What use are mathematics education researchers?

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What is the object of the encapsulation of a process [1]?
Say what!
What kind of silly title is that?
What on earth do the people who wrote that paper think they might have to say that's of any value to a mathematician?

Is that your reaction when you come across a mathematics education research paper? Silly, abstruse and impractical? What, then, is the use of mathematics education researchers [2]?

Categorising Research

Alan Schoenfeld breaks mathematics education research into three different categories: product oriented, social engineering and basic research into cognition [3]. Of these, the first is probably the most obviously of immediate use to a teacher. Mathematics education researchers can, and do, produce products for direct application to the classroom or the lecture theatre. This product oriented work, particularly in the context of the use of computer packages to aid understanding of mathematical concepts, has a good history. The graphics calculus packages of David Tall can give students insight into the nature of, say, differentiability as local straightness (which fit well either with non-standard approaches to analysis or with the development of the notion of the differential and the generalisation to higher dimensions) and is an idea that has been taken very much further by Matthias Kawski who looks at how much of multi-variable calculus can be visualised using computer graphics [4]. Of course, the most obvious form of product oriented research is the text book which even if not directly under the heading of mathematics education research, involves the author in making conscious decisions about the needs of the reader.

Schoenfeld's idea of social engineering research is not quite as disturbing as it sounds. This is research which we might otherwise classify as 'product oriented', but whose role is not to provide support for existing teaching methods, but to suggest radically different ways in which mathematics might be taught (or even radically different ways of thinking about what being a mathematics student might mean). The most obvious products which might make us think about the nature of mathematics teaching are computer based packages (many of which we have read about in the precursor to this newsletter). The advent of symbolic manipulation packages might allow us to spend less time on enhancing the fluency of manipulation in different contexts. Access to cd-roms full of information, or the data in the web means that we might put less emphasis on factual memory.

Of course, neither of these gets replaced: fluency is still important in doing a calculation and access to factual memory is still important in deciding which tools, skills or theorems one might bring to bear in a given situation. Technology, however, changes the balance between developing these skills and working on conceptual understanding of the mathematics. In particular, work done by Ed Dubinsky and the RUMEC group concentrates on students using technology (and particularly the act of programming in languages with a strong mathematical syntax) to enable students to construct mathematical objects for themselves [5].

Again, however, the social engineering work is not confined to that led by the technology. Work by Bob Burn in developing carefully structured sequences of questions are allowing teachers to radically rethink their approach to teaching traditional topics like analysis, number theory and group theory [6]. In one such project, the Warwick Analysis Project, an entire year group of single honours mathematicians work through sets of questions based on Burn's *Numbers and Functions*. Working in small groups with peer tutors and staff helpers, the students get only one, summary, lecture each week and are asked to generate almost all of the major results expected of a 30 hour first course in analysis for themselves. The results are startling: students have become more fluent in their use of mathematical language, more confident in their own abilities, take more ownership of the mathematics, are more comfortable with the notion of proof ... oh, and more of them pass the exams!

Similarly, work by Ken Houston and others is encouraging university teachers to consider the role of modelling in mathematics [7]: not as a separate subject, but as a philosophy for thinking mathematically across all subjects. In doing so, they are also encouraging university teachers to think about the other skills students might develop on mathematics courses: communication and presentation skills, team working, etc.

Thinking about Thinking

Schoenfeld's last category is 'basic research into cognition'. This is the category that most in mathematics education would most readily associate with the word 'research' (and it is, after all, the main focus that we expect RAE panel to have). The main question researchers ask here is:

> What is it about the way in which the brain works that might cause students to learn (or fail to learn) mathematics?

Current work in neuropsychology (such as that reported by Stanislas Dehaene and Brian Butterworth [8]) is beginning to provide direct insight into this question: brain imaging techniques are just beginning to allow us to see which areas of the brain are activated in different situations (such as doing simple calculations). However, it may be some time before we are in a position to see the mathematician's mind 'doing mathematics' and compare it to a failing student's mind (and it will be longer still before we can use that information to tell us what to do about it).

Most of the 'basic research into cognition' category considers experiments to uncover the way in which students think about mathematical concepts and postulate larger scale cognitive theories which might explain that behaviour. For example, there is a long history in researching students understanding of the notion of a function. Some students seem to view a function as a non-arbitrary rule, or will only confirm drawings as graphs of valid functions if they represent differentiable real valued functions of the reals. One suggestion for how students may come to have this view, and to retain it long after having met a set-theoretic definition involves the theoretical construct 'concept definition / concept image' [9].

It is suggested that students have two cognitive entities: concept definition and concept image. The former entity (which may be empty) contains a form of words which specify the concept exactly (eg "a subset f of the cartesian product of sets A and B so that if (a,b) and (a,c) are elements of f, then $b = c$"). The latter entity, the concept image, is a set of ideas, impressions, experiences and representations which the learner associates with the concept. So, for many students this might include numerous examples of functions (written symbolically and represented graphically), experiences of producing tables, drawing graphs, finding roots, etc. The suggestion is that, when dealing with new experiences, the learners rely on comparison with their concept images to reason. Thus questions like "how many functions are there from a set T of three elements to a set S of three elements?" might cause great confusion, since the learner's concept image might predominantly consist of differentiable real valued functions of the reals.

Thus 'basic research into cognition' might provide the mathematics teacher with ways in which they can predict students difficulties and, perhaps, suggest where they might place the emphasis in their teaching. (In this example, the teacher might spend more time in thinking about the role of definitions and their importance in deductive reasoning).

We might think about the results of this 'basic research' as providing the mathematics teacher with a different pair of spectacles with which to view their classroom. Concept definition / image focuses our attention on the previous experiences students may have had and the misconceptions they may have built from abstracting properties their teachers did not intend (such as abstracting that graphs of continuous functions can be drawn without taking the pen off the paper).

A different focus is given by one of the theoretical perspectives described by Piaget: assimilation and accommodation. Piaget suggested that students may be able to cope with new experiences using their existing ways of thinking (assimilation). However, some new experiences require the learner to fundamentally reconstruct their ways of thinking to be able to reconcile the experience with their previous experience (accommodation). The transition from school mathematics to university mathematics is littered with problems of accommodation for students. Not only do students have to fundamentally change what they mean by the notion of function, they have to change what they mean by mathematics (to the extent that one student interviewed recently suggested that universities were dishonest in calling the subject 'mathematics' when it clearly bears so little resemblance to real [ie school] mathematics). This perspective might suggest to the teacher the importance of support while students are reconstructing, the need for some form of indication about how the reconstruction might take place and might predict the likely outcome for those who struggle to reconstruct.

One needs to approach the use of these theoretical perspectives with some caution, however. It is not enough to concentrate on the surface issues which 'basic research' papers examine. For example, a classic error explored in the literature is the status of 0.999.... Many students consider that this is a number less than one [10]. At a surface level we might be pleased that having focussed on this problem in a course, in an true/false test as part of the final examination, a student says '0.999... < 1' is false 'because 0.999... = 1'. We might then be quite upset when the next line reads, as one recent examination script did: 'Of course, the largest number less than 1 is 0.9999...8'.

Teachers as researchers

One category which Schoenfeld does not address significantly, and which cuts across his categorisation is the teacher as a researcher. One of the difficulties of reading the results of other people's research is that the effort involved to avoid the surface understanding indicated above, of translating it to one's own educational context and of turning it into a practical course is probably more than was involved in developing the research in the first place. So why not do the research in the first place?

In particular, much of the product-oriented and 'social engineering' research mentioned has come from teachers starting to examine their own teaching, trying new ideas, reading some of the 'basic research', applying it to their context and throwing away what doesn't work. Many of the articles which will appear in this newsletter (and appeared in its predecessor) detailed how teachers were using new ideas in their teaching. Every time a teacher makes a decision to use a new textbook for some reason, they are getting involved in the beginnings of educational research. Once you start questioning why you are doing something, you can begin to articulate what you believe the problems are. Then you can investigate whether there are some possible and practicable solutions. It might help to think about the problem from a particular perspective developed from reading some 'basic research'.

Some starting and finishing points.

The places to start thinking about how reflecting on mathematics teaching can help to become better at mathematics teaching (which we might take to be a draft definition of mathematics education research) is this newsletter and the Learning and Teaching Support Network as a whole. Similarly, conferences like the Undergraduate Mathematics Teaching Conference give reflective mathematics teachers the opportunity to share ideas. Some of the papers quoted in this article give a flavour of the different types of research and a good, if ageing, review of much of the work in university mathematics education research is given in *Advanced Mathematical Thinking [11]*.

As you do reflect on your own teaching and begin to articulate your approach and the reasons for your approach, the same forums, LTSN, this newsletter and UMTC might be the ideal outlet for sharing those ideas with others. Who knows, in a short while, you too could write papers with titles like *What is the object of the encapsulation of a process*?

Or maybe not!

References

[1] Tall D O, Thomas M, Davis G E, Gray, E M and Simpson, A P (2000) "What is the object of the encapsulation of a process?", *Journal of Mathematical Behavior*, 18(4)

[2] I'm convinced many readers will already be considering this question along the lines of Simon Bond's *101 Uses of a Dead Cat*.

[3] Schoenfeld A (1991) "On pure and applied research in mathematics education" *Journal of Mathematical Behavior*, 10, 263-276
[4] http://math.la.asu.edu/~kawski/
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[11] Tall D (ed) (1991) *Advanced Mathematical Thinking*, Dordrecht: Kluwer