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Plate Theories

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Different plate theories

ESL (**E**quivalent **S**ingle **L**ayer) plate theories:

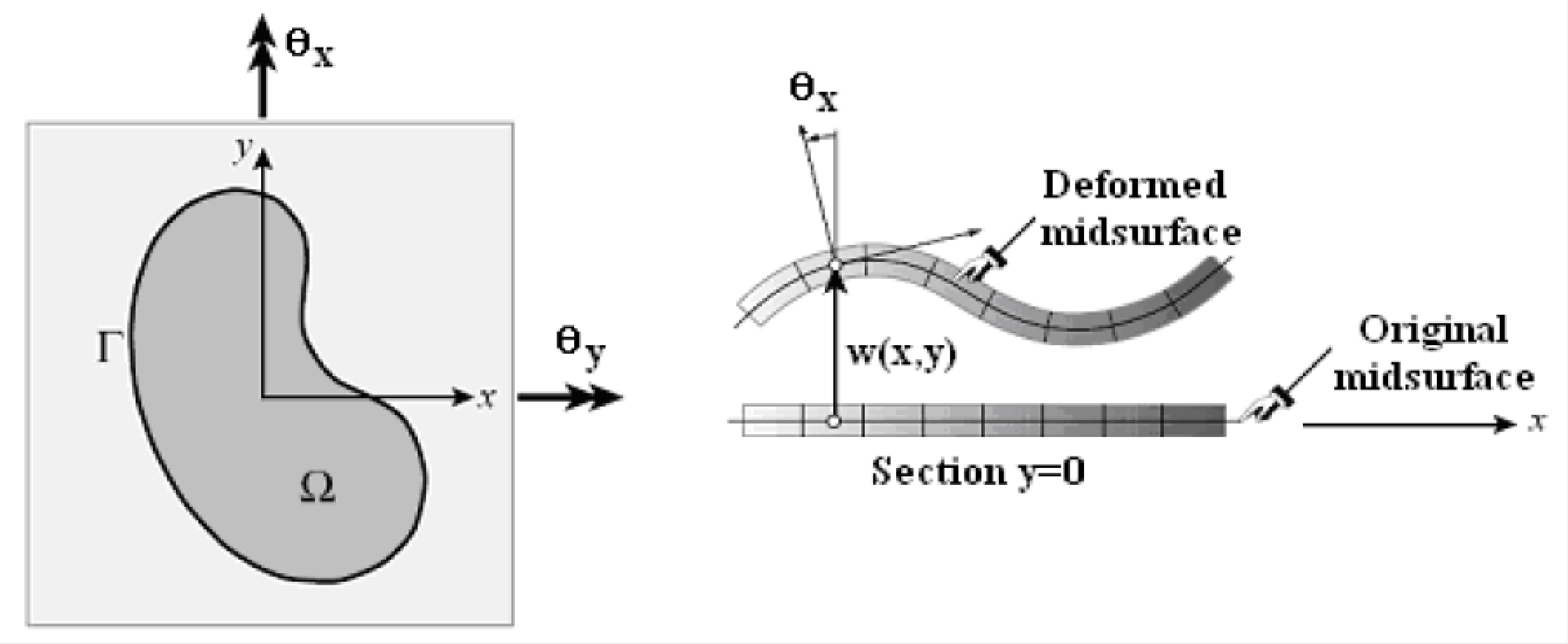
- **CLPT** (**CL**assical **P**late **T**heory)
- **FSDT** (**F**irst order **S**hear **D**eformation **T**heory)
- **HSDT** (**H**igher order **S**hear **D**eformation **T**heory)

Layerwise plate theories:

- Layerwise FSDT
- Layerwise HSDT

Kirchhoff hypotheses for CLPT

1. straight lines normal to the mid-surface remain straight after deformation
2. the thickness of the plate does not change during a deformation.
3. straight lines normal to the mid-surface remain normal to the mid-surface after deformation



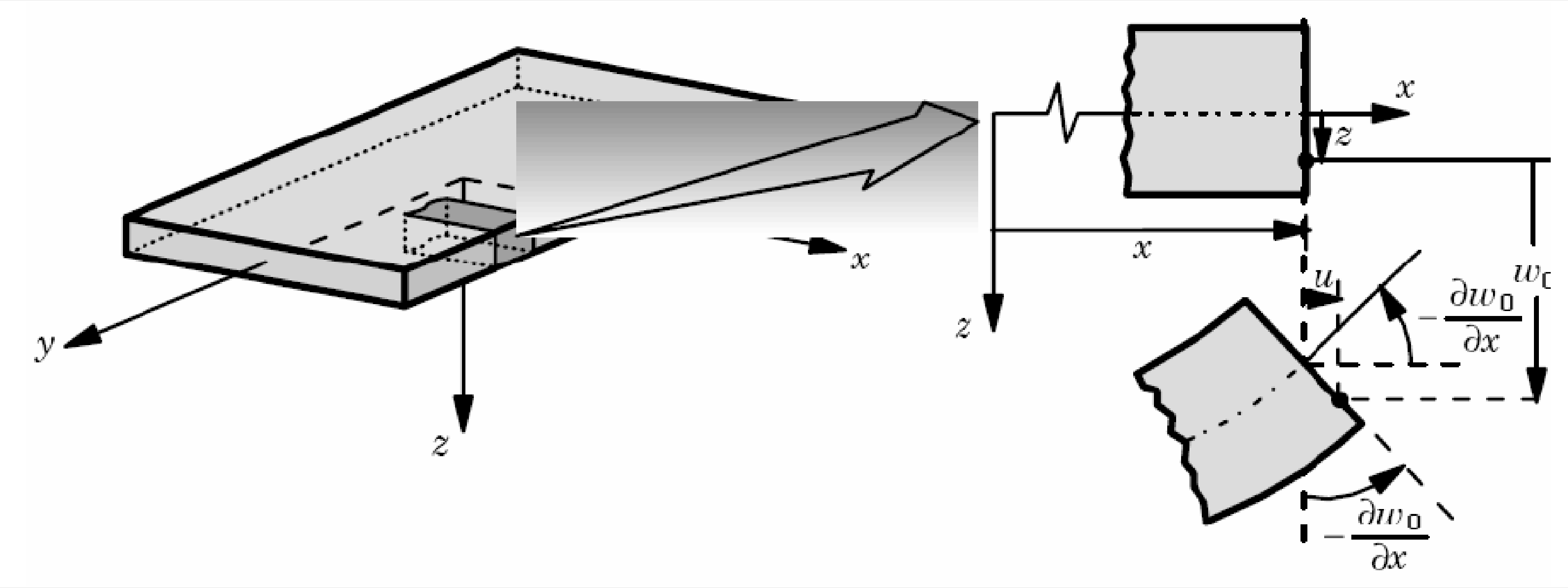
Displacement components for CLPT

From first and second Kirchhoff hypotheses:

$$\varepsilon_{zz} = \frac{\partial w}{\partial z} = 0$$

w component is an independent parameter from z.
From third Kirchhoff hypotheses:

$$\varepsilon_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = 0$$
$$\varepsilon_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} = 0$$



Displacement components using CLPT

Considering u_0 and v_0 as the values of the displacement on the mid-plane:

$$u(x,y,z) = u_0(x,y) - z\frac{\partial w_0}{\partial x}$$

$$v(x,y,z) = v_0(x,y) - z\frac{\partial w_0}{\partial y}$$

$$w(x,y,z) = w_0(x,y)$$

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Obtaining Strain components

Using Lagrange - Green strain tensor:

$$E_{jk} = \frac{1}{2}\left(\frac{\partial u_j}{\partial X_k} + \frac{\partial u_k}{\partial X_j} + \frac{\partial u_m}{\partial X_j}\frac{\partial u_m}{\partial X_k}\right)$$

$$E_{11} = \frac{\partial u}{\partial x} + \frac{1}{2}\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial x}\right)^2\right]$$

$$E_{22} = \frac{\partial v}{\partial y} + \frac{1}{2}\left[\left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right]$$

$$E_{33} = \frac{\partial w}{\partial z} + \frac{1}{2}\left[\left(\frac{\partial u}{\partial z}\right)^2 + \left(\frac{\partial v}{\partial z}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2\right]$$

$$E_{12} = \frac{1}{2}\left[\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\frac{\partial v}{\partial x} + \frac{\partial w}{\partial y}\frac{\partial w}{\partial x}\right]$$

$$E_{13} = \frac{1}{2}\left[\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} + \frac{\partial u}{\partial x}\frac{\partial u}{\partial z} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial z} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial z}\right]$$

$$E_{23} = \frac{1}{2}\left[\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} + \frac{\partial u}{\partial y}\frac{\partial u}{\partial z} + \frac{\partial v}{\partial y}\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\frac{\partial w}{\partial z}\right]$$

strain components using CLPT

Considering small rotation:

$$\frac{\partial u}{\partial x},\frac{\partial u}{\partial y},\frac{\partial v}{\partial x},\frac{\partial v}{\partial y},\frac{\partial w}{\partial z} = order(\varepsilon)$$

Considerable terms in the analysis:

$$\left(\frac{\partial w}{\partial x}\right)^2,\left(\frac{\partial w}{\partial y}\right)^2,\left(\frac{\partial w}{\partial x}\right)\left(\frac{\partial w}{\partial y}\right)$$

$$\varepsilon_x = \varepsilon^0{}_x + z\psi_x$$
$$\varepsilon_y = \varepsilon^0{}_y + z\psi_y$$
$$\varepsilon_{xy} = \varepsilon^0{}_{xy} + z\psi_{xy}$$

$$\varepsilon^0{}_x = \frac{\partial u_0}{\partial x} \qquad \psi_x = -\frac{\partial^2 w_0}{\partial x^2}$$

$$\varepsilon^0{}_y = \frac{\partial v_0}{\partial y} \qquad \psi_y = -\frac{\partial^2 w_0}{\partial y^2}$$

$$\gamma^0{}_{xy} = \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} + \frac{\partial w_0}{\partial x}\frac{\partial w_0}{\partial y} \qquad \psi_{xy} = -2\frac{\partial^2 w_0}{\partial x \partial y}$$

Strains, Stresses and resultant forces using CLPT

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_{xy} \end{Bmatrix} = \begin{Bmatrix} \varepsilon^0{}_x \\ \varepsilon^0{}_y \\ \varepsilon^0{}_{xy} \end{Bmatrix} + z \begin{Bmatrix} \psi_x \\ \psi_y \\ \psi_{xy} \end{Bmatrix}$$

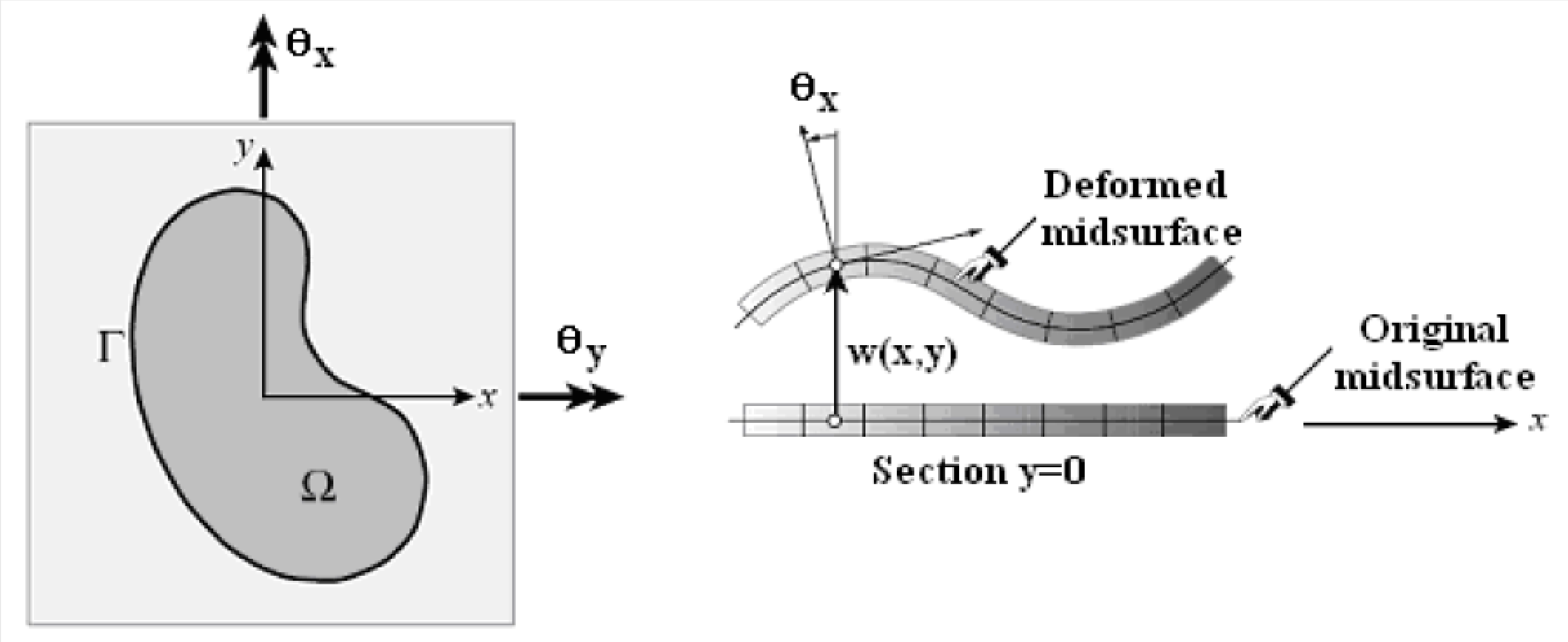
$$\begin{Bmatrix} \varepsilon^0{}_x \\ \varepsilon^0{}_y \\ \varepsilon^0{}_{xy} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial u_0}{\partial x} \\ \dfrac{\partial v_0}{\partial y} \\ \dfrac{\partial u_0}{\partial y} + \dfrac{\partial v_0}{\partial x} \end{Bmatrix} \qquad \begin{Bmatrix} \psi_x \\ \psi_y \\ \psi_{xy} \end{Bmatrix} = \begin{Bmatrix} -\dfrac{\partial^2 w_0}{\partial x^2} \\ -\dfrac{\partial^2 w_0}{\partial y^2} \\ -2\dfrac{\partial^2 w_0}{\partial x \partial y} \end{Bmatrix}$$

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_{xy} \end{Bmatrix}$$

$$\begin{Bmatrix} \{N\} \\ \{M\} \end{Bmatrix} = \begin{bmatrix} [A] & [B] \\ [B] & [D] \end{bmatrix} \begin{Bmatrix} \{\varepsilon^0\} \\ \{\psi\} \end{Bmatrix}$$

Kirchhoff hypotheses for FSDT

1. straight lines normal to the mid-surface remain straight after deformation
2. the thickness of the plate does not change during a deformation.



Displacement components for FSDT

From first and second Kirchhoff hypotheses:

$$\varepsilon_{zz} = \frac{\partial w}{\partial z} = 0$$

w component is an independent parameter from z.
From third Kirchhoff hypotheses:

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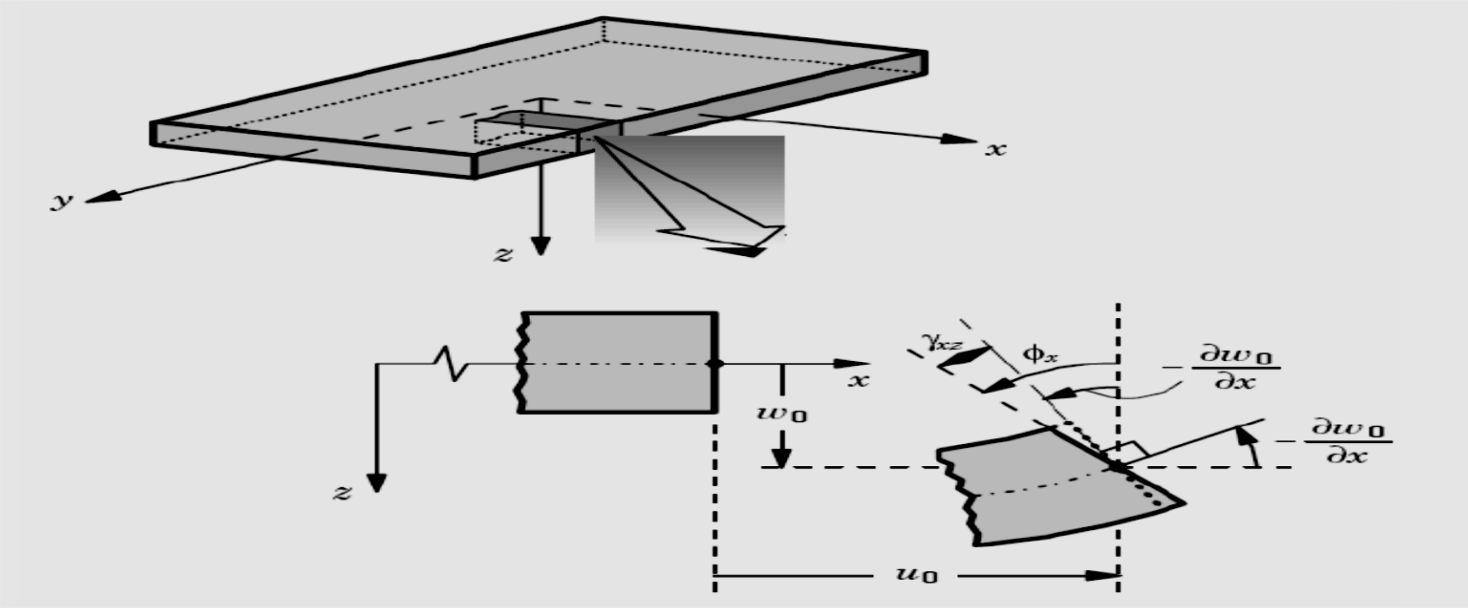

Displacement components using FSDT

Considering u_0 and v_0 as the values of the displacement on the mid-plane using FSDT:

$$u(x,y,z) = u_0(x,y) + z\phi_x$$
$$v(x,y,z) = v_0(x,y) + z\phi_y$$
$$w(x,y,z) = w_0(x,y)$$

ϕ_x and ϕ_y are independent parameters.

For CLPT we had:

$$u(x,y,z) = u_0(x,y) - z\frac{\partial w_0}{\partial x}$$
$$v(x,y,z) = v_0(x,y) - z\frac{\partial w_0}{\partial y}$$
$$w(x,y,z) = w_0(x,y)$$

In CLPT: $\phi_x = -\frac{\partial w_0}{\partial x}$ and $\phi_y = -\frac{\partial w_0}{\partial y}$

strain components using FSDT

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{yz} \\ \gamma_{xz} \\ \varepsilon_{xy} \end{Bmatrix} = \begin{Bmatrix} \varepsilon^0{}_x \\ \varepsilon^0{}_y \\ \gamma^0{}_{yz} \\ \gamma^0{}_{xz} \\ \varepsilon^0{}_{xy} \end{Bmatrix} + z \begin{Bmatrix} \psi_x \\ \psi_y \\ 0 \\ 0 \\ \psi_{xy} \end{Bmatrix}$$

$$\varepsilon^0{}_x = \frac{\partial u_0}{\partial x} \qquad \psi_x = \frac{\partial \phi_x}{\partial x}$$

$$\varepsilon^0{}_y = \frac{\partial v_0}{\partial y} \qquad \psi_y = \frac{\partial \phi_y}{\partial y}$$

$$\gamma^0{}_{xz} = \frac{\partial w_0}{\partial x} + \phi_x$$

$$\gamma^0{}_{yz} = \frac{\partial w_0}{\partial y} + \phi_y$$

$$\varepsilon^0{}_{xy} = \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} \qquad \psi_{xy} = \frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x}$$

$$\varepsilon_z = 0$$

Strains, Stresses and resultant forces using FSDT

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{yz} \\ \gamma_{xz} \\ \varepsilon_{xy} \end{Bmatrix} = \begin{Bmatrix} \varepsilon^0{}_x \\ \varepsilon^0{}_y \\ \gamma^0{}_{yz} \\ \gamma^0{}_{xz} \\ \varepsilon^0{}_{xy} \end{Bmatrix} + z \begin{Bmatrix} \psi_x \\ \psi_y \\ 0 \\ 0 \\ \psi_{xy} \end{Bmatrix} \quad where: \quad \{\varepsilon^0\} = \begin{Bmatrix} \frac{\partial u_0}{\partial x} \\ \frac{\partial v_0}{\partial y} \\ \frac{\partial w_0}{\partial y} + \phi_y \\ \frac{\partial w_0}{\partial x} + \phi_x \\ \frac{\partial v_0}{\partial x} + \frac{\partial u_0}{\partial y} \end{Bmatrix} \quad \{\psi\} = \begin{Bmatrix} \frac{\partial \phi_x}{\partial x} \\ \frac{\partial \phi_y}{\partial y} \\ 0 \\ 0 \\ \frac{\partial \phi_y}{\partial x} + \frac{\partial \phi_x}{\partial y} \end{Bmatrix}$$

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xz} \\ \tau_{yz} \\ \tau_{xy} \end{Bmatrix} = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & 0 & 0 & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & 0 & 0 & \bar{Q}_{26} \\ 0 & 0 & \bar{Q}_{44} & 0 & 0 \\ 0 & 0 & 0 & \bar{Q}_{55} & 0 \\ \bar{Q}_{16} & \bar{Q}_{26} & 0 & 0 & \bar{Q}_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{yz} \\ \gamma_{xz} \\ \varepsilon_{xy} \end{Bmatrix}$$

$$\begin{Bmatrix} \{N\} \\ \{M\} \\ \{Q\} \end{Bmatrix} = \begin{bmatrix} [A] & [B] & 0 \\ [B] & [D] & 0 \\ 0 & 0 & [A_s] \end{bmatrix} = \begin{Bmatrix} \{\varepsilon^0\} \\ \{\psi\} \\ \{\gamma^0\} \end{Bmatrix} \qquad [A_s] = K \int_{-\frac{h}{2}}^{\frac{h}{2}} \begin{bmatrix} \bar{Q}_{44} & 0 \\ 0 & \bar{Q}_{55} \end{bmatrix} dz$$

h is the thickness of the lamina and **k** is Shear energy correction coefficient and it is about **0.833**

ESL and layerwise theories