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FSI Lectures

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Tutorial I: Stiff Dirichlet-Neumann Coupling

Model problem. The purpose of this tutorial is to illustrate the stiff nature of the Dirichlet-Neumann coupling in incompressible fluid-structure interaction. As example, we consider the propagation of a pressure-wave within an elastic straight tube in two-dimensions (see Figure 1). Assuming that the displacements of the interface are infinitesimal and that the Reynolds number in the fluid is small, the

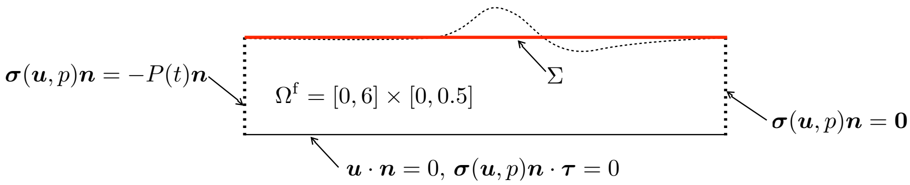

Figure 1: Geometrical configuration and external boundary conditions.

mechanical system can be modeled by a simplified linear model problem in which the fluid is described by the Stokes equations, in the fixed domain $\Omega^{\rm f}$, and the structure by a generalized string model in the fluid-structure interface Σ. All the geometrical and constitutive non-linearities are hence neglected. The resulting coupled problem reads as follows: find the fluid velocity $\boldsymbol{u} : \Omega\times\mathbb{R}^+ \rightarrow \mathbb{R}^2$, the pressure $p : \Omega\times\mathbb{R}^+ \rightarrow \mathbb{R}$ and the solid vertical displacement $\eta : \Sigma\times\mathbb{R}^+ \rightarrow \mathbb{R}$ and the solid vertical velocity $\dot\eta : \Sigma\times\mathbb{R}^+ \rightarrow \mathbb{R}$ such that

$$\left\{\begin{aligned} \rho^{\rm f}\partial_t\boldsymbol{u} - \mathbf{div}\boldsymbol{\sigma}(\boldsymbol{u},p) &= \boldsymbol{0} \quad \text{in} \quad \Omega^{\rm f}\times\mathbb{R}^+,\\ \operatorname{div}\boldsymbol{u} &= 0 \quad \text{in} \quad \Omega^{\rm f}\times\mathbb{R}^+,\\ \boldsymbol{u}\cdot\boldsymbol{n} = 0, \quad \boldsymbol{\sigma}(\boldsymbol{u},p)\boldsymbol{n}\cdot\boldsymbol{\tau} &= 0 \quad \text{on} \quad \Gamma_1\times\mathbb{R}^+,\\ \boldsymbol{\sigma}(\boldsymbol{u},p)\boldsymbol{n} &= -P\boldsymbol{n} \quad \text{on} \quad \Gamma_2\times\mathbb{R}^+,\\ \boldsymbol{\sigma}(\boldsymbol{u},p)\boldsymbol{n} &= \boldsymbol{0} \quad \text{on} \quad \Gamma_4\times\mathbb{R}^+, \end{aligned}\right. \tag{1}$$

$$\left\{\begin{aligned} \boldsymbol{u}\cdot\boldsymbol{n} = \dot\eta, \quad \boldsymbol{u}\cdot\boldsymbol{\tau} &= 0 \quad \text{on} \quad \Sigma\times\mathbb{R}^+,\\ \rho^{\rm s}\epsilon\partial_t\dot\eta - c_1\partial_x^2\eta + c_0\eta &= -\boldsymbol{\sigma}(\boldsymbol{u},p)\boldsymbol{n}\cdot\boldsymbol{n} \quad \text{on} \quad \Sigma\times\mathbb{R}^+,\\ \dot\eta &= \partial_t\eta, \quad \text{on} \quad \Sigma\times\mathbb{R}^+,\\ \eta &= 0 \quad \text{on} \quad \partial\Sigma\times\mathbb{R}^+, \end{aligned}\right. \tag{2}$$

and complemented with the given initial conditions $\boldsymbol{u}(0) = \boldsymbol{u}_0$, $\eta(0) = \eta_0$ and $\dot\eta(0) = \dot\eta_0$. The constants $\rho^{\rm f}$ and $\rho^{\rm s}$ denote, respectively, the fluid and solid densities and ϵ is the solid thickness. The fluid Cauchy-stress tensor is given by the relation $\boldsymbol{\sigma}(\boldsymbol{u},p) \stackrel{\rm def}{=} -p\boldsymbol{I} + 2\mu\boldsymbol{\varepsilon}(\boldsymbol{u})$, with $\boldsymbol{\varepsilon}(\boldsymbol{u}) \stackrel{\rm def}{=} \frac{1}{2}\left(\boldsymbol{\nabla}\boldsymbol{u} + \boldsymbol{\nabla}\boldsymbol{u}^{\rm T}\right)$ and where μ stands for the fluid dynamic viscosity. All units will be expressed in the CGS (centimetre-gram-second) system. The physical parameter for the fluid are $\rho^{\rm f} = 1.0$ and $\mu = 0.035$, and for the solid we take $\rho^{\rm s} = 1.1$, $c_1 \stackrel{\rm def}{=} \frac{E\epsilon}{2(1+\nu)}$, $c_0 \stackrel{\rm def}{=} \frac{E\epsilon}{R^2(1-\nu^2)}$, with $\epsilon = 0.1$, the Young modulus $E = 0.75\cdot 10^6$ and the Poisson ratio $\nu = 0.5$. The fluid domain is given by $\Omega^{\rm f} = [0,L]\times[0,R]$ and the fluid-solid interface by $\Sigma = [0,L]\times\{R\}$, with $L = 6$ and $R = 0.5$. A sinusoidal pressure-wave $P(t) = P_{\max}(1-\cos(2t\pi/T^*))/2$, with maximum $P_{\max} = 2\cdot 10^4$, is prescribed on the inlet boundary Γ_4 during $T^* = 5\cdot 10^{-3}$ seconds. Zero pressure is imposed on Γ_2 and a slip condition is enforced on the lower boundary Γ_1. For the solid we set $\eta = 0$ on the extremities $x = 0, L$.

Numerical methods. The spatial discretizations of the fluid and of the structure will be based on piece-wise affine continuous finite elements and compatible on the interface (matching meshes). The following Brezzi-Pitkäranta pressure stabilization operator $\gamma_{\rm p}\int_{\Omega^{\rm f}}\frac{h^2}{\mu}\boldsymbol{\nabla}p\cdot\boldsymbol{\nabla}q$, $\quad\gamma_{\rm p} = 10^{-3}$, is added to the Stokes bilinear form in order to avoid the *inf-sup* compatibility issues. Here, h stands for the spatial mesh parameter. With regard the time–discretisation, we will for simplicity consider a simple backward Euler time-stepping in the bulk terms of the fluid $(1)_{1,2}$ and of the solid $(2)_2$. The time–discretization of the interface coupling $(2)_{1,2}$ will be performed using either a implicit coupling scheme or a Dirichet-Neumann explicit coupling scheme. These two numerical methods are implemented in the `FreeFem++` script files `fsi-SI.edp` and `fsi-DN.edp`, respectively.

Exercise 1: Energy balance. We assume that $P = 0$ in (1)–(2). Show that the following energy identity holds for $t > 0$

$$\frac{\rho^{\rm f}}{2}\|\boldsymbol{u}\|^2_{0,\Omega^{\rm f}} + \frac{\rho^{\rm s}\epsilon}{2}\|\dot\eta\|^2_{0,\Sigma} + \frac{c_1}{2}\|\partial_x\eta\|^2_{0,\Sigma} + \frac{c_0}{2}\|\eta\|^2_{0,\Sigma} = \frac{\rho^{\rm f}}{2}\|\boldsymbol{u}_0\|^2_{0,\Omega^{\rm f}} + \frac{\rho^{\rm s}\epsilon}{2}\|\dot\eta_0\|^2_{0,\Sigma} + \frac{c_1}{2}\|\partial_x\eta_0\|^2_{0,\Sigma} + \frac{c_0}{2}\|\eta_0\|^2_{0,\Sigma} - 2\mu\int_0^t \|\boldsymbol{\varepsilon}(\boldsymbol{u})\|^2_{0,\Sigma}. \quad (3)$$

What does this identity guarantee?.

Exercise 2: Implicit coupling scheme. We consider a fully implicit Euler time-discretization of (1)-(2). Show that, assuming again $P = 0$, the resulting approximations $(\boldsymbol{u}^n, p^n, \dot\eta^n, \eta^n)$ satisfy a time–semidiscrete counterpart of (3). What does this guarantee?.

Exercise 3: Partitioned iterative solution of implicit coupling. The previous implicit coupling scheme requires the solution, at each time-step, of a heterogenous system of equations which couples the dynamics of $(\boldsymbol{u}^n, p^n)$ and $(\dot\eta^n, \eta^n)$. The `FreeFem++` script file `fsi-IMP.edp` implements three partitioned solution algorithms for this coupled system:

- Dirichlet-Neumann (DN) iterations with static relaxation (options `method=1` and `dymrel=0`);
- DN iterations with Aitken's dynamic relaxation (options `method=1` and `dymrel=1`);
- Robin-Neumann (RN) iterations with a fixed Robin coefficient (option `method=2`).

Te purpose of this exercise is to test and compare the efficiency of these approach and, in particular, their sensitivity to the amount of added-mass effect in the system, characterized by the relation

$$\frac{\rho^{\rm f}\mu_{\max}}{\rho^{\rm s}\epsilon} > 1, \quad (4)$$

with $\mu_{\max} \approx L^2/(\pi^2 R)$.

1. DN iterations with static relaxation:
   a) How many iterations are approximately needed at each time-step?.
   b) Tune the relaxation parameter `omega0` in order to improve the convergence speed of the iterations.
   c) For a given value of `omega0` which guarantees convergence, investigate the impact of $\rho^{\rm f}$, $\rho^{\rm s}$,ϵ, L and R on the convergence speed. Explain the results.
2. Repeat points 1(a) and 1(c) with the dynamic relaxation variant. Does this improves the situation?.
3. Repeat points 1(a) and 1(c) with the RN iterations. Which are the benefits of this approach?.
4. How are the interface Dirichlet conditions enforced in `fsi-IMP.edp`?.
5. How are the interface fluid stresses evaluated in `fsi-IMP.edp`?.

Exercise 4: Explicit coupling scheme. We consider now an explicit Dirichlet–Neumann coupling scheme for the time–discretization of (1)–(2). The interface coupling $(2)_{1,2}$ is hence discretized in time as follows

$$\begin{aligned} \boldsymbol{u}^n\cdot\boldsymbol{n} = \dot\eta^{n-1}, \quad \boldsymbol{u}^n\cdot\tau = 0 \quad &\text{on} \quad \Sigma,\\ \rho^{\rm s}\epsilon\partial_\tau\dot\eta^n - c_1\partial_x^2\eta^n + c_0\eta^n = -\boldsymbol{\sigma}(\boldsymbol{u}^n,p^n)\boldsymbol{n}\cdot\boldsymbol{n} \quad &\text{on} \quad \Sigma. \end{aligned}$$

A salient feature of this time-stepping scheme is that it splits the computation of $(\boldsymbol{u}^n, p^n)$ and $(\dot\eta^n, \eta^n)$. Assuming that $P = 0$, try to derive an energy estimate for the resulting approximation $(\boldsymbol{u}^n, p^n, \dot\eta^n, \eta^n)$ as in Exercise 2.

Exercise 5: Explicit coupling scheme. The `FreeFem++` script file `fsi-EXP.edp` implements the explicit Dirichlet–Neumann coupling scheme. Te purpose of this exercise is to illustrate numerically that the stability of this splitting scheme is dictated by the amount of added-mass effect in the system (i.e., relation (4)) and not by the discretization parameters.

1. Run the script `fsi-EXP.edp` with `FreeFem++`. What do you observe?. Are the results obtained similar to those provided by `fsi-IMP.edp` for the same set of physical and discretization parameters?.
2. Try reducing the time-step length `tau`, does this cure the problem?.
3. Try reducing the fluid density `rhof` or increasing the solid density `rhos`. Explain the results.