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FSI Lectures

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Tutorial II: Robust and accurate splitting schemes

Model problem. We consider the model problem of the Tutorial I, that we repeat here for the sake of convenience: find the fluid velocity $\boldsymbol{u} : \Omega \times \mathbb{R}^+ \to \mathbb{R}^2$, the pressure $p : \Omega \times \mathbb{R}^+ \to \mathbb{R}$, the solid vertical displacement $\eta : \Sigma \times \mathbb{R}^+ \to \mathbb{R}$ and the solid vertical velocity $\dot{\eta} : \Sigma \times \mathbb{R}^+ \to \mathbb{R}$ such that

$$\begin{cases} \rho^{\mathrm{f}} \partial_t \boldsymbol{u} - \mathbf{div} \boldsymbol{\sigma}(\boldsymbol{u}, p) = \boldsymbol{0} & \text{in} \quad \Omega^{\mathrm{f}} \times \mathbb{R}^+, \\ \mathrm{div} \boldsymbol{u} = 0 & \text{in} \quad \Omega^{\mathrm{f}} \times \mathbb{R}^+, \\ \boldsymbol{u} \cdot \boldsymbol{n} = 0, \quad \boldsymbol{\sigma}(\boldsymbol{u}, p)\boldsymbol{n} \cdot \boldsymbol{\tau} = 0 & \text{on} \quad \Gamma_1 \times \mathbb{R}^+, \\ \boldsymbol{\sigma}(\boldsymbol{u}, p)\boldsymbol{n} = -P\boldsymbol{n} & \text{on} \quad \Gamma_2 \times \mathbb{R}^+, \\ \boldsymbol{\sigma}(\boldsymbol{u}, p)\boldsymbol{n} = \boldsymbol{0} & \text{on} \quad \Gamma_4 \times \mathbb{R}^+, \end{cases} \tag{1}$$

$$\begin{cases} \boldsymbol{u} \cdot \boldsymbol{n} = \dot{\eta}, \quad \boldsymbol{u} \cdot \boldsymbol{\tau} = 0 & \text{on} \quad \Sigma \times \mathbb{R}^+, \\ \rho^{\mathrm{s}} \epsilon \partial_t \dot{\eta} - c_1 \partial_x^2 \eta + c_0 \eta = -\boldsymbol{\sigma}(\boldsymbol{u}, p)\boldsymbol{n} \cdot \boldsymbol{n} & \text{on} \quad \Sigma \times \mathbb{R}^+, \\ \dot{\eta} = \partial_t \eta & \text{on} \quad \Sigma \times \mathbb{R}^+, \\ \eta = 0 & \text{on} \quad \partial\Sigma \times \mathbb{R}^+, \end{cases} \tag{2}$$

with the given initial conditions $\boldsymbol{u}(0) = \boldsymbol{u}_0$, $\eta(0) = \eta_0$ and $\dot{\eta}(0) = \dot{\eta}_0$.

Numerical methods. In this tutorial, the time–discretization of the interface coupling in (1)-(2) will be performed using a projection based semi-implicit coupling scheme and a Robin-Neumann explicit coupling scheme. These two numerical methods are implemented in the `FreeFem++` script files `fsi-SI.edp` and `fsi-RN.edp`, respectively.

Part I: Projection based semi-implicit coupling

Applied to the coupled problem (1)-(2) this time–splitting method reads as follows: for $n \geq 1$, find $\boldsymbol{u}^n : \Omega \to \mathbb{R}^2$, $p^n : \Omega \to \mathbb{R}$, $\eta^n : \Sigma \to \mathbb{R}$ and $\dot{\eta}^n : \Sigma \to \mathbb{R}$ such that

1. Explicit part (fluid viscous step):

$$\begin{cases} \rho^{\mathrm{f}} \partial_\tau \boldsymbol{u}^n - 2\mu \mathbf{div} \boldsymbol{\varepsilon}(\boldsymbol{u}^n) = -\boldsymbol{\nabla} p^{n-1} & \text{in} \quad \Omega^{\mathrm{f}}, \\ \boldsymbol{u}^n \cdot \boldsymbol{n} = 0, \quad 2\mu \boldsymbol{\varepsilon}(\boldsymbol{u}^n)\boldsymbol{n} \cdot \boldsymbol{\tau} = 0 & \text{on} \quad \Gamma_1, \\ 2\mu \boldsymbol{\varepsilon}(\boldsymbol{u}^n)\boldsymbol{n} = \boldsymbol{0} & \text{on} \quad \Gamma_2 \cup \Gamma_4. \\ \boldsymbol{u}^n \cdot \boldsymbol{n} = \dot{\eta}^{n-1}, \quad \boldsymbol{u}^n \cdot \boldsymbol{\tau} = 0 & \text{on} \quad \Sigma. \end{cases} \tag{3}$$

2. Implicit part:

   a) Fluid projection step:

$$\begin{cases} -\Delta p^n = -\dfrac{\tau}{\rho^{\mathrm{f}}} \mathrm{div} \boldsymbol{u}^n & \text{in} \quad \Omega^{\mathrm{f}}, \\ \dfrac{\partial p^n}{\partial \boldsymbol{n}} = 0 & \text{on} \quad \Gamma_1, \\ p = P(t_n) & \text{on} \quad \Gamma_2, \\ p = 0 & \text{on} \quad \Gamma_4, \\ \dfrac{\partial p^n}{\partial \boldsymbol{n}} = -\dfrac{\rho^{\mathrm{f}}}{\tau} \left(\dot{\eta}^n - \dot{\eta}^{n-1} \right) & \text{on} \quad \Sigma. \end{cases} \tag{4}$$

   b) Solid step:

$$\begin{cases} \rho^{\mathrm{s}} \epsilon \partial_\tau \dot{\eta}^n - c_1 \partial_x^2 \eta^n + c_0 \eta^n = -\boldsymbol{\sigma}(\boldsymbol{u}^n, p^n)\boldsymbol{n} \cdot \boldsymbol{n} & \text{on} \quad \Sigma, \\ \dot{\eta}^n = \partial_\tau \eta^n & \text{on} \quad \Sigma, \\ \eta^n = 0 & \text{on} \quad \partial\Sigma. \end{cases} \tag{5}$$

Here, the symbol $\partial_\tau x^n \stackrel{\text{def}}{=} (x^n - x^{n-1})/\tau$ stands for the first-order backward difference. This coupling scheme requires the solution, at each time-step, of the heterogenous system of equations (4)-(5) which couples the dynamics of p^n and $(\dot{\eta}^n, \eta^n)$. Note, however, that the computation of fluid velocity $\boldsymbol{u}^n$ through step (3) is uncoupled with (4)-(5). This explains the terminology *semi-implicit*. The `FreeFem++` script file `fsi-SI.edp` implements three partitioned solution algorithms for the coupled system (4)-(5):

- Fixed-point iterations with static relaxation (options `method=1` and `dymrel=0`);
- Fixed-point iterations with Aitken's dynamic relaxation (options `method=1` and `dymrel=1`);
- Robin-Neumann (RN) iterations with a fixed Robin coefficient (option `method=2`).

Exercise 1: Robustness. The purpose of this exercise is to illustrate numerically that, in spite of not being fully implicit, the stability of the above semi–implicit scheme is robust with respect to the added-mass effect. The reason for this is that all these stability issues are circumvent via the coupling between (4) and (5). In this exercise we will choose the option `method=2`.

1. Run the command `FreeFem++ fsi-SI.edp`. Do you observe a stable or unstable approximation?. Did you get the same kind of behavior with the script `fsi-EXP.edp` in Tutorial I?. Explain the answer.
2. Investigate the impact of ρ^{f} , ρ^{s} ,ϵ, L and R on the stability of the approximations. What do you observe?. Explain the results.

Exercise 2: Accuracy. The purpose of this exercise is to illustrate numerically that, in spite of not being fully implicit, the above semi–implicit scheme delivers a comparable accuracy. In this exercise we will choose the option `method=2`. The scripts `FreeFem++ fsi-SI.edp` and `FreeFem++ fsi-IMP.edp` generate, respectively, the `ascii` files `plot_si.gp` and `plot_imp.gp` which contain the displacement approximation at the final time. Using `gunplot` (or `Matlab`) you can compare both approximations. We will also compare the accuracy of these approximation with the reference solution given in the `ascii` file `plot_ref.gp`. For the following set of space–time refinement

$$\tau = 2 \cdot 10^{-4}/2^i, \quad h = 0.1/2^i, \quad i = 0, 1, 2, 3,$$

compare the approximations obtained with `fsi-SI.edp` and `fsi-IMP.edp`. What do you observe?

Exercise 3: Efficiency. Compare the efficiency of the partitioned solution algorithms for the coupled system (4)-(5):

1. Fixed-point iterations with static relaxation:
   a) How many iterations are approximately needed at each time-step?.
   b) Tune the relaxation parameter `omega0` in order to improve the convergence speed of the iterations.
   c) For a given value of `omega0` which guarantees convergence, investigate the impact of ρ^{f} , ρ^{s} ,ϵ, L and R on the convergence speed. Explain the results.
2. Repeat points 1(a) and 1(c) with the dynamic relaxation variant. Does this improves the situation?.
3. Repeat points 1(a) and 1(c) with the RN iterations. Which are the benefits of this approach?.
4. The Robin coefficient is `alphaf=rhof/(rhos*eps)`, why?.

Part II: Robin-Neumann explicit coupling

Applied to the coupled problem (1)-(2) this explicit splitting scheme reads as follows: for $n \geq 1$, find $\boldsymbol{u}^n : \Omega \to \mathbb{R}^2$, $p^n : \Omega \to \mathbb{R}$, $\eta^n : \Sigma \to \mathbb{R}$ and $\dot{\eta}^n : \Sigma \to \mathbb{R}$ such that

1. Fluid step (interface Robin condition):

$$\left\{\begin{aligned}
\rho^{\mathrm{f}} \partial_\tau \boldsymbol{u}^n - \mathbf{div} \boldsymbol{\sigma}(\boldsymbol{u}^n, p^n) = \boldsymbol{0} \quad & \text{in} \quad \Omega^{\mathrm{f}},\\
\mathrm{div} \boldsymbol{u}^n = 0 \quad & \text{in} \quad \Omega^{\mathrm{f}},\\
\boldsymbol{u}^n \cdot \boldsymbol{n} = 0, \quad \boldsymbol{\sigma}(\boldsymbol{u}, p)\boldsymbol{n} \cdot \boldsymbol{\tau} = 0 \quad & \text{on} \quad \Gamma_1,\\
\boldsymbol{\sigma}(\boldsymbol{u}^n, p^n)\boldsymbol{n} = -P(t_n)\boldsymbol{n} \quad & \text{on} \quad \Gamma_2,\\
\boldsymbol{\sigma}(\boldsymbol{u}^n, p^n)\boldsymbol{n} = \boldsymbol{0} \quad & \text{on} \quad \Gamma_4,\\
\boldsymbol{\sigma}(\boldsymbol{u}^n, p^n)\boldsymbol{n} \cdot \boldsymbol{n} + \frac{\rho^{\mathrm{s}}\epsilon}{\tau} \boldsymbol{u}^n \cdot \boldsymbol{n} = \frac{\rho^{\mathrm{s}}\epsilon}{\tau} \left(\dot{\eta}^{n-1} + \tau \partial_\tau \dot{\eta}^{n,\star}\right) + \boldsymbol{\sigma}(\boldsymbol{u}^{n,\star}, p^{n,\star})\boldsymbol{n} \cdot \boldsymbol{n} \quad & \text{on} \quad \Sigma,\\
\boldsymbol{u}^n \cdot \boldsymbol{\tau} = 0 \quad & \text{on} \quad \Sigma.
\end{aligned}\right. \tag{6}$$

2. Solid step (standard "Neumann"):

$$\begin{cases} \rho^{\mathrm{s}}\epsilon\partial_\tau\dot{\eta}^n - c_1\partial_x^2\eta^n + c_0\eta^n = -\boldsymbol{\sigma}(\boldsymbol{u}^n, p^n)\boldsymbol{n}\cdot\boldsymbol{n} & \text{on} \quad \Sigma,\\ \dot{\eta}^n = \partial_\tau\eta^n & \text{on} \quad \Sigma,\\ \eta^n = 0 & \text{on} \quad \partial\Sigma. \end{cases}$$

Here, we used the notation

$$x^{n,\star} \stackrel{\text{def}}{=} \begin{cases} 0 & \text{if} \quad r = 0,\\ x^{n-1} & \text{if} \quad r = 1,\\ 2x^{n-1} - x^{n-2} & \text{if} \quad r = 2, \end{cases}$$

for the interface extrapolations of order r.

Exercise 1: Robustness. The purpose of this exercise is to illustrate numerically that the stability of the above explicit scheme is robust with respect to the added-mass effect. The reason for this is that the Robin condition $(6)_6$ guarantees that the inertia of the solid is implicit coupled with the fluid.

1. Run the command `FreeFem++ fsi-RN.edp` with different values of the extrapolation order r. Do you observe a stable or unstable approximation?. Did you get the same kind of behavior with the script `fsi-EXP.edp` in Tutorial I?.
2. Investigate the impact of ρ^{f} , ρ^{s} ,ϵ, L and R on the stability of the approximations. What do you observe?.
3. For the variants with $r = 0$ and $r = 1$, take different values of τ and h. Does this compromise the stability of the method?. Do you observe the same for the variant with $r = 2$?.

Exercise 2: Accuracy. The purpose of this exercise is to investigate numerically the accuracy of the above Robin-Neumann explicit scheme. The scripts `FreeFem++ fsi-RN.edp` generates the `ascii` files `plot_exp0.gp`, `plot_exp1.gp` and `plot_exp2.gp` containing the displacement approximation at the final time, respectively, for each extrapolation variant $r = 0, 1, 2$. For the following set of space–time refinement

$$\tau = 2\cdot 10^{-4}/2^i, \quad h = 0.1/2^i, \quad i = 0, 1, 2, 3,$$

display the approximations obtained with `fsi-RN.edp`, for all the values of r, and compare them with the approximation provided by `fsi-IMP.edp` and the reference solution `plot_ref.gp`. What do you observe?. What can be concluded on the Robin-Neumann scheme with $r = 1$?.