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ORIGINAL ARTICLE

Collectives of humans-with-media in mathematics education: notebooks, blackboards, calculators, computers and … notebooks throughout 100 years of ICMI

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Abstract The main aim of this study was to present evidence of the ways in which different media have conditioned and dramatically reorganized education, in general, and mathematics education, in particular. After an introduction of the theme, we discuss the epistemological perspective that provides the foundation for our analysis: the notion of humans-with-media. Then, we briefly illustrate how the medium is related to the scientific production of mathematical knowledge. We take a detour into the world of art to examine how devices and instruments have historically been associated with the production of mathematical knowledge. Then, we review studies on the history of education to show how traditional media were introduced into schools and have influenced education. In particular, we examine how devices such as blackboards and notebooks, which were novelties a 100 years ago, came to be accepted in schools and the mathematical activities that were promoted with their use. Finally, we discuss how information technology has changed education and how the Internet may have an impact on mathematics education comparable to that of the notebook over a century ago.

Keywords Notebook · Blackboard · Textbook · Humans-with-media · Information and communication technology · History of ICMI

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1 Introduction

Much has been said about the mediating role of technologies in human life, the production of knowledge and education. Kenski (2007), for example, asserts that "man moves through life mediated by the technologies that are contemporary to his time. They transform his way of thinking, feeling and acting" (p. 21). We would add that technologies not only transform humans' way of thinking, but that humans think with technologies. In Borba and Villarreal (2005), we have argued that collectives of *humans-with-media*[1] are responsible for the production of knowledge.

In this study, we will try to analyze and show evidence, throughout history, of the presence and influence of different collectives of humans-with-media related to the production of mathematical knowledge, education in general and mathematics education, in particular, and of the transformations that such media have produced in those contexts. We consider not only new information and communication technologies, but also old and traditional media. Take notebooks, for example. Are we talking about laptops or about a stack of paper bound together? It is important to note how a word such as notebook can have such a different meaning in the classroom after a period of only a 100 years. These days, when we talk about media, new information and communication technologies immediately come to mind: Internet, avatar, artificial intelligence, applets, multimodal language, and laptops. Three decades ago, when we heard about technology related to education, calculators, scientific calculators or maybe even

[1] In this paper, when we refer to media, we mean any kind of tool, device, equipment, instrument, artifact or material as a result of technological developments.

graphing calculators might have come to mind, and definitely not notebooks as most people would understand them today. The acceptance, use and role of different kinds of technologies in educational settings have varied and changed over time. In the same way, technology has affected the production of scientific knowledge.

In this study, we will reflect, based on a partial review of literature, on how media has been intertwined with mathematics education. We will consider a scope that goes beyond the history of ICMI, but which in our view will help us to see how different artifacts have been transforming mathematics education over these 100 years of ICMI history. In reviewing a significant part of the literature that deals with the role of artifacts in the production of knowledge, teaching and learning, we hope to show how some technologies have been the focus of controversy and discussion in ICMI meetings, while others seem to be treated as transparent or non-controversial. To add to the historical perspective of this association between artifacts and knowledge production, we also take a brief detour into the world of art to observe the portrayal, in famous paintings hundreds of years old, of artifacts and tools in association with important historical intellectual figures and scientists.

To develop our analysis, we first present our epistemological perspective regarding the relationship between humans and media.

2 Technology and cognition: an epistemological view

The analysis presented in this study is supported by an epistemological view that attributes an important role to artifacts and to a particular way of seeing the history of technology. In Borba and Villarreal (2005), we presented an analysis of different philosophy dictionaries, showing how technology is seen as separate from humans. In most of these dictionaries, humans are viewed as essentially being the opposite of technology and technique. While the former are seen as creative and warm, the sole role attributed to technology is that of repetition. It is viewed as something mechanical that can be reproduced by others. Lalande (1996), for example, defines technics as:

> a set of well-defined and transmittable procedures aimed at producing certain useful results. Another noteworthy characteristic of this initial technics, which was the infrastructure on which physical science rests, is its permanence over the centuries… It is an institution, probably the oldest of the institutions, and still remains, with the same characteristics today that it had at its beginnings… they are traditions that are passed from generation to generation by way of individual teaching, learning, and oral transmission… (p. 1109).

It should be noted that humans appear in the citation only as agents of the process of "education", or transmission, as the dictionary's author prefers to call it. Procedures or techniques are on one side; humans on the other! An alternative to this view is presented by authors such as Kenski (2007) and Lévy (1993). Kenski (2007) asserts that the concept of technology is associated not only with the development of sophisticated equipment, but goes further to include all the things and constructions that human inventiveness could create at different times, the ways of using them and their applications. Lévy (1993), for example, claims that technology and artifacts should be seen as intertwined with humans as we produce knowledge. He says that libraries, cities and artifacts are part of the way we know. Moreover, he argues that libraries and artifacts, in general, play an important role in the way knowledge is produced. He claims that one can observe how different technologies have shaped the way different people have produced knowledge throughout history. He sees part of the history of knowledge production as intertwined with the history of technologies of intelligence: orality, writing and informatics.

From this perspective, orality, writing and informatics are qualitatively different ways of extending our memory, and as each comes into play, they help to shape humans. Orality shapes the knowledge impregnated in the myths of people who have no developed system of writing. Orality and myths are associated, in the same way that writing is associated with demonstrations in mathematics. Similarly, we could say that simulation is associated with the availability of computers.

Lévy's (1993) idea is consistent with the notion developed by Borba (1993) that computers shape the knower at the same time that the knower shapes computers. Borba presented a substantial amount of data related to the way students dealt with function software to support the notion that students used the software in ways other than those anticipated by the software design team, of which he was part. Conversely, he showed that there were features in the design that shaped the actions and thinking of students. Borba (1993) named this phenomenon the "intershaping relationship", as he pointed out the important role of computers in the way that students coordinated multiple representations. Borba and Confrey (1996), Noss and Hoyles (1996) and Kaput (1989), among others, have claimed that the availability of computers has helped to change the way students know. These authors have emphasized the need to explore the potential of computers so as to avoid using new technologies in an old way.

These led Borba and Villarreal (2005) to propose that knowledge is actually produced by collectives of *humans-*

with-media. This notion is based on two main ideas: that cognition is not an individual enterprise, but rather collective in nature; and that cognition includes tools, devices, artifacts and media with which knowledge is produced. The media are components of the epistemic subject, being neither auxiliary nor supplementary, but an essential, constitutive part of it. They are so relevant that different media lead to the production of different knowledge. The media we work with alter, redefine and reorganize practices and contents. Old and new media have usually been considered useful tools to support learning and improve teaching, but we want to emphasize that media transform and reorganize those activities. It is in this sense that we assert that media cannot be considered as transparent.

This does not mean that material devices prepared for teaching are carriers of predetermined knowledge, but they can be used to produce knowledge according to the way people employ them. In this sense, specific material devices enable the development of certain activities and the production of ideas, which would not be possible if the devices were not present. We assert that if the use of the devices becomes routine, their presence seems natural; they become transparent, and it appears that the device makes no difference to the activity being developed. In this sense, the notion of humans-with-media is compatible with that of Meira (1998), who suggests:

> …that (a) the transparency of instructional displays is something achieved through a process of use and (b) that this process is mediated by users' participation in specific sociocultural practices (e.g., practices in mathematical classroom) (p. 139).

One of the purposes of the notion of humans-with-media is to emphasize the notion that technology is not transparent, but at the same time it opens and closes possibilities for knowing in a given way. For instance, dynamic geometry software opens the possibility for more experimentation in mathematics education, but it also depends on the way knowers use them, or the way the mediation takes place, as emphasized by Meira.

Meira's work refers to instructional displays such as computer microworlds or physical devices, but in our work, we also refer to tools that are not specifically designed for educational goals, such as calculators or computers.

We consider that the notion that knowledge is produced by collectives of humans-with-media is relevant not only to the use of computers, but also to other artifacts. In our previous work (e.g., Borba and Villarreal 2005), we have discussed how the arrival of new media is often met with resistance, and how this sometimes translates into the domestication of the new media, in the sense that it comes to be used in much the same way as those that preceded it. We believe that new media lead to the transformation of collectives of humans-with-media that produce knowledge. In our previous research on educational settings (Borba and Villarreal 2005), we documented how information and communication technology transformed the way students do mathematics, leading them to produce mathematics in new and non-domesticated ways.

In the sections that follow, we will discuss how humans and media are intertwined in the production of mathematical knowledge in scientific, as well as educational, environments. We will analyze how particular collectives of humans-with-media have altered life in schools and reorganized mathematical classrooms.

3 Media and mathematics

What are the materials necessary to produce mathematics? Many mathematicians would say paper, pencil and access to the library to get articles and books. For instance, Davis and Hersh (1981) assert that:

> Some mathematicians like to think that it [mathematics] could even be done in a dark closet by a solitary man drawing on the resources of a brilliant platonic intellect" (p. 13).

These authors talked about the auxiliary tools or equipment needed to produce mathematics. They asserted that, in the past, primitive mathematics, as well as religion and great epics, were probably transmitted via oral tradition, but it later became evident that the use of devices for writing, recording, and duplication were essential for doing mathematics. Davis and Hersh (1981) go further when they highlight that

> The ruler and the compass are built into the axioms at the foundation of Euclidean geometry. Euclidean geometry can be defined as the science of ruler-and-compass construction" (p. 13).

In a sense, these authors consider the media as intrinsic parts of the foundation of Euclidean geometry. From our theoretical perspective, we would rewrite this sentence and say that Euclidean geometry is produced by collectives of humans-with-ruler-and-compass.

It is an interesting exercise to observe one of the most famous paintings portraying the important Greek philosophers and ancient scientists, the fresco known as "School of Athens" by the Italian artist Raphael Sanzio (Fig. 1). It was painted between 1509 and 1512 to decorate one of the rooms at the Vatican Palace, the *Stanza della Segnatura*. This is a nice artistic image of what we refer to as the collective thinking of humans-with-media.

In the center of the fresco we see the figures of Plato, on the left, and Aristotle, on the right, gesturing and

Fig. 1 The school of Athens (free photograph from http://en.wikipedia.org/wiki/The_School_of_Athens#cite_note-8)

dialoguing with each other. Both of them carry books, with Plato holding *Timaeus* and Aristotle his *Ethics* (see details in Fig. 2).

According to Bell (1995), a specialist in art history,

> Not every figure in this work is worthy of speculation, but it seems that the men whom Raphael clearly intended his audience to recognize are linked to specific iconography. These form the most reliable identifications: Plato and Aristotle indisputably are here, each holding a titled work; Pythagoras is in the lower left studying his tablet of harmonic proportions; Euclid is in the lower right area with his compass; near him, Ptolemy wears his crown and holds a terrestrial globe; and Zoroaster holds his starry globe (p. 640).

Fig. 2 Plato and Aristotle. A detail in Raphael's "School of Athens"

It is interesting to note that the use of different iconography is related to the activity that each scholar developed. Focusing on mathematicians, we can see that Euclid makes a drawing on a slate using a compass, probably showing a geometrical construction to his disciples (see details in Fig. 3), and Pythagoras uses a kind of pencil and a book (see Fig. 4).

Another interesting and nice pictographic evidence of the media that have been associated historically with mathematics comes from Domenico Fetti's painting, *Archimedes Thoughtful* (Fig. 5), which currently hangs in the Alte Meister Museum in Dresden (Germany). This

Fig. 3 Euclid, a detail in Raphael's "School of Athens"

Fig. 4 Pythagoras, a detail in Raphael's "School of Athens"

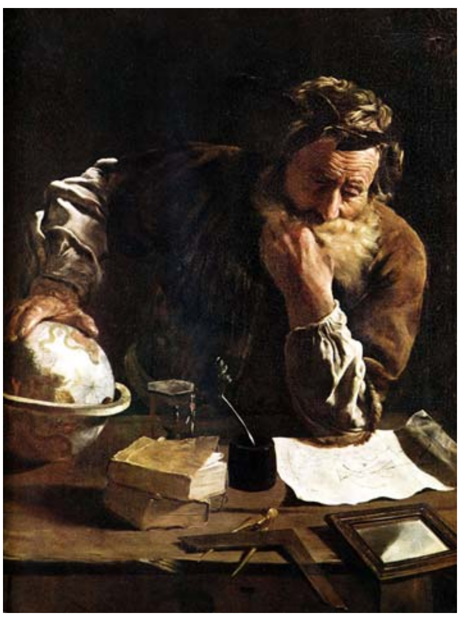

Fig. 5 Archimedes thoughtful

painting of the 1620s shows an image of Archimedes reflecting on a piece of paper with a geometrical drawing surrounded by many tools: compass, paper, pen, books, a set square and a terrestrial globe.

This detour into the realm of art illustrates how devices and instruments have historically been associated with the production of knowledge, particularly in mathematics.

While we can identify the presence of the media associated with the production of knowledge in ancient mathematics, this pictorial evidence does not mean that the tools represented there (paper, pencil, compass, set square, etc.) were seen by mathematicians at that time, or even nowadays, as fundamental tools for producing mathematics.

As we approached and entered the new millennium, new technologies continued to appear on the scene, and debates on their contribution to mathematical production started. Devlin (1997), for example, refers to the transformations that computers have brought to the activities of mathematicians:

> Over the past decade or so, the professional mathematician has changed from being a person who sits at a desk working with a paper and pencil to a person who spends a lot of time sitting in front of a computer terminal. The paper and pencil are still there, but a lot of the mathematician's activities now involve use of the computer…. This rapid transformation of mode of working has changed the nature of *doing* mathematics in a fundamental way. Mathematics done with the aid of a computer is qualitatively different from mathematics done with paper and pencil alone. The computer does not simply 'assist' the mathematician in doing business as usual; rather, it changes the nature of what is done (p. 632, italics in original).

Devlin (1997) believes that the computer can play a significant role in the process of reasoning of the mathematician. But, many authors indicate that this trend, when associated with mathematical proof, is still highly polemic in the mathematical community: the computer is seen as a "stranger in the nest" (Domingues 2002) and "the pure mathematical community by and large still regards computers as invaders, despoilers of the sacred ground" (Mumford[2] (quoted in Horgan 1993) who is critical of that position among mathematicians, p. 76). These references are only a sample of different positions on the use of new technologies in the mathematical community. There is obviously a fear of attributing a relevant role to computers in mathematics, but that fear is not insignificant, since we are entering a philosophical discussion that deals with the nature of mathematics itself.

Although this is a brief discussion considering the media related to the scientific production of mathematical knowledge, it is apparent from the examples that the role of media continues to be controversial and opinions vary. From our epistemological perspective, we have tried to present evidence that knowledge is always produced by collectives of human-with-media.

4 "Traditional" media in (mathematics) education

We hope that we have convinced the reader that tools such as ruler and compass and computers are active actors in the

[2] David Mumford was granted the Fields Medal in 1974 for his research in pure mathematics.

production of mathematical knowledge. We would now like to focus on other artifacts that are very important for education in schools.

According to Kenski (2007), technologies are so close to us and present in our lives that we fail to perceive that they are not natural things, and that they have not always been there. Pencils, notebooks, pens, blackboards, chalk or textbooks are technological products "that allow mathematical learning and comprehension, but they are so incorporated into school activities that their influences on the construction of mathematical knowledge are almost imperceptible or invisible" (Borba and Villarreal 2005, p. 92); these are media, the presence of which in schools today has become transparent.

In this section, we will refer to some of these media: blackboards, notebooks, graph paper and textbooks. We found no discussion regarding the role of (paper) notebooks in mathematics education throughout ICMI history. It seems that tools associated with writing were not problematized in ICMI history, and phenomena such as the introduction of the blackboard were not topics of discussion. Although blackboards, notebooks and textbooks are not associated solely with mathematics teaching and learning, they have been the classical media in mathematics classrooms for many years.

The blackboard is characterized by its ubiquity in schools. We cannot imagine a classroom without a blackboard, but this device was not always present. Different sources indicate that the blackboard was used for the first time in schools around 1800. An instructor at West Point Military Academy, Mr. George Baron, is considered to be the first American instructor to use a large blackboard to teach mathematics in 1801 (Kidwell et al. 2008). A contemporary of Baron, the Headmaster of the Old High School of Edinburgh (Scotland), James Pillans, is also credited with inventing the blackboard and colored chalk, which he used to teach geography. In France, since 1882, the blackboard was considered a teaching material that every teacher should have in primary school, and a dogma of the modern school was: "the best teacher is the one who uses chalk the most" (Bastos 2005, p. 136). The arrival and definitive adoption of the blackboard in Latin American schools occurred at the end of the nineteenth century (Bastos 2005), shortly before ICMI was created, when public systems of elementary education were being consolidated.

The blackboard in the school gave rise to new educational practices. Prior to its presence in the classroom, students used a handheld slate where they wrote the assignments. The teacher had to go from one student to another copying the tasks on each student's individual slate. The use of these slates meant that there were no permanent records of the school tasks. According to Bastos (2005):

> It may be affirmed that the pedagogical centrality of the blackboard is due to the absence of school manuals and other visual learning resources, and also results in centralizing the pedagogical process in the figure of the teacher (p. 133).

One of the main practices that the large blackboard enabled was the simultaneous teaching of reading and writing lessons for the whole class. From that moment on, the blackboard and the teachers became central actors in classroom life, and copying from the blackboard was the main activity of the students. However, the blackboard is also used to share students' solutions to a given problem and, in this sense, can also become a medium for hearing the students' voices. Kidwell et al. (2008) report on some practices related to the use of the blackboard in mathematics classrooms that were considered novel by the time the blackboard was a new medium in schools. For example, in an 1820 American school, a teacher reported that she asked her students to sit in a circle and tell her:

> …how they do the sums -& following their directions with a piece of chalk upon a black board, so that every error can be made manifest to the whole class, who have the privilege of correcting each other (p. 26).

The 1841s statutes of the State of New York relating to common schools say:

> Large blackboards in frames are indispensible [*sic*] to a well conducted school. The operations in arithmetic performed on them, enable the teacher to ascertain the degree of the pupils' acquirements, better than any result exhibited on the slates. He sees the various steps taken by the scholar and can require him to give the reason for each. It is in fact an exercise for the entire class and the whole class by this public process insensibly acquires a knowledge of the rules and operations in this branch of study (p. 27 in Kidwell et al. 2008).

The public character of the activities developed on the blackboard seemed to be an important characteristic emphasized in both excerpts. The possibility of exhibiting errors and misunderstandings is also mentioned.

The author of *Slate and Blackboard exercises for common schools*, William Alcott, a Massachusetts teacher, claimed in 1842:

> Students who copied drawings, made measurements, and worked arithmetic problems following instructions on the blackboard would learn more than those who simple recited by rote (p. 28 in Kidwell et al. 2008).

Nowadays, such practices on the blackboard are standard, and they are not mentioned as didactical orientations or suggestions in any document for teachers.

The individual handheld slates were the only personal devices that students used in the school before the arrival of notebooks, when the production of paper increased and the costs decreased. Certainly, the historical data on the use of notebooks in schools differ from place to place. In France, for example, the use of the notebook became common in high school (10–14-year-olds) in the sixteenth century and was obligatory in the teaching of calligraphy in the seventeenth century, but its generalized use in the elementary schools dates from the first third of the nineteenth century (Hébrard 2001). The cost of paper limited its use until the more advanced grades, restricting children's learning in the area of literacy to reading. Hébrard reports that, around 1833, the use of the notebook in elementary education was considered by the Ministry of Elementary Instruction to be a sign of pedagogical modernity. In Argentina, the notebook was introduced into the classroom around 1920 and was closely linked to the so-called new school movement, which recommended the use of a single class notebook as an organizing tool for schoolwork (Gvirtz 1999).

When notebooks were introduced into the schools, everyday life inside as well as outside the school was transformed. These transformations affected the activities of the students. The use of the notebook implied knowing not only how to copy, write dictation, do arithmetic exercises or solve arithmetic problems, but also how to organize and present them, making the notebook a "small theatre of school knowledge" (Hébrard 2001, p. 137). The presence of the notebook in the school introduced a series of changes into the day-to-day activities in the classroom, whose importance equals that of broader administrative and curricular changes. This aspect was raised by Gvirtz (1999), who asserts that:

> The notebook is not a mere physical support… On the contrary, it is a device whose articulation generates effects: in more concrete terms, the notebook constitutes, together with other elements, a shaper of the classroom (p. 160).

According to Gvirtz, considering the changes introduced by the notebook, its use cannot be seen as a simple change in the technology used to report school activities, but as a re-organizer of life in the classroom.

As in the case of notebooks in Argentina, the decreasing cost of paper made it possible for another medium to enter the mathematics classrooms at the beginning of the twentieth century: *graph paper*. It is interesting to consider the way *graph paper* arrived in American mathematics classrooms. Its use was common in scientific and engineering applications in Europe in the nineteenth century (Brock and Price 1980). According to Kidwell et al. (2008), graph paper started to be used in educational settings during the last decades of that century, when influential British scholars advocated its use in education. The authors assert that in the USA, the confluence of several changes in mathematical pedagogy stimulated the use of graph paper in the early 1900s: the increased promotion of visualization as a teaching tool, the efforts to make mathematics education more helpful for science and engineering, and the "increased recognition of the value of the function concept for unifying large portions of the subject" (p. 196). The availability of low-cost graph paper made it possible for those pedagogical changes to be implemented in the schools. Ruthven (2008) historically situated a discussion about the use of graph paper in mathematics education. By the beginning of the twentieth century, the recognition of the role of intuition and experimentation in mathematics education at the secondary school level was considered a foundational theme for ICMI. One of the areas associated with this trend was the graphical method, and graph paper was the material support to implement such a trend. This same author reported, using Smith's 1913[3] words, about the value of squared millimeter paper at that time.

The history of books and textbooks offers another example that shows how media can influence humans' thinking and behavior in general, and shape educational settings, in particular. In 1958, Febvre and Martin published the book entitled *L'apparition du livre*. In this book, the authors refer to the changes that the advent of printed books entailed for Europe during the first few centuries after its invention: the economic changes associated with printed books in terms of jobs and business and also the deep intellectual changes introduced into the European culture. In Paul Chalus' preface to the book by Febvre and Martin (1958), he acknowledges, in a sense, the influence of media on *human lives:*

> Thus, the men made the books, and the books, in their turn, made the men (p. 33, our translation).

Chalus' words echo the concept of Borba (1993) on the "intershaping relationship" mentioned earlier, in the sense that men shape books and, at the same time, books shape men. It is also in this sense that we say that the construct of humans-with-media stresses the notion that media developed by humans also permeate humans.

Consistent with this perspective, an interesting example of the power of textbooks in mathematics education is provided by Kidwell et al. (2008). These authors examined "…how Americans consciously shaped textbooks into mathematical teaching apparatus" (p. 4) and reflected on "the potential that textbooks held as teaching tools for

[3] Smith (1913).

bringing structure and uniformity to the classroom" (p. 6) in American schools at the beginning of the nineteenth century.

An example of the way in which textbooks influenced the structure of Brazilian mathematics classrooms, as vehicles of a curricular reform initiated around 1930,[4] can be found in Braga (2006). This author refers to the way that the reformist ideas of Félix Klein, with regard to the teaching of mathematics at the secondary level at the beginning of the twentieth century, were implemented in Brazil. He analyzes several collections of secondary mathematics textbooks that were being used at that time. Braga (2006) asserts that, prior to the reform, the common practice in the classroom was to use a notebook to make annotations about theory and examples and, at most, a book with exercises. In 1940, the author of the bestselling collection of mathematics textbooks critiqued this practice in the preface of the first volume of his collection, saying:

> Let's get rid of the notebook, which is the main cause of the failure of high school teaching in Brazil (Stávale, apud Braga 2006, p. 93, our translation).

According to Braga (2006), this assertion showed the author's concern, as well as the editor's interests, regarding the introduction of textbooks into the day-to-day activities of school. At the same time, this quotation offers clues on the use of notebooks in Brazilian secondary schools at the beginning of the twentieth century.

Just as collectives of humans-with-notebooks and humans-with-textbooks were foreign to schools at different times and places in the past, humans-with-computers are absent in many educational settings today. As discussed earlier in this section, artifacts such as blackboards and notebooks changed the role of the teacher and made possible "new" formats of teaching, which are considered standard today. When we think of notebooks today, understood as laptops or netbooks, and their arrival in schools, together with wireless Internet and smart boards, one may ask whether we are going through a change similar to the one that took place in schools a 100 years ago in countries such as Brazil and Argentina, when blackboards and notebooks started to become active actors in schools.

5 "New" media in mathematics education

In this section, we will refer to media that are closer to us than the traditional ones analyzed in the previous section, with respect to the time they entered the school, and their relationships with mathematics education.

[4] Such reform was called Francisco Campos Reform and was decreed in April 1931.

In the context of mathematical education, the use of manipulative materials has been a frequent recommendation for teaching and learning mathematics at different times in the history of mathematics education. We can count or solve arithmetic calculations using pebbles, abacus, counters, fingers, paper and pencil, mechanical adding machines or handheld electronic calculators. According to Davis and Hersh (1981): "Each one of these modes leads one to a slightly different perception of, and different relationship to, the integers" (p. 33). Such words seem to be in resonance with the notion that media are essential for knowledge production. However, other points of view regarding the use of different manipulative materials with children (logical blocks, matches, etc.) can also be found. For instance, in the Proceedings of the Second International Congress on Mathematical Education (ICME), held in Exeter in 1972 (Howson 1973), we found a warning saying that "generalisations about material and methods could be dangerous" (p. 24) due to the complexity of the development of children's thinking and the fact that little was known about it. Another section of these proceedings was devoted to the mathematics workshop or laboratory where the use of games, apparatuses or manipulative materials was analyzed in terms of advantages and disadvantages for the teaching of mathematics. In this case, the materials were considered to be the means for evoking the students' interest or leading them to discover a mathematical fact, concept or generalization. At that time, this method appeared to be more widely accepted at the primary level, and recommendations regarding their use included the following:

> Each lesson should have an underlying framework of objectives: both subject oriented and student oriented. <u>The hardware and the equipment are not the essence</u>; more important than the physical facilities are the attitudes of the children, the atmosphere of the classroom, and the objectives of the teacher (p. 54, our emphasis).

While the attitudes of the children, the atmosphere of the classroom and the objectives of the teacher are considered paramount in the classroom, the teaching materials are not seen as main actors in knowledge production, and they may become "transparent" media or "co-adjuvant actors". However, as we have argued, we believe they shape the classroom and the possibilities for the kind of mathematics produced.

The use of calculators in schools has also provoked many debates. In this sense, it is worth noting, for example, an interesting excerpt from the Proceedings of ICME 2 (Howson 1973), where the contributions to mathematics education of the first ICMI president, Felix Klein, were referred to:

> His books on Elementary Mathematics from an Advanced Standpoint are still read throughout the world—indeed, some of his suggestions, such as the use of calculating machines for teaching arithmetic to children, still have a modern ring about them! (p. 11)

Klein died in 1925 and became an icon in mathematics education. In light of this quote, he could also be credited with being a pioneer in proposing the use of technology in schools.

The Proceedings of ICME 2 include other references to the use of calculators and computers. Programmable calculators in school were "seen both as an aid for the teacher when introducing mathematical concepts and also as a calculating aid for the pupils" (p. 57). Computers were seen as motivating for students and promising for mathematics: "Numerical and graphical methods will undoubtedly receive impetus from the advent of computers" (p. 26). Clearly, the media were considered mainly as a sort of scaffolding at that time to help in the construction of mathematics, but not as a fundamental actor in the production of mathematical knowledge in school.

> According to Kidwell et al. (2008), calculating machines dated from the mid-nineteenth century and were used for commercial endeavors, but it was in the 1970s with the decreasing cost that handheld electronic calculators entered into educational environments. At that time "mathematics teachers, parents and school officials disagreed about how much, if at all, the new tools should be allowed in more elementary courses" (p. 245).

These same authors present a report on opinions regarding the role of electronic calculators in schools, coming from a sample of teachers, mathematicians and laymen in the USA in 1974. Most of the respondents (66.6%) believed that the major goal of elementary and junior high school mathematics teaching was arithmetic computation; 87% believed that speed and accuracy were essential for business and industrial workers and intelligent consumers. Referring to the calculators specifically, Kidwell et al. (2008) noted that some of the respondents:

> …worried about cost, others thought calculators should be provided only when students had demonstrated their proficiency in paper and pencil computations, and others thought providing calculators would promote laziness and inefficiency (p. 256).

More recently, Ruthven (2008), based on an article he published in 1999, asserted that:

> …at primary level, where the curriculum has traditionally been organised around highly valorised methods of written and mental calculation, the pedagogical benefits of calculator use remain controversial, and appropriate forms of curricular (re)organisation underdeveloped (p. 6).

Nowadays, the situation is not so different. Although the opinions previously referred to were made over 30 years ago, and Ruthven's analysis is 10 years old, there are still many doubts on the use of calculators in Argentina and Brazil. For example, many educators consider that the use of calculators by students will prevent them from using mathematical reasoning.

The following anecdote illustrates a position of resistance toward the use of calculators. This fact occurred with Emilia, an Argentinean student who was attending the first year of secondary school (12–13 years old). During a mathematics class, the teacher posed the following task: *If the 9 key on your calculator doesn't work, how would you use it to compute 9393–1439?* Emilia asked for help from her aunt, and when her aunt told her to get the calculator, she responded that the teacher had said that the exercise was to be done without the calculator. Apparently, the task was aimed at applying properties of numbers and operations using the calculator, but since the teacher did not allow the students to use it, the task became meaningless and boring for them, leading them to do the calculations with paper and pencil. Obviously, the teacher did not restrict the use of paper and pencil. These devices have come to be seen as transparent, neutral tools in the act of doing mathematics; no one questions their use to solve a mathematical exercise. It seems as though paper and pencil, and the paper version of notebooks, have always been present together with mathematics. In the same classroom where the previous episode occurred, the illustration shown in Fig. 6 appeared in the students' mathematics textbook alongside an exercise on prime numbers: a rather unintelligent-looking man holding a calculator.

Many interpretations can be made of this image, but, in a sense, it illustrates the opinion of the textbook author on

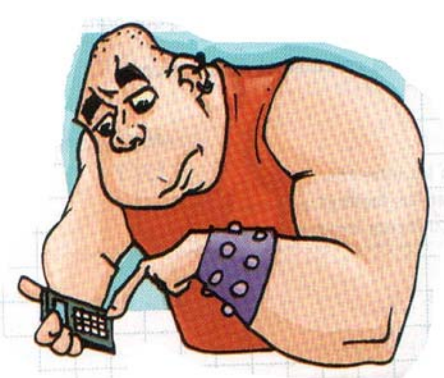

Fig. 6 Illustration from a mathematics textbook of a man using a calculator

the use of calculators in the mathematics classroom. Illustrations in books should not be taken as mere ornamentation; they are not naïve.

Concerns and interest regarding the influence of technologies in teaching and learning mathematics have always been present on the ICMI agenda. Evidence of this interest, for example, is a round table that took place during the Third International Congress on Mathematical Education, held in Karlsruhe in 1976, organized around the theme: what role will computers and calculators play in the future teaching of mathematics? The table was coordinated by H. Freudenthal, and the participants were U. D'Ambrosio, A. Engel, M. Meissner, J. Nievergelt, S. Papert and H. Pollak (Roulin 1976). In 1980, during ICME 4, held at the University of California, Berkeley, Seymour Papert delivered a plenary speech on the computer as carrier of mathematical culture (Hilton 1981).

In 1983, ICMI proposed the establishment of a series of studies, now known as the ICMI studies, with the aim of promoting and assisting discussions and actions through:

> The identification of key problems within the subject area and the provision of a framework which will facilitate further study, research, development and/or decision-taking; the provision of up-to-date accounts of relevant thought, research and practice (Howson 1983, p. 349).

By this time, interest in the use of computers was increasing in the mathematics education community, and the first ICMI study was related to this theme: *The Influence of Computers and Informatics on Mathematics and its Teaching*. The Study Conference was held in Strasbourg, France, in March 1985. The work in this particular study focused on three issues: the effects of computers and informatics on mathematics; their effects on the design of new curricula; and the uses of computers as aides in the teaching of mathematics. In the discussion document that was prepared prior to the conference, when referring to the effects of computers on mathematics, it was recognized that "Only on rare occasions has it been allowed to influence and infiltrate our teaching" (Churchhouse et al. 1984, p. 164).

Concern regarding the use of technology in mathematics education has been increasing rapidly in our community. We found mention of technology in several ICMI studies, although technology was not always their main focus. For example, in the questions related to teaching methods posed in the discussion document of ICMI Study 3: *Mathematics as a service subject*, computers were mentioned (Howson et al. 1986). The authors referred to the new possibilities offered by computers (rapid computation, graphics, experimentation) and the changing needs caused by their introduction (curricular content, desirable qualities to be developed in students).

In the discussion document of ICMI Study 5: *The popularization of mathematics*, Howson et al. (1988) wrote:

> …new technologies provided new stimulation and new tools. Computer graphics have enabled new and advanced mathematics to be introduced to vast numbers of people… A new range of mathematical activities can also be introduced through the computer. How can the micro best be used in the popularization of mathematics?… (p. 211)

Twenty years ago, mathematics educators advocated that tools could change the nature of mathematical activity as well as its relation to the majority of the population whose jobs did not directly involve mathematics. About 10 years later, similar concerns could be found in the fourth chapter of the book *Perspectives on the Teaching of Geometry for the twenty-first Century*, corresponding to ICMI Study 9, which refers to computer technology and the teaching of geometry. According to Villani (2000), who reported on the contents of this volume, the dynamical possibilities that some geometrical software offer and their demonstrative functions lead to:

> …a more functional role of the computer as an explorative tool, making intuition, construction, and spatial sense more important factors, but also providing ways to link them to the theoretical aspects (p. 413).

An examination of the programs of ICMEs since their first edition reveals that working groups, discussion groups, topic study groups, survey teams and plenary as well as regular lectures have addressed many aspects related to different media (technologies, devices) in mathematics education. In 1996, during ICME 8 in Seville (Spain), nearly 300 participants attended working group 16: *The role of technology in the mathematics classroom,* making it one of the largest groups in the conference, if not the largest. Below we present two of the many concerns that Borba et al. (1997) presented in the final report of the group.

> The issue was raised whether technology was just a complementary tool to be used by students when thinking mathematically or whether technology, such as computers, would actually transform (or reorganize) the activity of doing mathematics (p. 10).

> A major issue was raised […] regarding student thinking: could students transfer what they learn while using computers to occasions in which they were not using this kind of technology? Some suggested that students could not and that we should tackle this problem. Others thought this was not the correct question, since mathematics was being

> transformed as we used computers and therefore we should be thinking more about mathematics in computer environments without having the "measuring stick" of paper and pencil mathematics (p. 10).

The first quote refers to the role of technology in the thinking process, and the second one to the transformation of the mathematics mediated by the computer. These issues continue to be concerns among mathematics educators today.

Inside the field of mathematics education, research has provided evidence of the transformation that the use of computers brings to the teaching and learning of mathematics. One such transformation was the creation of environments where mathematics could be experienced as an experimental science, through tools to generate conjectures and check their validity: a mathematical laboratory where "educated trial and error" was permitted and visualization was an ally to understanding mathematics. These environments make some old practices obsolete, uninteresting and boring. For instance, making the graph of the function $y = 3x^2 - 5x + 8$ using a plotter is a mechanical exercise, but exploring the effects of parameters a, b and c in the graphs of equation $y = ax^2 + bx + c$ can generate many interesting conjectures, as shown in research presented in Borba and Villarreal (2005).

In spite of such research findings, there is continued resistance to the use of technology in educational environments. For example, at some Argentinean and Brazilian universities, where pre-service mathematics teachers study, computers are used only in a "domesticated way", as aides to show a graph or a dynamic geometrical construction. Students have no opportunities to learn mathematics with the computer, because that is not the usual way to learn mathematics and because teachers do not believe that the media is part of the collective that produces knowledge.

It is possible to see that issues discussed in meetings organized by ICMI are still part of our agenda. So software is still trying to become popular in classrooms. Although a lot of research has been developed on the use of software in the classroom, the blackboard seems to be the main medium that structures classroom activities. Our previous discussion about the blackboard suggests that the timing of the acceptance of blackboards in the classroom varied from country to country, and this may well be the case with computers. On the other hand, overcoming obstacles to the use of the Internet in classrooms may be an even greater challenge, as this discussion is still absent in conferences.

The ubiquity of technology in our society constantly renews the concern regarding the use of technology in educational settings among mathematics educators. In 2006, this topic was revisited when a new ICMI Study Conference was held in Hanoi, Vietnam: *Digital Technologies and Mathematics Teaching and Learning: Rethinking the Terrain*. Among the 72 contributions that were presented, we could find only five abstracts related to the use of the Internet ("Seventeenth ICMI Study", 2006). We would like to make a short reference to the Internet as being a communication technology that has grown considerably in recent years. Particularly, the Internet has created new possibilities for distance education, and this situation constitutes a new research scenario for mathematics education. Again, in this case, the media play a fundamental role. For instance, proposals for teacher education activities in online courses have fostered the development of communities that discuss issues related to mathematics and mathematics education. There is a variety in the types of virtual learning environments, depending on the technological resources these rely on. We would like to refer to an experience with distance education that has been ongoing since 2000 by members of the Brazilian research group, called GPIMEM[5] (Borba et al. 2007). They have developed online courses aimed at mathematics teachers where issues of mathematics and mathematics education are discussed. All these courses were structured with the conception that interaction, dialogue and collaboration are factors that condition the nature of learning and the quality of the distance education. The diverse technological possibilities available in the virtual learning environments enabled the appearance of different styles of mathematical knowledge production. The learning possibilities in an environment where the interactions occur through chat alone are different from those using videoconference to communicate.

Although the use of the Internet in online classes is fundamental for communication, and the attempt to prohibit it from entering the class is almost impossible, the situation is quite different in the case of face-to-face classes. Borba (2007) has questioned whether the Internet will be accepted in the classroom. According to him,

> …access to Internet may not be allowed based on arguments such as: you can find answers for the problems given, students may get distracted, or it will privilege students who know how to navigate better on the Internet (p. 4).

In spite of these arguments, or maybe because of them, he argues that maybe the Internet, with its overwhelming informational role, will have the effect of changing what will be considered a mathematical problem in schools. Collectives of humans-with-Internet transform the very notion of problem. A problem also depends on the media

[5] Grupo de Pesquisa em Informática, outras Mídias e Educação Matemática: Technology, other media and Mathematics Education Research Group.

available to solve it. For instance, a problem given with a motion detector attached to a graphing calculator is shaped by the availability of such a tool. Thirteen-year-old students would probably not know how to approach a problem in which they are asked to associate a given body movement, such as walking toward a wall, with the Cartesian graph of time versus distance displayed by the graphing calculator. However, with the availability of such a device, a student could start to create a notion about Cartesian graph, linear function and other concepts with the help of a teacher in a reasonable amount of time, as reported in studies such as the one discussed in Borba & Villarreal (2005).

It is not quite clear whether calculators will be accepted or not in classrooms around the world; but it is quite likely that the Internet will participate in one way or another in schools. We believe that disagreement regarding the use of the Internet and other technologies in mathematics classrooms will continue, but there is a major concern that, as educators, we cannot elude the students' right to have access to information and communication technologies as new tools in our culture and as new actors in knowledge production. We believe that it is important to examine the history of the use of technical devices throughout the history of ICMI, and also to look at the history of media in education to recognize the crucial role of media in the production of knowledge.

6 Notebooks and notebooks

The image in Fig. 7 was taken from a recently published Argentinean magazine, *El monitor de la educacio´n* (2008), edited by the National Ministry of Education. The picture shows a classroom with a notebook (laptop) for each student, but one of them is using "another kind" of notebook. A classmate is pointing an accusing finger and telling the teacher that "González is using pencil and paper!" This is a futuristic picture for Argentinean schools in which it would be considered cheating, or at least inappropriate, to use paper and pencil, though we can note that the blackboard is still part of the classroom furniture. The old and the new media co-exist in a manner similar to that described by Lévy (1993), who argued that one medium does not suppress another. A collective of humans-with-media in school is shown in the picture in which kids and computers are the main actors.

It would be interesting to imagine the mathematics that is being produced in a classroom such as the one in that picture. Our focus in this particular paper, however, is to look back. We have discussed how the word notebook has changed its meaning in less than 100 years of ICMI history. The meaning has evolved from a stack of papers

Fig. 7 "Teacher, González is using pencil and paper!" (a picture in the humor section of the magazine *El monitor de la Educacio´n* 2008)

bound together to one of the main symbols of the "Internet culture". The word notebook has become a word (without translation) that identifies laptops in many languages such as Portuguese and Spanish. Much has changed since the first notebooks (copybooks) were introduced and became the norm in Argentinean schools (we have little documentation regarding the Brazilian case). The current policy of the Brazilian government, now in its experimental phase, is: *a notebook (laptop) for every student*. The Brazilian government has been studying, and is currently carrying out pilot studies in several schools, to provide every public school student with a laptop, just as textbooks have been provided to them. This notebook would look much more like a netbook, the latest "evolution" of the notebook, which privileges access to the Internet and uses the metaphor "computer in the clouds" to express the idea that software and data will be stored on the Internet and not in a computer.

There are similarities regarding the introduction of both kinds of notebooks: notebooks in Argentina at the beginning of the twentieth century, and laptops in Brazil at the beginning of the twenty-first century. Both, for example, generated crisis and resistance. In the twenty-first century, the introduction of notebooks in Brazil was met with resistance from the private schools (where most middle class youth study), as they would no longer be able to make the argument that they were better equipped and more modern than public schools. Some questionable studies carried out in the USA and in Brazil regarding how computers could be a problem to learning have made it to the front page of some publications.

Not everything that happens in education in different countries is reported in ICMEs and ICMI studies. We attempted in this paper to bring to light the way different artifacts, from manipulatives to the Internet, have been portrayed (or not) in our 100 years of history. We have shown that some have been the theme of papers, working groups or even entire conferences, as in the case of the two ICMI studies on computer technology, while others, such as paper and pencil and blackboards, are absent, though they have helped to shape what we understand as the classroom today. Collectives of humans-with-notebooks (laptops) were present in our conferences, but not collectives of humans-with-notebooks (copybooks). Notebooks, books, compasses or set squares were present, however, in sacred Italian art! We would like to raise the conjecture that the absence of these material cultural artifacts in ICMI conferences helped to shape the notion that knowledge is constructed only by collectives of humans, or, as has been very popular throughout ICMI history, by a "lone individual knower". Here, we have tried to show evidence that knowledge is constructed by collectives of humans-with-media. Restivo (2007) has proposed that individuals express social ideas, and that the basic units of analysis should all be social. This view is consistent with our view (Borba and Villarreal 2005) that humans are infused with different media. Once students or teachers interact with a function software, their view of function changes. When an individual expresses a given mathematical idea, he or she is infused with culture, people and material culture, and this is why we state that collectives of humans-with-media should be considered as the epistemological subject, as a unit of analysis in the production of knowledge. Notebooks helped to shape mathematics in schools in the early 1900s and now in the early twenty-first century. Different technologies, created and developed by humans, have become co-actors of qualitatively different ways of knowing.

In this paper, we presented a review of documents from ICMI conferences, literature on arts, literature on the history of education that deals with material culture, as well as literature in mathematics and mathematics education. In summary, those media that we have called "traditional" in this study privilege orality and writing as the main activities: learning by rote, solving equations or doing computations with paper and pencil or on the blackboard. Meanwhile, the manipulatives, or calculators and computers, privilege visual and explorative activities. The method of doing, teaching and learning mathematics are different for each one of the collectives of humans-with-media that we have considered in this paper.

Using the final words of Kindell et al. (2008) from the book *"Tools of American mathematics teaching, 1800–2000"*, to understand how objects have become standard components of classrooms,

> one should consider not only the history of electronic devices but also the stories of older products—such as textbooks, blackboards, graph paper,…—which may have proved so practical and durable as to be taken for granted… these instruments reflect new educational ideas and technical capabilities as well as changing views of mathematics and education throughout … culture (p. 312).

We think that the analysis developed in this study can help us to understand mathematics education in the last century, as much as the analysis of Gvirtz (1999) helped some of us to understand how schools were structured, in the material sense, in classrooms with blackboards, desks and notebooks!

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