period such as the bat hits a baseball) or the force is on and off for some regular or irregular period of time.

#### 1. The Laplace Transform

If  $f(t)$  is defined over interval  $[0, \infty)$ , the Laplace transform of  $f$ , denoted as  $\hat{f}(s)$ , is

$$\mathcal{L}(f) = \hat{f}(s) = \int_0^{\infty} e^{-st} f(t) dt$$

Our first theorem states when Laplace transform can be performed,

**THEOREM 1.1.** *If  $f(t)$  is (piecewise) continuous and there are positive numbers  $M, a$  such that*

$$|f(t)| \leq Me^{at}$$

*for all  $t \geq c$  Then  $\hat{f}(s)$  is defined for all  $s > c$*

The next result shows that Laplace transform is unique in the sense that different continuous functions will have different Laplace transform.

**THEOREM 1.2.** *If  $\hat{f}(s) = \hat{g}(s)$  for all  $s > c$ , then  $f(t) = g(t)$  at all  $t$  where both are continuous.*

Notice if  $f(t)$  and  $g(t)$  are piecewise continuous (continuous except at finite points where left and right limits exists,) their Laplace transforms can be same for all  $s > c$  even if they are different at the isolated discontinuous point. Since solutions of ordinary equations must be continuous, so this is of no important concern.

Due the uniqueness, we can define the inverse Laplace transform  $\mathcal{L}^{-1}$  as

$$\mathcal{L}^{-1}(\widehat{f})(t) = f(t).$$

**THEOREM 1.3.** *If both  $\widehat{f}(s)$  and  $\widehat{g}(s)$  exist for all  $s > c$ , then  $af(t) + bg(t)$  has Laplace transform for all constant  $a$  and  $b$  and*

$$\widehat{af + bg}(s) = a\widehat{f}(s) + b\widehat{g}(s), \text{ for all } s > c$$

So to find Laplace transform of summation, we just need to find Laplace transform of each term.

The next result shows that Laplace transform changes derivative into scalar multiplication, it is this property enable Laplace transform to change ODE into algebraic equation.

**THEOREM 1.4.** *Suppose that the function  $f(t)$  is continuous and piecewise smooth ( $f'(t)$  is piecewise continuous) for all  $t \geq 0$  and there are constants  $M, a$  such that*

$$|f(t)| \leq Me^{at} \text{ for } t \geq T,$$

Then  $\widehat{f}'(s)$  is defined for all  $s > a$  and

$$\widehat{f}'(s) = s\widehat{f}(s) - f(0).$$

Now if nth derivative  $f^n(t)$  is piecewise smooth, then

$$\widehat{f^{(n)}}(s) = s^n \widehat{f}(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \dots - f^{(n-1)}(0).$$

For example,

- **When  $n = 2$**

$$\widehat{f^{(2)}}(s) = s^2 \widehat{f}(s) - sf(0) - f'(0).$$

- **When  $n = 3$**

$$\widehat{f^{(3)}}(s) = s^3 \widehat{f}(s) - s^2 f(0) - sf'(0) - f''(0).$$

- **When  $n = 4$**

$$\widehat{f^{(4)}}(s) = s^4 \widehat{f}(s) - s^3 f(0) - s^2 f'(0) - sf''(0) - f^{(3)}(0).$$

The following table list most commonly used functions, and

$$u_a(t) = u(t - a) = \begin{cases} 0 & \text{if } t < a \\ 1 & \text{if } t \geq a \end{cases}$$

is the Heaviside function.

$f = \mathcal{L}^{-1}(\widehat{f})$	$\widehat{f}$	$f = \mathcal{L}^{-1}(\widehat{f})$	$\widehat{f}$
1	$\frac{1}{s}(s > 0)$	t	$\frac{1}{s^2}(s > 0)$
$t^n (n \geq 0)$	$\frac{n!}{s^{n+1}}(s > 0)$	$t^a (a > -1)$	$\frac{\Gamma(a+1)}{s^{a+1}}(s > 0)$
$e^{at}$	$\frac{1}{s-a}(s > 0)$	$\sin(kt)$	$\frac{k}{s^2+k^2}(s > 0)$
$\cos(kt)$	$\frac{s}{s^2+k^2}(s > 0)$	$\sinh(kt)$	$\frac{k}{s^2-k^2}(s >  k )$
$\cosh(kt)$	$\frac{s}{s^2-k^2}(s >  k )$	$u(t-a)$	$\frac{e^{as}}{s}(s > 0)$

EXAMPLE 1.1. Find  $\widehat{x}(s)$  if  $x'' + 2x' - 3x = e^t$ ,  $x(0) = 1$ ,  $x'(0) = -1$ .

**Solution** Apply Laplace transform on both side of the equation,

$$\mathcal{L}(x'' + 2x' - 3x)(s) = \mathcal{L}(e^t)(s)$$

Using the linear property  $\widehat{af + bg}(s) = a\widehat{f}(s) + b\widehat{g}(s)$  we have

$$\mathcal{L}(x'' + 2x' - 3x)(s) = \widehat{x}''(s) + 2\widehat{x}'(s) - 3\widehat{x}(s).$$

Together with  $\widehat{x}''(s) = s^2\widehat{x}(s) - sx(0) - x'(0)$  and  $\widehat{x}'(s) = s\widehat{x}(s) - x(0)$

We have, due to  $x(0) = 1$ ,  $x'(0) = -1$ ,

$$\begin{aligned} \mathcal{L}(x'' + 2x' - 3x)(s) &= \widehat{x}''(s) + 2\widehat{x}'(s) - 3\widehat{x}(s) \\ &= [s^2\widehat{x}(s) - sx(0) - x'(0)] + 2[s\widehat{x}(s) - x(0)] - 3\widehat{x}(s) \\ &= (s^2 + 2s - 3)\widehat{x}(s) - s - 1 \end{aligned}$$

So we have an algebraic equation for  $\widehat{x}(s)$ ,

$$(s^2 + 2s - 3)\widehat{x}(s) - s - 1 = \widehat{e^t}(s) = \frac{1}{s-1}$$

Solve this equation for  $\widehat{x}(s)$ ,

$$\begin{aligned} (s^2 + 2s - 3)\widehat{x}(s) - s - 1 &= \widehat{e^t}(s) = \frac{1}{s-1} \\ (s^2 + 2s - 3)\widehat{x}(s) &= \frac{1}{s-1} + s + 1 \\ (s^2 + 2s - 3)\widehat{x}(s) &= \frac{1}{s-1} + \frac{(s+1)(s-1)}{s-1} \\ (s^2 + 2s - 3)\widehat{x}(s) &= \frac{1+(s+1)(s-1)}{s-1} \\ (s^2 + 2s - 3)\widehat{x}(s) &= \frac{1+(s^2-1)}{s-1} \\ (s^2 + 2s - 3)\widehat{x}(s) &= \frac{s^2}{s-1} \end{aligned}$$

we get

$$\widehat{x}(s) = \frac{s^2}{(s-1)(s^2+2s-3)}$$

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## 2. Using Mathcad

Mathcad can help us in find both Laplace transform and inverse Laplace transform.

To use Mathcad to find Laplace transform, we first enter the expression of the function, then press [Shift][Ctrl][.], in the place holder type the key word **laplace** followed by comma(,) and the variable name. For example, to find the Laplace of  $f(t) = t^2 \sin(at)$ , you first enter the expression  $t^2 \sin(at)$  by typing,

$$t^2 * \sin(a * t),$$

then press [Shift][Ctrl][.], and entering keyword **laplace** followed by comma(,) and t, you will get

$$t^2 \sin(at) \text{laplace, } t \rightarrow \frac{8as^2}{(s^2 + a^2)^3} - \frac{2a}{(s^2 + a^2)^2}$$

If you want a simplified result, you type comma(,) after entering the variable name and keyword **simplify**, so the following input

$$t^2 * \sin(a * t) [\text{Shift}] [\text{Ctrl}] [.] \text{laplace, } t, \text{simplify}$$

will produce the result

$$t^2 \sin(at) \text{laplace, } t, \text{simplify} \rightarrow -\frac{2a(a^2 - 3s^2)}{(s^2 + a^2)^3}$$

Similarly, to use Mathcad to find inverse Laplace transform, we first enter the expression, then press [Shift][Ctrl][.], in the place holder type the key word **invlaplace** followed by comma(,) and the variable name. For example, to find the inverse Laplace of  $\frac{8as^2}{(s^2+a^2)^3}$ , you first enter the expression  $\frac{8as^2}{(s^2+a^2)^3}$  as,

$$8a * s^2 / (s^2 + a^2)^3,$$

then press [Shift][Ctrl][.], and entering keyword **invlaplace** followed by comma(,) and s, and for simplifying the result, by another comma(,) and keyword **simplify**, we get

$$\frac{8as^2}{(s^2 + a^2)^3} \text{invlaplace, } s, \text{simplify} \rightarrow \frac{1}{a^2} \left( \sin(at) - ta \cos(at) + t^2 a^2 \sin(at) \right)$$

When using Mathcad together with Laplace transform to solve an ODE

$$a_n x^{(n)} + a_{n-1} x^{(n-1)} + \cdots + a_1 x' + a_0 x = f(t)$$

we follow these steps,

- **Step One:** Apply Laplace to both sides of equation. Using Mathcad to find Laplace transform of  $f(t)$ .

- **Step Two:** Using the linear property and

$$\widehat{f^{(n)}}(s) = s^n \widehat{f}(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - f^{(n-1)}(0).$$

to find an algebraic equation for  $\widehat{x}(s)$ ,

- **Step Three:** Solve the equation obtained in **Step Two** for  $\widehat{x}(s)$  and using Mathcad to find inverse transform which will be the solution  $x(t)$ .

EXAMPLE 2.1. Find general solution to  $x'' + 2x' + 3x = t^2 e^t$ .

**Solution** Since we want to find general solution, we set  $x(0) = a$  and  $x'(0) = b$

- **Step One** Apply Laplace transform to both sides of the equation and find Laplace transform for  $t^2 e^t$ .

$$\mathcal{L}(x'' + 2x' + 3x)(s) = \mathcal{L}(t^2 e^t)(s)$$

Type,

`t^2*e^t[Shift][Ctrl][.]laplace,t,simplify`

we find

$$\mathcal{L}(t^2 e^t)(s) = \frac{2}{(s-1)^3}$$

- **Step Two:** Using linear property to find an equation for  $\widehat{x}(s)$ .

$$\begin{aligned} \mathcal{L}(x'' + 2x' + 3x)(s) &= (\widehat{x}'' + 2\widehat{x}' + 3\widehat{x})(s) \\ &= s^2 \widehat{x}(s) - sa - b + 2(s\widehat{x}(s) - a) + 3\widehat{x}(s) \\ &= (s^2 + 2s + 3)\widehat{x}(s) - sa - b - 2a \end{aligned}$$

The equation for  $\widehat{x}(s)$  is

$$(s^2 + 2s + 3)\widehat{x}(s) - sa - b - 2a = \frac{2}{(s-1)^3}$$

Hence,

$$\widehat{x}(s) = \frac{sa+b+2a}{(s^2+2s+3)} + \frac{2}{(s-1)^3(s^2+2s+3)}$$

- **Step Three:** Using Mathcad to find inverse Laplace transform and  $x(t)$ , we enter,

`s*a+b+a/(s^2+2s+3)+2/(s-1)^3(s^2+2s+3)[Shift][Ctrl][.]laplace,s`

and get

$$x(t) = ae^{-t} \left( \frac{\sqrt{2}}{2} \sin(\sqrt{2}t) + \cos(\sqrt{2}t) \right) + b \frac{\sqrt{2}}{2} e^{-t} \sin(\sqrt{2}t) \\ + e^t \left( \frac{1}{3}t + \frac{2}{9} + \frac{2}{9} \cos(\sqrt{2}t) \right) + \frac{\sqrt{2}}{18} e^{-t} \sin(\sqrt{2}t)$$

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**2.1. Solving System of equations.** We can use Laplace transform method to solve system of differential equations. The procedure is the same as solving a higher order ODE . But, after applying Laplace transform to each equation, we get a system of linear equations whose unknowns are the Laplace transform of the unknown functions. The following example shows how we can use Laplace method with Mathcad to solve system of differential equations.

EXAMPLE 2.2. Find solution to

$$\begin{aligned} x' &= 3x - 4y + t^2 \\ y' &= -x + 5y + e^t \\ x(0) &= 1, \quad y(0) = -1 \end{aligned}$$

**Solution**

- **Step One:** Apply Laplace transform to both sides of the equations and use Mathcad to find Laplace transform of  $t^2$  and  $e^t$ ,

$$\begin{aligned} \mathcal{L}(x') &= \mathcal{L}(3x - 4y + t^2) \\ \mathcal{L}(y') &= \mathcal{L}(-x + 5y + e^t) \end{aligned}$$

and  $\mathcal{L}(t^2)(s) = \frac{6}{s^3}$  and  $\mathcal{L}(e^t)(s) = \frac{1}{s-1}$ ,

- **Step Two:** Apply linear property to get a system of equations for  $\hat{x}(s)$  and  $\hat{y}(s)$ , due to  $\mathcal{L}(x') = s\hat{x}(s) - x(0)$  and  $\mathcal{L}(y') = s\hat{y}(s) - y(0)$ ,

$$\begin{aligned} s\hat{x}(s) - 1 &= 3\hat{x}(s) - 4\hat{y}(s) + \frac{6}{s^3} \\ s\hat{y}(s) + 1 &= -\hat{x}(s) + 5\hat{y}(s) + \frac{1}{s-1} \end{aligned}$$

from which we get

$$\begin{aligned} (s-3)\hat{x}(s) + 4\hat{y}(s) &= 1 + \frac{6}{s^3} \\ \hat{x}(s) + (s-5)\hat{y}(s) &= -1 + \frac{1}{s-1} \end{aligned}$$

Define  $\hat{\mathbf{x}}(s) = \begin{bmatrix} \hat{x}(s) \\ \hat{y}(s) \end{bmatrix}$ ,  $\mathbf{A} = \begin{bmatrix} s-3 & 4 \\ 1 & s-5 \end{bmatrix}$ , and  $\mathbf{b}(s) = \begin{bmatrix} 1 + \frac{6}{s^3} \\ -1 + \frac{1}{s-1} \end{bmatrix}$ , we have, in matrix form,

$$\mathbf{A}\hat{\mathbf{x}}(s) = \mathbf{b}(s)$$

- **Step Three:** Solve the algebraic equation and apply the inverse Laplace transform, using Mathcad we have

$$\hat{\mathbf{x}}(s) = \mathbf{A}^{-1}\mathbf{b}(s) = \frac{1}{s^2 - 8s + 11} \begin{bmatrix} \frac{(s-5)(s^3+6)}{s^3} - 4\frac{2-s}{s-1} \\ \frac{(s-3)(2-s)}{s-1} - \frac{s^3+6}{s^3} \end{bmatrix}$$

$$\mathbf{x}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} -\frac{15}{11}t^2 - \frac{174}{121}t - \frac{1062}{1331} + e^{4t} \left( \frac{3742}{1331} \cosh(\sqrt{5}t) - \frac{2334}{6655} \sinh(\sqrt{5}t) \right) - e^t \\ -\frac{3}{11}t^2 - \frac{48}{121}t - \frac{318}{1331} - e^{4t} \left( \frac{695}{2662} \cosh(\sqrt{5}t) + \frac{8143}{1331} \sinh(\sqrt{5}t) \right) - \frac{1}{2}e^t \end{bmatrix}$$

### 3. The Piecewise defined Functions

When solving an ODE  $a_n x^{(n)} + a_{n-1} x^{(n-1)} + \dots + a_1 x' + a_0 x = f(t)$  with multiple defined function  $f(t)$ , we need to change  $f(t)$  into linear combinations of step functions  $u_a(t)$ .

Here  $u_a(t) = u(t - a) = \begin{cases} 0 & \text{if } t < a \\ 1 & \text{if } t \geq a \end{cases}$

and

$$u(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t \geq 0 \end{cases}$$

is called **Heaviside** function or **unit step function**.

We have graphed some step functions,

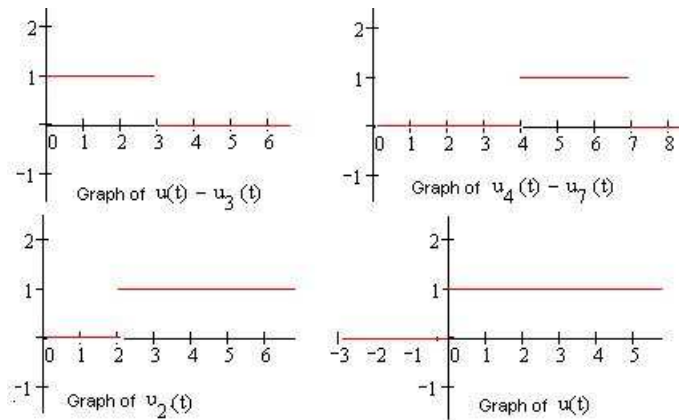


FIGURE 1. Step Functions

$$\text{In general, if } b > a, u_a(t) - u_b(t) = \begin{cases} 0 & \text{if } 0 < t < a \\ 1 & \text{if } a < t \leq b \\ 0 & \text{if } t > b \end{cases}$$

To express a multiple defined function as linear combination of step functions, we need, for each finite interval  $[a, b]$  on which  $f$  has a different expression, multiply the expression by the difference of step functions  $u_a(t) - u_b(t)$ ; for the expression defined on an infinite interval  $[c, \infty)$  we just multiply the expression by  $u_c(t)$ .  $f(t)$  is the summation of all the products, as illustrated in the following examples,

EXAMPLE 3.1. *Express*

$$f(t) = \begin{cases} t^2 - t + 3 & \text{if } 0 < t < 2 \\ e^t & \text{if } 2 \leq t < 5 \\ 2t \sin(t) & \text{if } t \geq 5 \end{cases}$$

as linear combination of step functions.

**Solution** Notice  $f(t)$  has three different expresses over interval  $[0, 2)$ ,  $[2, 5)$  and  $[5, \infty)$ , so we have two difference,  $u(t) - u_2(t)$  and  $u_2(t) - u_5(t)$ , and three products,

- $(t^2 - t + 3)(u(t) - u_2(t))$  for  $(t^2 - t + 3)$  defined on  $[0, 2)$ ;
- $e^t(u_2(t) - u_5(t))$  for  $e^t$  defined on  $[2, 5)$ ;
- $2t \sin(t)u_5(t)$  for  $2t$  defined on  $[5, \infty)$ .

$f(t)$  is sum of these three products,

$$\begin{aligned} f(t) &= (t^2 - t + 3)(u(t) - u_2(t)) + e^t(u_2(t) - u_5(t)) + 2t \sin(t)u_5(t) \\ &= (t^2 - t + 3)u(t) + (e^t - t^2 + t - 3)u_2(t) + (2t \sin(t) - e^t)u_5(t). \end{aligned}$$

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EXAMPLE 3.2. *Express*

$$f(t) = \begin{cases} t^2 e^t & \text{if } 0 < t < 3 \\ \cos(t) & \text{if } 3 \leq t < 6 \\ 2t & \text{if } t \geq 6 \end{cases}$$

as linear combination of step functions.

**Solution** Notice  $f(t)$  has three different expresses over interval  $[0, 3)$ ,  $[3, 6)$  and  $[6, \infty)$ , so we need to have two difference,  $u(t) - u_3(t)$  and  $u_3(t) - u_6(t)$  and

$$\begin{aligned} f(t) &= t^2 e^t(u(t) - u_3(t)) + \cos(t)(u_3(t) - u_6(t)) + 2tu_6(t) \\ &= t^2 e^t u(t) + (\cos(t) - t^2 e^t)u_3(t) + (2t - \cos(t))u_6(t). \end{aligned}$$

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**3.1. Heaviside Function in Mathcad.** You can get Heaviside function from Insert—>functions, click on **all** and the box of the function list, type heaviside, Click **Ok**, to insert the function.

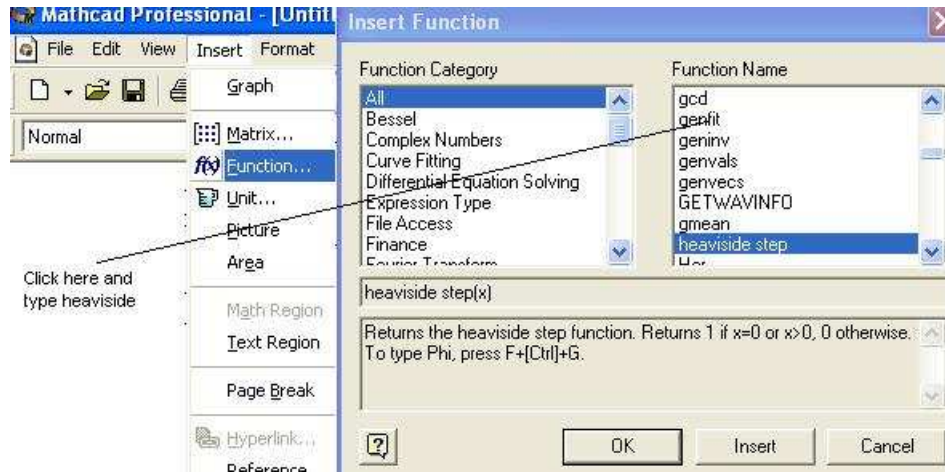


FIGURE 2. Insert Heaviside Function

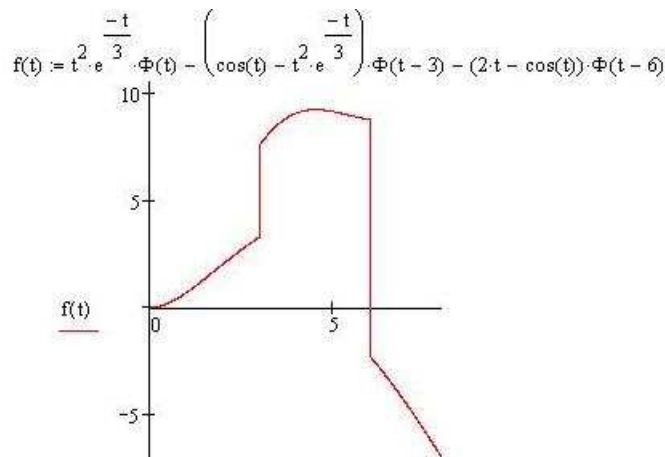
The step function is given by  $u_a(t) = \Phi(t - a)$ . Hence the function

$$f(t) = \begin{cases} t^2 e^{-\frac{t}{3}} & \text{if } 0 < t < 3 \\ \cos(t) & \text{if } 3 \leq t < 6 \\ 2t & \text{if } t \geq 6 \end{cases}$$

can be defined in Mathcad as

$$f(t) = t^2 e^{-\frac{t}{3}} \Phi(t) + (\cos(t) - t^2 e^{-\frac{t}{3}}) \Phi(t - 3) + (2t - \cos(t)) \Phi(t - 6).$$

The following graph shows this function in Mathcad and its graph,



### 3.2. Solve ODE With Piecewise Input Function.

EXAMPLE 3.3. Find solution to  $x'' + 4x = f(t)$ ,  $x(0) = 0$ ,  $x'(0) = -1$ , where  $f(t) = \begin{cases} \cos(2t) & \text{if } 0 \leq t < \pi \\ 0 & \text{if } t \geq \pi \end{cases}$

**Solution** First we express  $f(t)$  as linear combination of step function,  $f(t) = \cos(2t)(u(t) - u_{2\pi}(t))$  Apply Laplace transform to both sides of the equation and using Mathcad we get

$$(s^2 + 4)\hat{x}(s) = -1 + (1 - e^{-\pi s})\frac{s}{s^2 + 4}$$

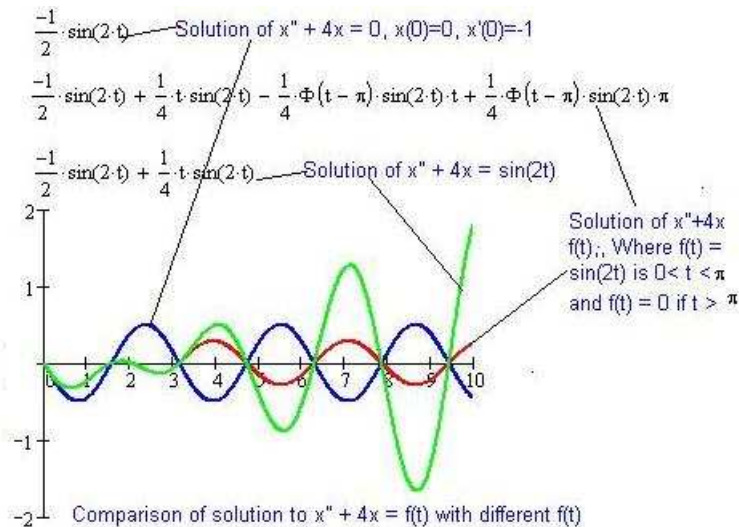
$$\hat{x}(s) = \frac{-1}{s^2 + 4} + (1 - e^{-\pi s})\frac{s}{(s^2 + 4)^2}$$

Apply inverse Laplace transform, we get,

$$x(t) = \begin{cases} -\frac{1}{2}\sin(2t) + \frac{1}{4}t\sin(2t) & \text{if } 0 \leq t < \pi \\ 0 & \text{if } \frac{\pi-2}{4}\sin(2t) \geq \pi \end{cases}$$

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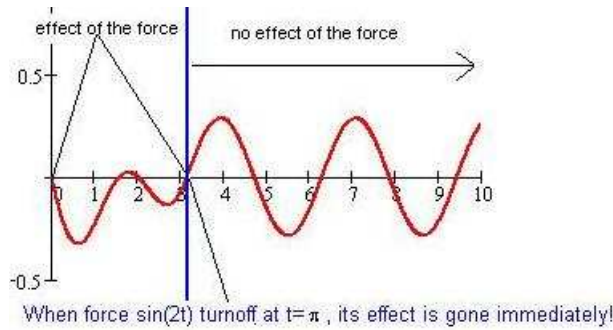
The equation  $x'' + 4x = f(t)$  models a spring system that with one spring, one end of the spring is fixed and a object with 1 unit mass attached at one end. The Hook's constant  $k = 4$ .  $f(t)$  is external force applied to the system. In the Example 3.3, the force  $\sin(2t)$  only applied at first  $\pi$  unit of time (second). The following diagram compares the solution of this equation with different input  $f(t)$ .



The graph contains solutions for the following three cases,

- (1) No external force is applied (blue curve).
- (2) Force of  $\sin(2t)$  is applied at first  $\pi$  seconds and turnoff (red curve).
- (3) Force of  $\sin(2t)$  is constantly applied (green curve).

The initial configuration is that the spring is at equilibrium position  $x(0) = 0$  and is compressed at 1 unit speed (feet/s)  $x'(0) = -1$ . From the graph we can clearly see that when the force is turnoff at  $t = \pi$  its effect is immediately gone (as shown below)!



#### 4. Delta Function

Delta function is one of so-called generalized functions, which are not functions in ordinary sense but as an operators that sometimes can be represented by ordinary functions.

DEFINITION 4.1. *The Dirac delta function at  $a$ ,  $\delta_a(t)$ , is an operator that satisfies*

$$\int_0^{\infty} g(t)\delta_a(t) dt = g(a)$$

for any continuous function  $g(t)$ .

In Physics, if  $f(t)$ ,  $a \leq t \leq b$  is a force that acts only during a short period of time interval  $[a, b]$ , the **impulse**  $p$  of force  $f(t)$  is computed as

$$\int_a^b f(t) dt.$$

$\delta_a(t)$  can be viewed as an instantaneous unit impulse that occurs precisely at the instant  $t = a$ . And  $p\delta_a(t)$  is an instantaneous  $p$  units impulse that occurs precisely at the time  $t = a$ .

$\delta_a(t)$  is an important function used in modelling real phenomena, for example,

- In modelling the movement of a baseball after being hit by bat.
- In spring-mass system, where a object with given mass it attached to one end of a spring whose other end is attached to a steady object (such as wall), modelling the movement when hits the object with a hammer.
- Modelling the current in a closed circuit when the switch is turn on and off instantly.

It turns out that we can stretch ourself a little and compute the Laplace transform of  $\delta_a(t)$ . Let  $\delta(t) = \delta_0(t)$  then  $\delta_a(t) = \delta(t - a)$ .

**THEOREM 4.1.**  $\widehat{\delta}(s) = 1$  and  $\widehat{\delta}_a(s) = e^{-as}$ .

**EXAMPLE 4.1.** A mass  $m = 1$  is attached to a spring with constant  $k = 4$ ; there is not dashpot. The mass is released from the rest with  $x(0) = 3$ . At the instant  $t = \pi$  the mass is struck with a hammer, providing an impulse  $p = 8$ . Determine the motion of the mass.

**Solution** In general  $mx'' + kx = f(t)$  models the movement of a mass attached to the end of spring with no dashpot. So we need to solve

$$x'' + 4x = 8\delta_\pi(t), \quad x(0) = 3, x'(0) = 0.$$

Applying Laplace transform to both sides of the equation,

$$\begin{aligned} \widehat{x}''(s) + 4\widehat{x}(s) &= 8\delta_\pi(s) \\ s^2\widehat{x} - sx(0) - x'(0) + 4\widehat{x}(s) &= 8e^{-\pi s} \\ (s^2 + 4)\widehat{x}(s) - 3s &= 8e^{-\pi s} \end{aligned}$$

Solve for  $\widehat{x}(s)$ , we get

$$\widehat{x}(s) = \frac{3s}{s^2 + 4} + \frac{8}{s^2 + 4}e^{-\pi s}$$

Using Mathcad , we can find inverse Laplace transform,

$$x(t) = 2 \cos(2t) + 4u_\pi(t) \sin(2t) = \begin{cases} 2 \cos(2t) & \text{if } 0 \leq t < \pi \\ 2 \cos(2t) + 4 \sin(2t) & \text{if } t \geq \pi \end{cases}$$

So clearly the impact of the impulse is felt by the spring and change the its movement immediately. The solution curve is shown below,

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#### 4. DELTA FUNCTION

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$x(t) := 3 \cdot \cos(2t)$  Solution with no impulse,  $x'' + 4x = 0$ ,  $x(0) = 3$ ,  $x'(0) = 0$

$x(t) := 3 \cdot \cos(2t) + 4 \Phi(t - \pi) \cdot \sin(2t)$  Solution of 8 impuls at time  $t = \pi$ ,  
 $x'' + 4x = 8\delta_x(t)$ ,  $x(0) = 3$ ,  $x'(0) = 0$

