

# GeoPDEs tutorial

Rafael Vázquez \*

## 1. Understanding $h$ -, $p$ - and $k$ -refinement.

- (a) Use the NURBS toolbox to define a NURBS parameterization of one quarter of a ring, with internal radius  $R_i = 1$  and external radius  $R_o = 2$ . Use degree elevation to set the degree to 2 in both directions. Plot the geometry and the control points with `nrbctrlplot`. (Hint: you can use either `nrbline` and `nrbrevolve`, or `nrbcirc` and `nrb ruled`. For degree elevation use `nrbdegelev`).
- (b) Check the knot vector of your geometry. Insert new knots to get a  $10 \times 10$  mesh. Plot the mesh with `nrbkntplot`. (Hint: use `kntrefine` to compute the new knots, and `nrbkntins` for knot insertion).
- (c) Perform  $h$ -refinement: starting from the geometry of point (a), modify the example `ex_article_15lines` to solve in a sequence of meshes with 1, 2, 4, 8, 16, 32 and 64 elements in each parametric direction. Check the convergence rate plotting the error vs. the degrees of freedom. Do the same with  $C^0$  continuity, inserting the knots twice, and compare the two curves of convergence.
- (d) Perform  $p$ -refinement: starting from the geometry of point (b), modify the example `ex_article_15lines` to solve with degree from 2 to 8. In this case, the continuity will be  $C^1$  in all the cases. Plot the error vs. the degrees of freedom. (Hint: use `nrbdegelev` for degree elevation).
- (e) Perform  $k$ -refinement: starting from the geometry of point (a), first raise the degree (from 2 to 8), and then insert new knots to get a  $10 \times 10$  mesh. Solve the same problem as before. In this case, for degree  $p$ , the continuity will be  $C^{p-1}$ . Compare the curve of the error with the one in (d).
- (f) **Optional:** Using a coarse mesh, you can compare the knot vectors, the basis functions and the `msh` and `space` structures for  $C^0$  and  $C^{p-1}$  continuity. (Hint: use `msh_precompute` and `sp_precompute`).

**Note:** In most of the examples of GeoPDEs the refinement is done automatically, giving the degree, the continuity, and the number of subdivisions as data. You can see how the data is given in `ex_laplace_iso_ring`, for instance, and how the refinement is performed in `solve_laplace_2d_iso`.

---

\*You can send your questions to the GeoPDEs mailing list: [geopdes-users@lists.sourceforge.net](mailto:geopdes-users@lists.sourceforge.net)

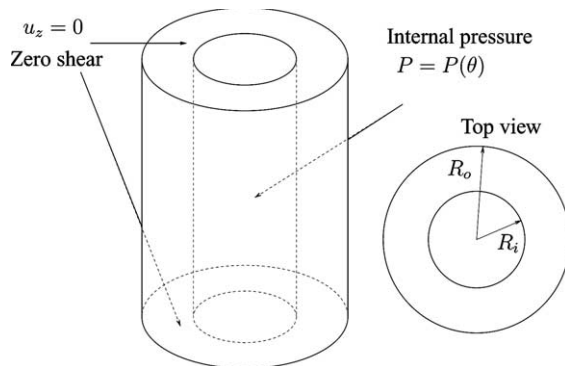


Figure 1: Thick cylinder subjected to internal pressure.

## 2. Solid circular cylinder subjected to internal pressure.

This is an example from the first paper on IGA [2]. The geometry is a hollow cylinder, subjected to a constant pressure on its internal side. The setting of the problem is given in Fig. 1. The same example is already solved with a plane strain formulation in the files `ex_plane_strain_ring` and `solve_plane_strain_2d`. We are going to solve the example in 3D in one eighth of the cylinder, using symmetry conditions.

- Extrude the geometry from exercise 1 (a), with the function `nrboxtrude`. Set the length of the cylinder equal to 5. Save the geometry in a `.txt` file, with the `nrboxexport` command.
- With the help of `ex_plane_strain_ring` and `ex_lin_elast_horseshoe`, write an example to solve the linear elasticity problem in the cylinder. Set the degree to 3 and  $C^2$  continuity in the three directions, and refine to have a  $5 \times 5 \times 5$  mesh. Apply a constant pressure on the inner boundary, symmetry conditions on the three symmetry boundaries, and homogeneous Neumann conditions on the other boundaries. (Hint: check the numbering of the boundaries, as they may be different from the 2D case).
- Using ParaView, compare the horizontal displacements and stress for the 2D and the 3D problems. For the 3D problem, check also that the displacement in the  $z$ -direction is null.
- Solve the same problem removing the symmetry condition on the bottom boundary, and setting a zero  $z$ -displacement condition on the top and bottom boundaries. The code as it is only allows to impose Dirichlet conditions for all the components at the same time, but you can impose the condition by modifying the function `solve_linear_elasticity`. Use the fields in `space.boundary` to impose the condition on the right dofs.
- Optional:** Write a new function, similar to `sp_l2_error` and `sp_h1_error`, to compute the error in  $L^2$ -norm for the stress. The exact solution for  $\nu = 0$  in cylindrical coordinates is

$$u_r = \frac{PR_i^2}{E(R_o^2 - R_i^2)} \left( r + \frac{R_o^2}{r} \right), \quad (1)$$

$$\sigma_{rr} = \frac{PR_i^2}{R_o^2 - R_i^2} - \frac{PR_i^2 R_o^2}{r^2(R_o^2 - R_i^2)}, \quad \sigma_{\theta\theta} = \frac{PR_i^2}{R_o^2 - R_i^2} + \frac{PR_i^2 R_o^2}{r^2(R_o^2 - R_i^2)}. \quad (2)$$

### 3. Vibrations of a clamped thin circular plate.

This is an example taken from [1]. We study the vibrations of a clamped, thin circular plate, modeled as a three-dimensional solid. The difference with the example in the article is that we will use the parameterization of the circle as the one in Fig. 2.

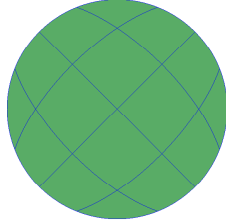


Figure 2: Top view of the parameterization of the thin circular plate.

The problem, in variational formulation, consists of finding the eigenpairs  $(\omega_n, \mathbf{u}_n) \in \mathbb{R} \times (H_0^1(\Omega))^3$ , such that

$$\int_{\Omega} (2\mu \varepsilon(\mathbf{u}_n) : \varepsilon(\mathbf{v}) + \lambda \operatorname{div} \mathbf{u}_n \operatorname{div} \mathbf{v}) = \omega_n^2 \int_{\Omega} \rho \mathbf{u}_n \cdot \mathbf{v}, \quad \forall \mathbf{v} \in (H_0^1(\Omega))^3, \quad (3)$$

where  $\varepsilon(\mathbf{u})$  is the symmetric part of  $\nabla \mathbf{u}$ ,  $\rho$  is the mass density, and  $\lambda, \mu$  are the Lamé parameters. The problem can also be written in the form

$$\mathbf{K} \mathbf{u}_n = \omega_n^2 \mathbf{M} \mathbf{u}_n, \quad (4)$$

where  $\mathbf{K}$  and  $\mathbf{M}$  are the stiffness and mass matrices, respectively.

- (a) Define the 2D geometry with the control points and weights given in Table 1, and use a linear transformation to make it of radius 2 m. Extrude the geometry to create the 3D plate, with a thickness of 0.02 m. (Hint: use `nrbmak` and `nrbtform` to define the geometry. Notice that the NURBS toolbox works with weighted control points, defined as  $B_{i,j}^w = B_{i,j} w_{i,j}$ .)
- (b) Create a function, similar to `solve_linear_elasticity_3d`, to solve problem (3), computing the first six eigenvalues and their corresponding eigenvectors. The function should also return the **geometry**, **msh**, **space** structures (objects). (Hint: use the command `eigs` to solve the generalized eigenvalue problem for sparse matrices.)
- (c) Solve the problem in a  $6 \times 6 \times 1$  mesh, with degree 2. Set  $E = 30 \cdot 10^6 \text{ KN}/m^2$ ,  $\nu = 0.2$  and  $\rho = 2.320 \text{ KN}s}/m^4$ . Compare the eigenvalues with the results in [1]. (Hint: plot the eigenvectors to be sure that you are comparing the right eigenvalues).
- (d) Raise the degree (up to 5) in the first two directions, and compare your results with the ones in [1]. Do the same using a coarser mesh.
- (e) **Optional:** solve the same problem with the parameterization of [1], which you can easily construct by revolution. Notice that in this case there are repeated control points, and it is necessary to identify the dofs associated to these control points as one single dof.

i	$B_{i,1}^w$	$B_{i,2}^w$	$B_{i,3}^w$	$w_{i,1}$	$w_{i,2}$	$w_{i,3}$
1	(1, 0)	$(1/\sqrt{2}, 1/\sqrt{2})$	(0, 1)	1	$1/\sqrt{2}$	1
2	$(1/\sqrt{2}, -1/\sqrt{2})$	(0, 0)	$(-1/\sqrt{2}, 1/\sqrt{2})$	$1/\sqrt{2}$	$\sqrt{2} - 1$	$1/\sqrt{2}$
3	(0, -1)	$(-1/\sqrt{2}, -1/\sqrt{2})$	(-1, 0)	1	$1/\sqrt{2}$	1

Table 1: Control points and weights for the circle of radius 1.

## References

- [1] J. A. Cottrell, A. Reali, Y. Bazilevs, and T. J. R. Hughes. Isogeometric analysis of structural vibrations. *Comput. Methods Appl. Mech. Engrg.*, 195(41-43):5257–5296, 2006.
- [2] T. J. R. Hughes, J. A. Cottrell, and Y. Bazilevs. Isogeometric analysis: CAD, finite elements, NURBS, exact geometry and mesh refinement. *Comput. Methods Appl. Mech. Engrg.*, 194(39-41):4135–4195, 2005.