to mean careless or incorrect teaching. I make a strong point of never teaching anything a student will ever have to unlearn. However, generality can be sacrificed with little loss at this stage. The special cases of theorems that are going to be needed are quite sufficient. Second, long and detailed proofs can be omitted; careful statements of the theorems can be substituted. With these two methods of condensation (limited generality and few proofs), we cover in one year the following topics: probability, infinite series, complex numbers, determinants and matrices, partial differentiation, multiple integrals, vector algebra and calculus, Fourier series, calculus of variations, transformation theory, diagonalization of matrices and applications, tensors, complex variables, special functions, Laplace and Fourier transforms, and partial differential equations.

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THE SMALL GROUP-DISCOVERY METHOD AS APPLIED IN CALCULUS INSTRUCTION

NEIL DAVIDSON, University of Maryland

Is there a way to learn mathematics that involves student pacing, active learning, thinking, and interpersonal communication? These criteria can be met by combining a small group method [9] with discovery learning [1, 7, 8]. In the small group-discovery method, the student discusses challenging problems with a few of his colleagues. The author first used this method in 1967–1968 in a one-year pilot study with a freshman calculus class at the University of Wisconsin.

During the pilot study, the students learned mathematics by doing mathematics. They formulated some definitions, stated most of the theorems, proved the theorems, constructed some examples and counterexamples, and developed techniques for solving various classes of problems. The students sometimes learned new concepts by discussing open-ended questions. For example: How can you find the area under a given curve? What is meant by a tangent to a curve? What happens at a high or low point on a graph? What can you conclude if a function vanishes at the endpoints of an interval? Discussion of the questions led to the statement of definitions or theorems.

The students worked together at the blackboard in small groups, with three or four members per group. The teacher stated the following guidelines for the small groups: (1) The students work together cooperatively and achieve a group solution to the problem. (2) Everybody understands the solution before the group tackles a new problem. (3) People listen carefully and try, whenever possible, to build upon the ideas of others. (4) There is no specified leader of the group. (5) Everybody participates and no one dominates the discussion. (6) People take turns writing solutions on the board.

The teacher selected the content and arranged it for small group learning. Since existing textbooks were not suitable for that purpose, he prepared a set of dittoed notes. He sometimes talked with the entire class at the beginning of the period, usually for no more than five or ten minutes per day. During these brief class discussions he presented new concepts, raised questions for investigation, proposed problems, and so forth.

The teacher spent most of the class period with the small groups. He observed the progress of the groups and visited particular groups as needed. In these visits he checked solutions, made corrections, gave hints, clarified notation, provided encouragement, and tried to help the groups function more smoothly.

The teacher used a democratic style of leadership [10] which involved considerable respect and friendliness toward the students. He did not give orders or disrupting commands. Instead, he offered guiding suggestions when the students wanted or clearly needed them. He used a minimal amount of constructive praise and criticism, usually directed to a group as a whole. Basically, he helped students to learn, rather than forcing them to learn.

Interest in the mathematical discussions was to be the major motivation; this required an increase in student freedom and a reduction in pressure. The students were free to explore mathematical questions that arose in their groups. The students decided whom to work with and when to change groups. The teacher used an A-B grading scale, and the students discussed grading policies and voted for take-home exams.

The discovery class met five periods per week for two semesters. The twelve students were all volunteers with A or B grades in high school mathematics and at least a mild interest in that subject. The students performed well on seven take-home examinations. In a final examination on basic facts and skills, the discovery class performed slightly better than a control class taught by the lecture-discussion system. However, the difference was not statistically significant, and it might have resulted from the special entrance requirements for the discovery class.

The students in the discovery class responded to an open-ended questionnaire, with the following results. On the negative side, most students were concerned for varying periods of time about covering enough material. Students sometimes became frustrated or angry, particularly when the mathematical problems were too hard. The students had difficulty at first in forming effective working groups. On the positive side, the pilot class had positive or null effects on each student's interest in mathematics and estimate of his problem solving skill. Almost all of the students had a closer, more personal relationship with their mathematics teacher than with their other teachers. Most students found their calculus class more stimulating than their other classes, and everyone's attitude toward the class either stayed the same or improved during the year.

The students' attitudes can be conveyed more vividly by quoting some questionnaire responses; no student is quoted more than once. (1) "Other students, no matter who, force you to learn more." (2) "Most classes stress being able to

use formulas while this stresses total understanding." (3) "It is my most interesting and liked class. I enjoy coming to it." (4) "I think I learned a lot more this year than I did in all three years of high school math." (5) "It showed me that I can do things that before looked impossible. All it takes is a little understanding. Math doesn't scare me as much now." (6) "I simply feel it was a great experiment (and experience) and more subjects should be adapted to this general method." (7) "This type of class was, in my estimation, the closest possible setup to an ideal learning situation."

After the pilot study, the author made three changes in his small group classes. First, the course grade was based largely on homework. The teacher checked some problems, and class members took turns checking the others [6]. Secondly, the teacher introduced new concepts and problems in written form, rather than in class discussions. Dittoed work sheets allowed each group to set its own pace. Finally, the teacher held a presession with one small group before planning each class meeting. He could then design work sheets that were interesting, challenging, and reasonable for students.

The author and other teachers have used the small group-discovery method in honors calculus, abstract algebra, and Euclidean geometry. These classes had roughly twenty-five students apiece. Conceivably, a small group approach with easier problems might be a realistic way to handle large enrollments without mass lectures. Such an approach would entail a special textbook, limited teacher guidance, and problems of suitable difficulty for the intended population.

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