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ASPECTS OF THE NATURE AND STATE OF RESEARCH IN MATHEMATICS EDUCATION ⋆

ABSTRACT. This paper offers an outline and a characterisation of the didactics of mathematics, alias the science of mathematics education, as a scientific and scholarly discipline, and discusses why its endeavours should be of interest to research mathematicians (and other mathematics professionals). It further presents and discusses a number of major, rather aggregate findings in the discipline, including *the astonishing complexity of mathematical learning*, *the key role of domain specificity*, *obstacles produced by the process-object duality*, *students' alienation from proof and proving*, and *the marvels and pitfalls of information technology in mathematics education*.

1. INTRODUCTION

During the last three decades or so mathematics education has become established as an academic discipline on the international scene. To show this we need only refer to a number of sociological facts, such as the existence of a multitude of departments in universities and research institutions; research grants and projects; academic programmes and degrees; international scientific organisations and bodies; journals and publication series; hosts of conferences; and so forth, all devoted to research in mathematics education. The discipline is given slightly different names in different quarters, which is mainly due to the fact that mathematics education has a dual and hence ambiguous meaning, in that it may refer both to something provided to students (for simplicity, throughout this paper we shall use 'student' as the general term for the learner, irrespective of educational level), and to the field in which this 'something' is made subject of research (and development). In order to avoid misunderstandings caused by this duality the discipline is sometimes called *mathematics education research* or *the science of mathematics education*, although *mathematics education* probably remains predominant in everyday usage. In Europe, there seems to be a preference for using the label *the didactics of mathematics*, inspired by names such as 'Didaktik der Mathematik' (German), 'didactique des mathématiques' (French), 'didáctica de las matemáticas'

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(Spanish), 'matematikdidaktik' (Scandinavian languages), and their analogues in most European languages, in spite of the slightly oblique connotations attached to the term 'didactical' in English. In the following I shall use the names interchangeably.

The sociological aspects aside, what are the issues and research questions of the didactics of mathematics, what are its methodologies, and what sorts of results or findings does it offer? In this paper attempts will be made to characterise this discipline, in particular as regards its nature and state, and to present and discuss some of its major findings. Key sections of this paper have been greatly inspired by a number of the world's leading researchers in mathematics education who were consulted during the preparation of this paper. My sincere thanks go to all of them (cf. 'Acknowledgements'). It is important to underline that these scholars hold a variety of different views and perspectives of the discipline, and many of them are likely to disagree with my exposition of it. Also, needless to say, the responsibility for the entire paper, especially for any flaws or biases it may contain, is mine alone.

Before undertaking the attempt just outlined it may be in order to ask why it would/should be of interest not only to mathematics educators but also to research mathematicians (and other categories of mathematics professionals too, for that matter) to become acquainted with the nature and state of research in mathematics education, i.e. a discipline which is not quite their own and towards which they may hold various degrees of skepticism. Well, let me offer an answer to this question. The answer consists of a number elements most of which are related to the fact that the majority of research mathematicians are also – and in some cases perhaps even primarily – university teachers of mathematics.

The first element is to do with changes in the boundary conditions for the teaching of mathematics at university level, changes which are, in turn, linked to major changes in the role, place and functioning – and financing! – of universities in society. In former times, say thirty-forty years ago, the situation was more or less the following (in condensed and simplistic terms). University students of mathematical topics were expected to assume all responsibility for their own studies and for their success or failure. Students who passed the exams had 'it' (i.e. necessary prerequisites, mathematical talent, and diligence), and those who failed lacked 'it', and apart from working hard there wasn't much one could do about that. Universities mainly had to pay attention to the former category, except that they also had a task in identifying members of the latter at an early stage and in pointing the exit from mathematics out to them. This implied that lecturers of mathematics could concentrate on the delivery of their teaching, whereas the individual student's learning of what was taught was not the business of the lecturers but entirely of the student him/herself. The outcome of learning was finally gauged in tests and examinations, and students were filtered accordingly. In those days not many question marks were placed against this way of operation. Universities were not blamed for students' failures, and enrolment and pass rates only influenced the marginals of institution and department budgets. Against this background it is not surprising that the typical university mathematician took no deeper interest in students' learning processes, especially not of those who were unsuccessful in their studies, or in devising innovative formats and ways of teaching or new kinds of student activity. By no means does this imply that teaching was generally neglected (although sometimes it was). But the focus was on the selection and sequencing of the material to be taught, and on the clarity and brilliance of its presentation, all of which was considered from the point of view of one-way communication. These deeply rooted traditional conditions and circumstances of university teaching of mathematics may well account for parts of the widespread, yet far from universal, absence of interest amongst research mathematicians in the didactics of mathematics.

But, whether or not it ought be deplored, these conditions and circumstances are no longer with (most of) us. Universities can no longer afford to concentrate their main efforts on students who can, and want to, stand the type of diet that used to be served in the past. Today, we have to cater for students who are actually able to learn mathematics, if properly assisted, but who would be likely to either not enroll at all in studies with a non-negligible component of mathematics, or to leave or fail the studies should they enroll, if no didactical or pedagogical attention were paid to their backgrounds, situations, prerequisites and needs. First of all, apart from the scarce 'happy few', these students are in fact the ones we get, and it is our professional (and moral) duty to look after them as best we can. Should we forget this ourselves our colleagues in other disciplines, deans and vice-chancellors/presidents, administrators, politicians and the public at large – and above all the students, by voting with their feet – will know to remind us and to blame us for our autistic arrogance and for our (co-)responsibility for waste of human potential. Besides, to an increasing extent the existence, position, and resources of departments which teach mathematics are strongly dependent not only on the number of students enrolled and taught, but also on the proportion who succeed in their courses and finish with a degree. Whether we perceive these as facts in a hostile or ill-informed world that have to be counteracted, or as a genuine challenge that has to be met, this – second element – points to the need try to understand what it is and what it takes to learn mathematics, including the processes involved therein, in particular for students who experience difficulties in this endeavour, and to invent and investigate ways of teaching that are more beneficial and effective to average students than the ones traditionally employed.

Then, thirdly, if we understood the possible paths of learning mathematics, and the obstacles that may block these paths, for ordinary students, we would gain a better understanding of what mathematical knowledge, insight, and ability are (and are not), of how they are generated, stored, and activated, and hence of how they may be promoted (and impeded) for other categories of students, including those with severe learning difficulties, as well as those with a remarkable talent. As far as the latter category is concerned, we would come closer to specifying what mathematical talent is and subsequently, perhaps, to fostering it. Similarly, it might well happen that effective improvements of our modes of teaching ordinary students could be transferred to have a positive bearing on the teaching of exceptional students as well. This would not pertain to the university level only. If such improvements could be devised and brought about at all levels of the educational system, we would not only do important service to society at large, we would do important service to the mathematics research community, too.

Finally, to the extent we are able to shed light on what mathematical knowledge, insight, and ability are, we shall eventually contribute to shedding light on what *mathematics* is. For, none of the issues touched upon here can be dealt with without continuous implication of and reflection on the characteristics of mathematics as a discipline in all its manifestations.

This completes my arguments for the claim that matters pertaining to mathematics education research ought to be of interest also to research mathematicians, at least in principle. Assuming that this argument be accepted, new issues arise. Although the questions posed by the didactics of mathematics are important enough, to what extent is the didactics of mathematics able to give answers to them, and what is the nature of the answers actually given? This is the main issue of this paper. In order to consider it, I shall offer a definition of the field.

2. CHARACTERISING THE FIELD

Various researchers in mathematics education have given definitions of the field which have a considerable amount of overlap. Instead of reviewing the definitions put forward by others I shall offer my own as follows. It contains four components.

A definition

Subject: *The didactics of mathematics, alias the science of mathematics education, is the scientific and scholarly field of research and development which aims at* identifying, characterising, *and* understanding *phenomena and processes actually or potentially involved in the* teaching *and* learning of mathematics at any educational level.

Endeavour: *As particularly regards 'understanding' of such phenomena and processes, attempts to uncover and clarify* causal relationships *and* mechanisms *are the focus*.

Approaches: *In pursuing these tasks, the didactics of mathematics addresses* all matters *that are pertinent to the teaching and learning of mathematics, irrespective of which scientific, psychological, ideological, ethical, political, social, societal, or other spheres this may involve. Similarly, the field makes use of considerations, methods, and results from* other fields and disciplines *whenever this is deemed relevant*.

Activities: *The didactics of mathematics comprises different kinds of activities, ranging from theoretical or empirical* fundamental research, *through* applied research *and* development, *to* systematic, reflective practice.

The overall purposes of work in the field are not part of the definition proper as different agents, including researchers, pursue different aims and objectives. To quite a few researchers in mathematics education the perspectives of pure, fundamental research are predominant. However, it is fair to claim that the over-arching, ultimate end of the whole enterprise is to promote/improve students' learning of mathematics and acquisition of mathematical competencies. It is worth pointing out that the very specification of the terms just used ('promote', 'improve' 'students' (what students are being considered?) 'learning', 'mathematics', 'acquisition', 'mathematical competencies') is in itself a genuine didactic task.

It is important to realise a peculiar but essential aspect of the didactics of mathematics: its *dual nature*. As is the case with any academic field, the didactics of mathematics addresses, not surpisingly, what we may call *descriptive/explanatory* issues, in which the generic questions are 'what *is* (the case)?'(aiming at description) and '*why* is this so?' (aiming at explanation). Objective, neutral answers are sought to such questions by means of empirical and theoretical data collection and analysis without any explicit involvement of values (norms). This does not imply that values are not present in the choice and formulation of the problems to be studied, or –

in some cases – of the methods to be adopted. However, by their nature numerous issues related to education, including mathematics education, imply the fundamental, explicit or implicit, presence of values and norms. In other words, in addition to its descriptive/explanatory dimension, the didactics of mathematics also has to contain a *normative* dimension, in which the generic questions are 'what *ought to be* the case?' and '*why* should this be so?'. It may come as a surprise to some that issues such as these are considered part of a scholarly and scientific discourse and are claimed to belong to the scope of research. But this is unavoidable in the same way as it is unavoidable to operate with the notion of 'good health' and 'sound treatment' in much medical research, or 'satisfactory functioning' of devices constructed in engineering. For normative issues to be subject of research it is necessary to reveal and explain the values implicated as honestly and clearly as possible, and to make them subject to scrutiny; and to undertake an analysis, as objective and neutral as possible, of the logical, philosophical, and material relations between the elements involved (cf. Niss, 1996). So, both dimensions are essential constituents of the science of mathematics education, both crucially relying on theoretical and empirical *analysis*, but they are not identical and should not be confused with one another.

It appears that in many respects the didactics of mathematics has a close analogue in the field of medicine which has the same duality between a descriptive/explanatory and a normative dimension as well as wide ranges of goals, methods, and activities. On the other hand, being hardly more than 30–40 years old, the didactics of mathematics is certainly not yet a mature, full-fledged discipline on a par with medicine.

In a brief outline of the main areas of investigation the two primary ones are, naturally, *the teaching of mathematics*, which focuses on matters pertaining to organised attempts to transmit or bring about mathematical knowledge, skills, insights, competencies, and so forth, to well-defined categories of recipients, and *the learning of mathematics*, where the focus is on what happens around, in and with students who engage in acquiring such knowledge, skills, etc., with particular regard to the processes and products of learning. A closely related area of investigation is the *outcomes* (results and consequences) of the teaching and the learning of mathematics, respectively.

We may depict, as in Figure 1, these areas as boxes in a 'ground floor' plane such that the 'teaching' and 'learning' boxes are disjoint and the 'outcomes' box intersects both of them. As the investigation of these areas leads to derived needs to investigate certain auxiliary areas related to the primary ones but not in themselves of primary didactic concern, such as

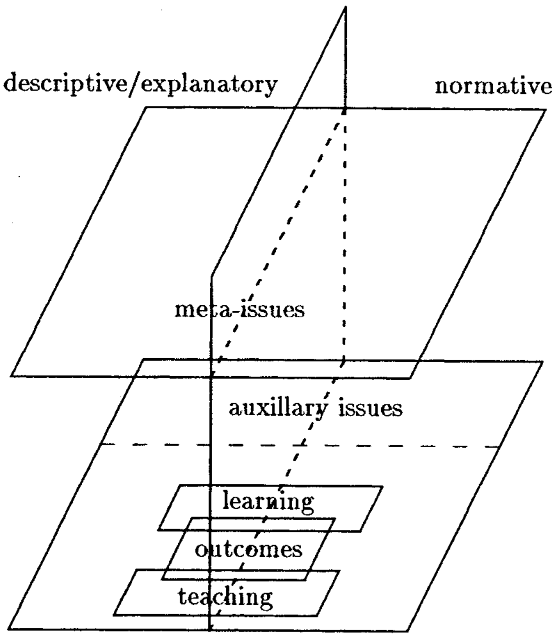

Figure 1. Survey map.

aspects of mathematics as a discipline, aspects of cognitive or learning psychology, aspects of curriculum design and implementation, and so on, we may place these auxiliary areas on the same plane as the primary areas but in a separate compartment at the back of the 'ground floor'. We may agree to call activities on the ground floor 'mathematical didactics of the first order'. Although the didactics of mathematics may be considered a mature discipline in a sociological sense (cf. the introduction), the same is not necessarily the case in a philosophical, a methodological, or a verificational sense. Thus, there is no universally established framework or consensus as regards schools of thought; research paradigms; methods; standards of verification, justification and quality, etc. This is one reason, among others, why a number of researchers in the field, during the last couple of decades, have been reflecting on its nature and characterstics, its issues, methods, and results (see, e.g., Grouws, 1992; Biehler et al., 1994; Bishop et al., 1996; Sierpinska and Kilpatrick, 1998). Theoretical or empirical studies in which the field as such is made subject of investigation do in fact form part of the field itself, although at a meta-level, which we may depict as an 'upper floor' plane parallel to the ground floor plane. We may think of it as being transparent so as to allow for contemplation of the ground floor from above. It seems natural to call such meta-activities 'mathematical didactics of the second order'.

Finally, for the survey picture being outlined to become complete, let us imagine a vertical plane cutting both floors as a common wall. On the ground floor, all three boxes, 'teaching', 'learning', and 'outcomes' are bisected by this wall. The two half-spaces thus created may be thought of as representing the descriptive/explanatory and the normative dimensions, respectively. These dimensions are then present at both floors. If we imagine the vertical wall to be transparent as well, it is possible to look into each half-space (dimension) from the perspective of the other.

Let us sum up, in a simplified and maybe also simplistic way, the ultimate (utopian?) goals of the didactics of mathematics as follows: We want to be able to specify and characterise *desirable or satisfactory learning* of mathematics, including the mathematical competencies we should like to see different categories of individuals possessing. We want to be able to devise, design and implement *effective mathematics teaching* (including curricula, classroom organisation, study forms and activities, resources and materials, to mention just a few components) that can serve to bring about satisfactory/desirable learning. We finally want to construct and implement valid and reliable *ways to detect and assess*, without destructive side effects, the results of learning and teaching of mathematics. Indicating and specifying these goals is a normative activity in the didactics of mathematics.

For all this to be possible we have to be able to identify and understand, in descriptive and explanatory terms, the role of mathematics in science and society; what learning of mathematics is/can be and what it is not, what its conditions are, how it may take place, how it may be hindered, how it can be detected, and how it can be influenced, all with respect to different categories of individuals. We further have to understand what takes place in existing approaches to and modes of mathematics teaching, and why, both as regards the individual student, groups of students and entire classrooms (in a general sense). We have to invent new modes of teaching and make similar investigations. We have to investigate the relationships between teaching modes and learning processes and outcomes. We have to investigate the influence of teachers' backgrounds, education, and beliefs on their teaching. We have to examine the properties and effects of current modes of assessment in mathematics education, with particular regard to the ability to provide valid insight into what students know, understand, and can do, as we have to devise and investigate, in the same way, innovative modes of assessment. All this points to endless multitudes of theoretical and empirical tasks of fundamental and applied research as well as of concrete development with practical aims.

If issues such as these are the ones we want to raise, what are the responses we can offer, and what is their nature? Let me deal with the latter question first, and devote the next section to the former.

Traditionally, fields of research within the sciences produce either *empirical findings* of objects, phenomena, properties, relationships, and causes – like in, say, chemistry – through some form of data collection guided or followed by theoretical considerations and interpretations, or they produce *theorems*, i.e. statements derived by means of logical deduction from a collection of 'axioms' (postulates, facts, laws, assumptions) that are taken as a (locally) undisputed basis for the derivations, like in mathematics and theoretical physics.

If we go beyond the predominant paradigms in the sciences and look at the humanities and the social sciences, other aspects have to be added to the ones just considered. In philosophical disciplines, the proposal and analysis of distinctions and concepts – sometimes sharp, but mostly somewhat fuzzy – and concept clusters, introduced to identify and represent matters from the real world, serve to create a platform for discourses on and investigations of these matters in an explicit, clear and systematic way. Such disciplines often produce *notions*, *distinctions*, *terms*, amalgamated into *concepts*, or extensive hierarchical networks of concepts connected by formal or material reasoning, called *theories*, which are meant to be stable, coherent and consistent. Or more simply put: tools for thought to assist the analysis of parts or aspects of the world. Disciplines dealing with human beings, their minds, types of behaviour and activity, as persons, members of different social and cultural groups, and as citizens, or with communities and societies at large, primarily produce *interpretations* and *models*, i.e. hypotheses of individual or social forces and mechanisms that may account for (explain) phenomena and structures observed in the human or societal domain under consideration, as encountered in, say, psychology, anthropology, or history. Sometimes sets of interpretations are organised and assembled into systems of interpretation that are, again, meant to be stable, coherent, and consistent, and are therefore often called 'theories', they too. We shall refer to such systems by the term *interpretative theories*. As most human behaviour is complex, and most of the time at best locally coherent, results in these disciplines cannot be expected to be simple and clear-cut. Finally, there are disciplines within all categories of science that produce *designs* (and eventually – *constructions*). For such products the ultimate test is their functioning and efficiency in the realm in which they are put into practice ('the proof of the pudding is in its eating'). However, as most important designs and constructions are required to have certain properties and meet certain specifications before the resulting constructions are installed, design disciplines are scientific only to the extent they can provide well-founded evidence and reasons to believe that their designs possess certain such properties to a satisfactory degree.

This is not the place to enter into classical philosophical (epistemological) discussions of the similarities and abundant differences between disciplines such as these, let alone of their well-foundedness and relative strengths and weaknesses. Suffice it, here, to note that irrespective of any dispute, all these types of disciplines are represented in academia with 'civil rights' of long standing.

Where is the didactics of mathematics situated in the discipline survey just sketched? In fact it contains instances and provides findings of most of the categories of disciplines mentioned, but to strongly varying degrees. There are empirical findings, like in chemistry or archaeology. There are terms, concepts and theories for analysis of a philosophical nature (e.g. Ernest, 1991; Skovsmose, 1994; Niss, 1994), and there are models, interpretations and interpretative theories of a psychological, sociological or historical nature. There are multitudes of designs and constructions of curricula, teaching approaches, instructional sequences, learning environments, materials for teaching and learning, and so forth. As to 'theorems', like in mathematics, the situation is different, though. In fact, the only theorems one may find in the didactics of mathematics are already theorems of mathematics itself. (For instance, in the 1960's and 1970's it was not unusual to encounter demonstrations that a given mathematical concept, theorem or topic could be introduced, in a consistent way, on some axiomatic basis taylored to a given educational level.)

However, as the teaching and learning of mathematics are always situated in context and time, and subjected to a multitude of conditions and circumstances, sometimes of a very specific nature, there will always be limits to the universality in space and time of the findings obtained in the didactics of mathematics.

When researchers in mathematics education are asked about the nature of their field, their answers point to some of these aspects but with varying perspectives and emphases. Some researchers are hesitant to use the term 'finding' in this context in order to avoid misunderstandings and too narrow expectations of what a scientific field should have to offer. They prefer to see the didactics of mathematics as providing generic tools – including conceptual apparatuses and models – for analysing teaching/learning situations, or as providing new questions, new ways of looking at things, new ideas inspired by other fields, etc. Others emphasise that the field offers illuminating case studies which are not necessarily claimed to be generalisable beyond the individual cases themselves, and hence should not be considered scientific findings in the classical sense, but are nevertheless stimulating for thought and practice. Still other didacticians give primary importance to the design aspects of the field (Wittmann, 1995; cf. also Artigue, 1987). However, as long as we keep in mind that the notion of finding is broader in disciplines not residing within the realm of classical empirical and theoretical sciences, I don't see any severe problems in using this term in the didactics of mathematics.

Although it is not an easy task to gauge the relative weights of the different categories of disciplines and findings across the entire field of the didactics of mathematics, it is probably fair to describe the situation as follows.

A major portion of research done during the last couple of decades has focused on students' *learning processes and products* as manifested on the individual, small group, and classroom levels, and as conditioned by a variety of factors such as mathematics as a discipline; curricula; teaching; tasks and activities; materials and resources, including text books and information technology; assessment; students' beliefs and attitudes; educational environment, including classroom communication and discourse; social relationships amongst students and between students and teacher(s); teachers' education, backgrounds, and beliefs; and so forth. The typical findings – of which examples will be given in the next section – take the shape of models, interpretations, and interpretative theories, but certainly often also of solid empirical facts. Today, we know a lot about the possible mathematical learning processes of students and about how these may take place within different areas of mathematics and under different circumstances and conditions, as we know a lot about factors that may hinder, impede or simply prevent successful learning.

We have further come to know a great deal about what happens in *actual mathematics teaching* in actual classrooms at different levels and in different places in the world (Cobb and Bauersfeld, 1995). Much of this knowledge is of a factual, descriptive nature. This has made it possible to describe and analyse various settings and forms of teaching, and the resulting teaching-learning situations. However, we are still left with hosts of unanswered questions as to how to design, stage, organise, implement, and carry through teaching-learning environments and situations addressing various categories of students, which to a reasonable degree of certainty and robustness lead to desirable or satisfactory learning outcomes, in a broad sense, for those students. Indeed this is not to say that we don't know anything in this respect. In fact we do, but as yet our knowledge is more fragmentary and scattered than is the case with our insights into the mathematical learning processes of students. This is to do with two factors.

Firstly, insights into such learning processes have turned out to be a prerequisite for insights into the outcomes of teaching. So, progress in the latter respect somehow has to await progress in the former. Secondly, as research on learning processes has revealed several variations, complexities and complications in students' learning of mathematics, traditional assessment modes and instruments to an increasing extent have proved insufficient, and sometimes outright misleading, in making well-founded inference of what students actually know, understand, and can achieve in different situations and contexts, especially when larger student populations are considered. In other words, it is far from a trivial matter to specify, detect, appraise, assess, and convincingly document the outcomes of teaching and learning in terms of students' mathematical knowledge, insights and competencies. A third factor that might have been expected to be in force here is disagreement about what desirable or satisfactory outcomes of mathematical learning are. Such disagreement on the goals would, of course, give rise to problems regarding what should be considered adequate modes of teaching. A considerable amount of literature has been devoted to the – normative – issue of the ends and aims of mathematics education (e.g., see Niss, 1996), and even though there is some variation in the views held by mathematics educators on these matters, in particular as regards details or terminology, a fair amount of agreement on the basics seems to prevail (with emphases on understanding, reasoning, creativity, problem solving, and the ability to apply mathematics in extra-mathematical contexts and situations, all under varying circumstances and in varying domains and contexts).

As I said, we know something about effective teaching modes in specific contexts (see, for instance, Leron, 1985; Tirosh, 1991; and, for an introduction to the idea of a 'scientific debate', Alibert and Thomas, 1991). In particular, based on our growing insight into mathematical learning processes and teaching situations, we know more and more about what *is not* effective teaching vis-à-vis various groups of recipients. At first sight such knowledge may appear to be a bit negative, but at closer reflection negative results are certainly valuable as they provide progress in the search for positive, definitive results. Moreover, the didactic literature displays numerous examples of innovative teaching designs and practices, many of which are judged highly successful. The fact that it is not always easy to analyse and document the success of an innovation in scientific terms, including to provide evidence of its transferability to other contexts and settings, does certainly not rule out that the innovation possesses highly valuable qualities of the kind claimed and experienced (Leron 1983). These qualities are just recognised and appraised at a more local or subjective level than asked for in research. By the way, wasn't it a renowned mathematician who said 'I cannot define my wife but I can recognise her when I see her'?

3. Examples of Major Findings

It follows from the previous section that findings in the didactics of mathematics only relatively seldom take the shape of empirical or experimental results in the traditional sense – and of mathematical results only to the extent they *are* already results in mathematics, just transferred to a didactic context. Nevertheless, findings in the field resulting from thorough theoretical or empirical analyses do give rise to solid insights of considerable significance to our understanding of processes and outcomes of mathematics teaching and learning, and hence for the ways in which mathematics may, or may not, be taught and learnt. This is not the place for a systematic review of the most important findings in the didactics of mathematics – in fact, no such single place can exist (for more comprehensive attempts in this direction, cf. the handbooks, Grouws, 1992, and Bishop et al., 1996). Instead, we shall consider a few selected, significant examples, of a fairly advanced level of aggregation, which can serve to illustrate the range and scope of the field. By the nature of this paper, it is not possible to provide detailed presentations or full documentation of the findings selected. A few references, mainly of survey or review type providing access to a broader body of primary research literature, have to suffice.

The astonishing complexity of mathematical learning: *An individual student's mathematical learning often takes place in immensely complex ways, along numerous strongly winding and frequently interrupted paths, across many different sorts of terrain. Many elements, albeit not necessarily their composition, are shared by large categories of students, whereas others are peculiar to the individual.*

Students' misconceptions (and errors) tend to occur in systematic ways in regular and persistent patterns, which can often be explained by the action of an underlying tacit rationality put into operation on a basis which is distorted or insufficient.

The learning processes and products of the student are strongly influenced by a number of crucial factors, including the epistemological characteristics of mathematics and the student's beliefs about them; the social and cultural situations and contexts of learning; primitive, relatively stable implicit intuitions and models that interact, in a tacit way, with new learning tasks; the modes and instruments by which learning is assessed; similarities and discrepancies between different 'linguistic re-

gisters', including everyday language and various language modes that are characteristic of mathematical discourses.

This over-arching finding is an agglomeration of several separate findings, each of which results from extensive bodies of research. The roles of epistemological issues and obstacles in the acquisition of mathematical knowledge have been studied, for instance, by Sierpinska (1994) and others (for an overview, see Sierpinska and Lerman, 1996). Social, cultural, and contextual factors in mathematical learning have been investigated from many perspectives, see for instance Bishop, 1988; Nunes et al., 1993; and Cobb and Bauersfeld, 1995. Schoenfeld (1983) and Pehkonen (e.g. Pehkonen and Törner, 1996), among others, have investigated students' (and teachers') belief's. Fischbein and his collaborators have studied the influence of tacit models on mathematical activity (see, e.g. Fischbein, 1989). The influence of assessment on the learning of mathematics has been subject of several theoretical and empirical studies (e.g. Leder, 1992; Niss, 1993a and b). The same is true with the role of language and communication (Pimm, 1987; and Ellerton and Clarkson, 1996, for an overview).

The studies behind these findings teach us to be cautious and not to jump to conclusions when dealing with students' learning of mathematics. Mathematical learning is not isomorphic to the edifice of mathematics to be learnt. Neither processes nor outcomes of learning are in general logically ordered, let alone globally deductive, at least not with respect to hierarchies that one might have thought of as natural or even canonical. For instance, research has shown that many students who are able to correctly solve an equation such as $7x - 3 = 13x + 15$ are unable to subsequently correctly decide whether $x = 10$ is a solution (Bodin, 1993). Normally, one would assume that knowing a complete solution to an equation, i.e. knowing exactly which elements are solutions and which are not, occupies a relatively high position in the logical hierarchy and hence will automatically lead to a correct answer to a question concerned with a special case. Apparently this need not be so. The explanation normally given to this phenomenon is that *solving* equations resides in one ('syntactic') domain, strongly governed by rules and procedures with no particular attention being paid to the objects involved in the procedures, whereas examining whether or not a given element solves the equation requires an ('semantic') understanding of what a *solution* means. Furthermore, checking directly, from scratch, whether a particular element is a solution usually involves procedures at variance with general solution algorithms. So, the two facets of the solution of equations, intimitely linked in the mind of the mature knower, need not even both exist in the mind of the novice mathematical learner, let alone be intertwined.

The key role of domain specificity: *For a student engaged in learning mathematics, the specific nature, content and range of a mathematical concept that he or she is acquiring or building up are, to a large part, determined by the set of specific domains in which that concept has been concretely exemplified and embedded for that particular student.*

For an illustration of what we are talking about, a large group of Danish 12th grade students who sat, a few years ago, the final national written examination in mathematics at the end of the most demanding mathematics course in upper secondary school, showed severe difficulties in recognising the object in 3-space given by the equation $z = 0$ as a plane. On closer inspection, the primary reason for this turned out to be that the equation was not explicitly stated in the standard form, $ax + by + cz = d$, the main problem being that x and y were absent in the equation. So, to these students, the concept of a general plane in the analytic geometry of 3-space did not comprise the x, y-plane in the form $z = 0$ as a special case, most certainly because such special cases had not received much attention, if any, in the teaching-learning activities on planes in which these students had been engaged.

The finding at issue is closely related to the finding that students' *concept images* are not identical with the *concept definitions* they are exposed to (Tall and Vinner, 1981; Vinner and Dreyfus, 1989; and for an overview, Vinner, 1991; and Tall, 1992; see also Robert, 1982). The concept images are generated by previous notions and experiences as well as by the examples against which the concept definitions have been tested. Several attempts have been made to construct general theoretical frameworks to elucidate these findings. One notable example is Vergnaud's notion of 'conceptual field' (Vergnaud, 1990).

At first sight, our finding may seem to be little more than a reformulation of a well-known observation belonging to the experience of any observant and reflective teacher of mathematics at whatever level. (If this is true, which it sometimes is, it is remarkable, though, how often the finding remains unemployed in actual teaching practice.) But, on closer inspection, the range and depth of the instances of this finding have far-reaching bearings on the teaching and learning of mathematics. Thus, not only are most 'usual' students unable to grasp an abstract concept, given by a definition, in and of itself unless it is elucidated by multiple examples (which is a well known fact), but, more importantly, the scope of the notion that a student forms is often barred by the very examples studied to support that notion. For example, even if students who are learning calculus or analysis are presented with full theoretical definitions, say of ϵ, δ-type, of function, limit, continuity, derivative, and differentiability, and even if it is explicitly stated in the textbook and by the teacher that the aim is to develop these concepts in a general form, and even if 'warning examples' meant to vaccinate against wrong conclusions caused by over-simplification are provided, students' actual notions and concept images will be shaped, and limited, by the examples, problems, and tasks on which they are actually set to work. If these are drawn exclusively from objects (sequences, functions) expressed as standard 'molecular' expressions composed of familiar, well-behaved standard objects on the shelves, 'atoms', the majority of students will gradually tie their notions more and more closely to the specimens actually studied, and aspects allowed by the general concepts but not exhibited by the specific specimens will wither or even, eventually, disappear. For instance, studies show that the number of calculus students who don't include, say, Dirichlet's function in their concept of function is legion. Instead, the general concept image becomes equipped with properties resulting from an over-generalisation of properties held by the collection of special cases but not implied by the general concept. Remarkably enough, this does not prevent many of the very same students from correctly remembering and citing general theoretical definitions without seeing any mismatch between these and properties characteristic of special cases only. These definitions seem to just be parked in mental compartments different and detached from the ones activated in the study of the cases. In other words, if average students are to establish a general notion of a mathematical concept and to understand its range, they have to experience this range by being given opportunities to explore a large variety of representative manifestations of the concept in various domains.

The danger of forming too restricted images of general concepts seems to be particularly manifest in domains – such as arithmetic, calculus, linear algebra, statistics – that lend themselves to an algorithmic 'calculus', in a general sense, i.e. a system of formalisable operations and manipulations in a symbolic setting, the virtue and strength of which exactly is to replace the continual, and often conceptually demanding, evocation of fundamental notions and concepts by algorithmic calculations based solely on selected aspects of the concepts. In such domains, algorithmic manipulations – procedures – tend to attract the main part of students' attention so as to create a 'concept filter': Only those instances (and aspects) of a general concept that are digestible by and relevant in the context of the 'calculus' are preserved in students' minds. In severe cases an overemphasis in instruction on procedures may even prevent students from developing further understanding of the concepts they experience through manipulations only (Hiebert and Carpenter, 1992).

The present finding shows that it is a non-trivial matter of teaching and learning to establish mathematical concepts with students so as to be both sufficiently general and sufficiently concrete. Research further suggests (see, e.g., Janvier, 1985) that for this to happen, several different *representations* (e.g. numerical, verbal, symbolic, graphical, diagrammatical) of concepts and phenomena are essential, as are the links and transitions between these representations.

There is a large and important category of mathematical concepts of which the acquisition becomes particularly complex and difficult, namely concepts generated by and *encapsulating* specific processes. Well-known examples of this are the concept of function as an *object*, encapsulating the mechanisms that *produce the values* of the function into an entity (which can further play the role of an element in some space of functions, or that of an unknown in a differential equation), and the concept of derivative, encapsulating the processes of differentiating a function pointwise, and of amalgamating the outcomes into a new function. Another example is the concept of quotient set (and structure) arising from an equivalence relation which in turn is an encapsulation of the process of determining whether or not given pairs of objects are equivalent in the original set. This *process-object duality*, so characterstic of many (but not all) mathematical concepts, is referred to in the research literature by different terms, such as ‘tool-object’ (Douady, 1991), ‘reification’ (Sfard, 1991, and Sfard and Linchevski, 1994), ‘procept’, a hybrid of process and concept, (Tall, 1991, Chapter 15). It constitutes the following finding:

Obstacles produced by the process-object duality: *The process-object duality of mathematical concepts that are constituted as objects by encapsulation/reification of specific processes typically gives rise to fundamental learning obstacles for students. They often experience considerable problems in leaving the process level and entering the object level. Some students are able to establish notions of both the processes underlying a certain concept and of that concept as an object, but are unable to establish links between the two.*

In addition to influencing the learning of mathematics, the syndrome uncovered in this finding gives rise to corresponding teaching difficulties as well. For example, many students conceive of an equation as signifying a prompt/request to perform certain operations, without holding any conception of an equation as such distinct from the operations to be performed. To them, an equation simply does not constitute a mathematical entity, such as a statement or a predicate – an issue which is, evidently, closely linked to other diffcult matters like variables, unknowns, the roles of the equality sign, and so forth. This undoubtedly accounts for large parts of the fact that equations of whichever type (algebraic or differential) constitute well-known hurdles in all teaching that focus on understanding of equations and not just on procedures to solve them.

Undoubtedly, the notions of mathematical proof and proving are some of the most crucial, demanding, complex, and controversial ones, in all of mathematics education. Deep scientific, philosophical, psychological, and educational issues are involved in these notions. Hence it is no wonder that they have been made subject of discussion and study in didactic research to a substantial extent over the years (for a recent discussion, see Hanna and Jahnke, 1996; see also Alibert and Thomas, 1991). Here, we shall confine ourselves to indicating but one finding pertinent to proof and proving in the teaching and learning of mathematics.

Students' alienation from proof and proving: *There is a wide gap between students' conceptions of mathematical proof and proving and those held in the mathematics community. Typically, at any level of mathematics education in which proof or proving are on the agenda, students experience great problems in understanding what a proof is (and is not) supposed to be, and what its purposes and functions are, as they have substantial problems in proving statements themselves, except in highly standardised situations. They tend to perceive proof and proving as strange freemasonry rituals into which mathematical professionals indulge but which are not really meant to be comprehended by ordinary human beings.*

Research further suggests that students' conceptions of what it means, to them, to convincingly establish the truth of a mathematical statement, are often centrered around either direct intuitive insight ('I can see it has to be true'), an amount of empirical evidence provided by special cases, or generic examples that 'contain it all in one'. Moreover, many students who are able to correctly reproduce a (valid) proof in oral or written form, do not see the proof to have, in itself, any bearing on the truth of the proposition arrived at by means of the proof.

The fact that proof and proving represent such great demands and challenges to the learning of mathematics implied that proof and proving have received, in the 1980's and 1990's, a reduced emphasis in much mathematics teaching. Rather than investing major efforts in training 'performing monkeys', with limited success, mathematics educators have concentrated on the provision of meaning and sense of mathematical ideas, notions, and activities to students. However, there seems to be a growing recognition that there is a need to revitalise (not just revive) proof and proving as central components in mathematics education – and not the least so in the light of the challenges to the classical notions of mathematical verification generated by modern computer systems. For instance, this recognition is the basis of a large ongoing research project in Italy ('Theorems in School: From History and Epistemology to Cognitive and Educational Issues'), directed by P. Boero, M. Bartolini Bussi, and others. Also there is growing evidence that it is possible to successfully meet, in the teaching of mathematics, parts of the demands and challenges posed by proof and proving, while at the same time furthering the fostering of mathematical meaning and sense-making with students. Literature on this topic also shows (see, for example, Alibert and Thomas, 1991) that it is possible to design and stage teaching–learning environments and situations that facilitate the bridging of the gap between students' conceptions of mathematical proof and proving and those characteristic of mathematics as a discipline. An extensive data base of literature on proof and proving, assembled and maintained by N. Balacheff, is accessible on the World Wide Web (International Newsletter on the Teaching and Learning of Mathematical Proof, at http://www-eiah.imag.fr/eiah/).

The last finding to be discussed here, briefly, is to do with the role and impact of information technology (calculators and computers and their software) on the teaching and learning of mathematics. As this is perhaps the single most debated issue in mathematics education during the last two decades, and one which has given rise to large amounts of research (for recent overviews, see Balacheff and Kaput, 1996; Ruthven, 1996; and Heid, 1997), we can touch upon one or two aspects only. Let us do this by formulating the following finding:

The marvels and the pitfalls of information technology in mathematics education: *Information technology gives rise to major transformations of mathematics education in all respects. Research shows that it has opened avenues to new ways of teaching and learning which may help to greatly expand and deepen students' mathematical experiences, insights, and abilities. However, it further shows that this does not happen automatically but requires the use of technology to be embedded with reflection and care into the overall design and implementation of teaching-learning environments and situations, of which IT-activities are but one amongst several components.*

The more students can do in and with information technology in mathematics, the greater is the need for their understanding, reflection, and critical analysis of what they are doing. So, in spite of what one might have expected because of the new opportunities offered by information technology, IT increases rather than decreases the demands on the teaching and learning of mathematics.

In other words, it is not a smooth and simple matter of ‘just doing it’ to make information technology assume a role in mathematics education which serves to extend and amplify students’ general mathematical capacities rather than replacing their intellects. There is ample research evidence for the claim that when it is no longer our task to train the ‘human calculator’ as was (also) the case in the past, parts of the traditional drill do become obsolete. But this does not imply that students’ no longer need to be able to perform basic operations themselves. We have yet to see research pointing out exactly what and how much procedural ability is needed for understanding the processes and products generated by the technology.

One other pitfall of information technology indicated in the research literature, is that the technological system itself (hardware and software) can form a barrier and an obstacle to learning, either by simply becoming a new and not necessarily easy topic in the curriculum, or by distracting students’ attention so as to concentrate on properties of the system rather than on the learning of mathematics. Once again, for this instance of ‘the tail wagging the dog’ to be avoided it is essential that information technology be assigned a role and place in the entire teaching-learning landscape on the basis of an overall reflective and analytic educational strategy. Where this happens, calculators and computers can give students access to mathematical experiences, insights, and abilities which otherwise demand years of dedication and hard work.

4. CONCLUSION

In a single paper it is not possible just to touch upon all major aspects and areas of the didactics of mathematics. So, it has been out of the question to do justice to the field, let alone to the thousands of researchers who have contributed to founding, shaping and developing it. Instead of the few findings put forward here, the selection of which was partly motivated by expectations concerning their potential interest to research mathematicians and other mathematics professionals, hosts of other findings could have been selected for discussion with no lesser right and relevance. Here is one:

There is no automatic transfer from a solid knowledge of mathematical theory to the ability to solve non-routine mathematical problems, or the ability to apply mathematics and perform mathematical modelling in complex, extra-mathematical contexts. For this to happen both problem solving and modelling have to be made object of explicit teaching and learning, and there is ample evidence that it is possible to design teaching settings so as to foster and solidify these abilities.

And here is another one:
Many of the assessment modes and instruments in current use in mathematics education fail to provide valid insight into what students know, understand, and can achieve, in particular as far as higher order knowledge, insight and ability are concerned. No single assessment instrument is sufficient for this purpose; balanced sets of instruments are needed. There is a general and increasing mismatch between established assessment modes and the ends and goals pursued by contemporary mathematics education. Nevertheless, appropriate (valid and reliable) assessment modes are *at our disposal, but are not put into large scale use because they tend to contradict external demands for inexpensive, fast, and easy assessment procedures that yield simple and summative results which are easy to record and communicate.*

Important findings concerning the values and efficiency of *collaborative learning* and *innovative teaching approaches and forms of study*, such as project work; the significance of carefully balanced, innovative *multifaceted curricula*, elucidating historical, philosophical, societal, applicational and modelling aspects of mathematics; the impact of *social, cultural and gender factors* on mathematics education; and many others, have not, regrettably, been given their due shares in this presentation. The same is true with the findings contributed by impressive bodies of research on the teaching and learning of specific mathematical *topics*, such as arithmetic, abstract and linear algebra, calculus/analysis, geometry, discrete mathematics, and probability and statistics, and with the findings represented by the instrumental *interpretative theories* of Brousseau (on ‘situations’, and ‘didactical contracts’ in mathematics education), of Chevallard (on the so-called ‘didactical transposition’), of Fischbein (on intuition), and of Mellin-Olsen (on ‘learning rationales’). Also the extensive and elaborate piece of didactical engineering (design and construction) contributed by the Freudenthal school (Freudenthal, de Lange, and several others) at the University of Utrecht (the Netherlands) has been left out of this survey.

Nevertheless, the findings which we have been able to present suffice to teach us two lessons which we might want to call *super-findings*. If we want to teach mathematics, with satisfactory or desirable results, to students other than the rather few who can learn mathematics without being taught, or the even fewer who cannot learn mathematics irrespective of what and how they are taught, two matters have to be kept in mind at all times:

1. We have to be infinitely careful not to jump to conclusions and make false inferences about the processes and outcomes of students' learning of mathematics. Wrong or simplistic assumptions and conclusions are always close at hand.

2. If there is something we want our students to know, understand, or be able to do, we have to make it the object of explicit and carefully designed teaching. Because of 1., there is no such thing as guaranteed transfer of knowledge, insight and ability from one context or domain to another. Transfer certainly occurs and can be brought about, but if it is to take place in a controlled way it has to be cultivated.

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