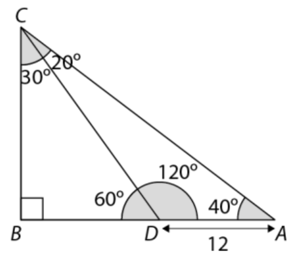
TESTE N.º 1 – Proposta de resolução

Caderno 1



1. α é o ângulo agudo tal que $\cos\alpha = \frac{1}{2}$.

Logo, $\alpha = 60°$.

Assim:

- $A\widehat{D}C = 180° - 60° = 120°$
- $D\hat{C}A = 180° - 120° - 40° = 20°$
- $B\hat{C}D = 180° - 90° - 60° = 30°$

1.º processo

Pela lei dos senos, tem-se que:

$$\frac{\text{sen}20°}{12} = \frac{\text{sen}40°}{\overline{CD}} \Leftrightarrow \overline{CD} = \frac{12\times\text{sen}40°}{\text{sen}20°}$$

Logo, $\overline{CD} \approx 22{,}553$.

Também se tem que:

$$\frac{\text{sen}90°}{22{,}553} = \frac{\text{sen}60°}{\overline{BC}} \Leftrightarrow \overline{BC} = \frac{22{,}553\times\text{sen}60°}{\text{sen}90°}$$

Logo, $\overline{BC} \approx 19{,}531$.

Assim, $\overline{BC} \approx 19{,}5$.

2.º processo

Seja $x = \overline{BC}$ e $y = \overline{BD}$.

Sabemos que:

$$\begin{cases} \text{tg}60° = \frac{x}{y} \\ \text{tg}40° = \frac{x}{12+y} \end{cases} \Leftrightarrow \begin{cases} \sqrt{3}y = x \\ \text{tg}40° = \frac{\sqrt{3}y}{12+y} \end{cases} \Leftrightarrow \begin{cases} \overline{} \\ 12\text{tg}40° + y\text{tg}40° = \sqrt{3}y \end{cases}$$

$$\Leftrightarrow \begin{cases} \overline{} \\ \sqrt{3}y - y\text{tg}40° = 12\text{tg}40° \end{cases}$$

$$\Leftrightarrow \begin{cases} \overline{} \\ y(\sqrt{3} - \text{tg}40°) = 12\text{tg}40° \end{cases}$$

$$\Leftrightarrow \begin{cases} \overline{} \\ y = \frac{12\text{tg}40°}{\sqrt{3}-\text{tg}40°} \end{cases}$$

Assim, $y \approx 11{,}276$.

Logo, $x = \sqrt{3}\,y \approx 19{,}531$.

Então, $\overline{BC} \approx 19{,}5$.

2.

2.1. $f(\alpha) = 2\left(\overline{OB} + \overline{OD} + \overline{BD}\right) = 2\left(1 + \text{sen}\alpha + (-\cos\alpha)\right) = 2(1 + \text{sen}\alpha - \cos\alpha)$

Observe-se que, como $\alpha \in \left]\frac{\pi}{2}, \pi\right[$, então $\cos\alpha < 0$. Logo, $\overline{BD} = -\cos\alpha$.

2.2. Sabe-se que, para um determinado valor de α, $f(\alpha + 1) = f(\alpha) - 0{,}2 \times f(\alpha)$, isto é, $f(\alpha + 1) = 0{,}8 \times f(\alpha)$.

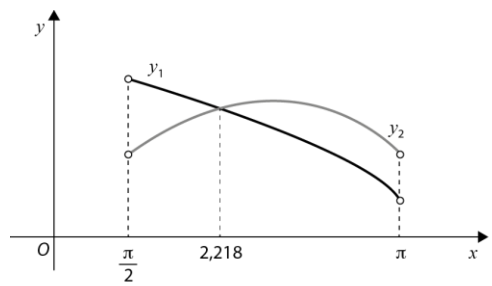
Pretende−se, então, resolver a equação:

$$2(1 + \text{sen}(\alpha + 1) - \cos(\alpha + 1)) = 0{,}8 \times 2(1 + \text{sen}\alpha - \cos\alpha)$$

Recorrendo à calculadora gráfica:

$y_1 = 2(1 + \text{sen}(x + 1) - \cos(x + 1))$

$y_2 = 0{,}8 \times 2(1 + \text{sen}x - \cos x)$



Assim, $\alpha \approx 2{,}22$ rad.

2.3. $\text{tg}(\pi - \beta) = \frac{1}{3} \Leftrightarrow -\text{tg}\beta = \frac{1}{3} \Leftrightarrow \text{tg}\beta = -\frac{1}{3}$

- Como $1 + \text{tg}^2\beta = \frac{1}{\cos^2\beta}$, tem-se que:

$$1 + \frac{1}{9} = \frac{1}{\cos^2\beta} \Leftrightarrow \frac{10}{9} = \frac{1}{\cos^2\beta} \Leftrightarrow \cos^2\beta = \frac{9}{10}$$

$$\Leftrightarrow \cos\beta = \pm\sqrt{\frac{9}{10}}$$

$$\Leftrightarrow \cos\beta = \pm\frac{3}{\sqrt{10}}$$

Como $\frac{\pi}{2} < \beta < \pi$, vem que $\cos\beta < 0$. Logo, $\cos\beta = -\frac{3\sqrt{10}}{10}$.

- Como $\text{tg}\beta = \frac{\text{sen}\beta}{\cos\beta}$, tem-se que:

$$-\frac{1}{3} = \frac{\text{sen}\beta}{-\frac{3\sqrt{10}}{10}} \Leftrightarrow \text{sen}\beta = \frac{1}{3} \times \frac{3\sqrt{10}}{10} \Leftrightarrow \text{sen}\beta = \frac{\sqrt{10}}{10}$$

Assim:

$$f(\beta) = 2(1 + \text{sen}\beta - \cos\beta) = 2\left(1 + \frac{\sqrt{10}}{10} - \left(-\frac{3\sqrt{10}}{10}\right)\right) =$$

$$= 2\left(1 + \frac{\sqrt{10}}{10} + \frac{3\sqrt{10}}{10}\right) =$$

$$= 2\left(1 + \frac{4\sqrt{10}}{10}\right) =$$

$$= 2 + \frac{4}{5}\sqrt{10}$$

2.4. Opção (B)

$$g(x)=F\left(\frac{\pi}{2}+x\right)-F(\pi+x)=$$

$$=2\left(1+\operatorname{sen}\left(\frac{\pi}{2}+x\right)-\cos\left(\frac{\pi}{2}+x\right)\right)-2\big(1+\operatorname{sen}(\pi+x)-\cos(\pi+x)\big)=$$

$$=2(1+\cos x-(-\operatorname{sen}x))-2(1-\operatorname{sen}x-(-\cos x))=$$

$$=2+2\cos x+2\operatorname{sen}x-2+2\operatorname{sen}x-2\cos x=$$

$$=4\operatorname{sen}x$$

3. Opção (C)

$$D=\left\{x\in\mathbb{R}:2x\neq\frac{\pi}{2}+k\pi,k\in\mathbb{Z}\right\}=$$

$$=\left\{x\in\mathbb{R}:x\neq\frac{\pi}{4}+\frac{k\pi}{2},k\in\mathbb{Z}\right\}=$$

$$=\mathbb{R}\backslash\left\{x:x=\frac{\pi}{4}+\frac{k\pi}{2},k\in\mathbb{Z}\right\}$$

<u>Caderno 2</u>

4. Opção (B)

$$\operatorname{sen}^2(15°)+\operatorname{sen}^2(75°)-\operatorname{tg}(10°)\times\operatorname{tg}(80°)=\operatorname{sen}^2(15°)+\cos^2(90°-75°)-\frac{\operatorname{sen}(10°)}{\cos(10°)}\times\frac{\operatorname{sen}(80°)}{\cos(80°)}=$$

$$=\operatorname{sen}^2(15°)+\cos^2(15°)-\frac{\operatorname{sen}(10°)}{\cos(10°)}\times\frac{\cos(90°-80°)}{\operatorname{sen}(90°-80°)}=$$

$$=1-\frac{\operatorname{sen}(10°)\times\cos(10°)}{\cos(10°)\times\operatorname{sen}(10°)}=$$

$$=1-1=$$

$$=0$$

5. $$\frac{(\operatorname{sen}x-\cos x)^2}{\cos^2x+\operatorname{sen}^2x}\times(1+\operatorname{tg}^2x)=\frac{\operatorname{sen}^2x-2\operatorname{sen}x\cos x+\cos^2x}{1}\times\frac{1}{\cos^2x}=\frac{\operatorname{sen}^2x+\cos^2x-2\operatorname{sen}x\cos x}{\cos^2x}=$$

$$=\frac{1-2\operatorname{sen}x\cos x}{\cos^2x}=$$

$$=\frac{1}{\cos^2x}-\frac{2\operatorname{sen}x\cos x}{\cos^2x}=$$

$$=\frac{1}{\cos^2x}-2\frac{\operatorname{sen}x}{\cos x}=$$

$$=1+\operatorname{tg}^2x-2\operatorname{tg}x=$$

$$=(1-\operatorname{tg}x)^2$$

6. Opção (D)

Como $x_2 \in]-\pi, 0[$, vem que $\text{sen}x_2 < 0$ e, como $\frac{\cos x_1}{\text{sen}x_2} < 0$, então $\cos x_1 > 0$.

Por outro lado, como $\text{tg}x_1 \times \text{sen}x_2 > 0$, então $\text{tg}x_1 < 0$.

Então, x_1 é tal que $\cos x_1 > 0$ e $\text{tg}x_1 < 0$, o que se verifica no 4.º quadrante.

7.

7.1. Opção (B)

$f(x) = 1 + 2\text{sen}\left(3x + \frac{\pi}{3}\right)$

• $f\left(x + \frac{\pi}{3}\right) = 1 + 2\text{sen}\left(3\left(x + \frac{\pi}{3}\right) + \frac{\pi}{3}\right) = 1 + 2\text{sen}\left(3x + \frac{4\pi}{3}\right)$, logo não é verdade que para todo o x se verifique $f\left(x + \frac{\pi}{3}\right) = f(x)$

• $f\left(x + \frac{2\pi}{3}\right) = 1 + 2\text{sen}\left(3\left(x + \frac{2\pi}{3}\right) + \frac{\pi}{3}\right) = 1 + 2\text{sen}\left(3x + 2\pi + \frac{\pi}{3}\right) =$

$= 1 + 2\text{sen}\left(3x + \frac{\pi}{3}\right) =$

$= f(x), \forall x \in \mathbb{R}$, logo f é periódica de período $\frac{2\pi}{3}$.

• $f(x + 3\pi) = 1 + 2\text{sen}\left(3(x + 3\pi) + \frac{\pi}{3}\right) = 1 + 2\text{sen}\left(3x + 9\pi + \frac{\pi}{3}\right) =$

$= 1 + 2\text{sen}\left(3x + \pi + \frac{\pi}{3}\right) =$

$= 1 + 2\text{sen}\left(3x + \frac{4\pi}{3}\right)$, logo não é verdade que para todo o x se verifique $f(x + 3\pi) = f(x)$.

• $f(x + \pi) = 1 + 2\text{sen}\left(3(x + \pi) + \frac{\pi}{3}\right) = 1 + 2\text{sen}\left(3x + 3\pi + \frac{\pi}{3}\right) =$

$= 1 + 2\text{sen}\left(3x + \pi + \frac{\pi}{3}\right) =$

$= 1 + 2\text{sen}\left(3x + \frac{4\pi}{3}\right)$, logo não é verdade que para todo o x se verifique $f(x + \pi) = f(x)$.

7.2. Sabemos que:

$-1 \leq \text{sen}\left(3x + \frac{\pi}{3}\right) \leq 1 \Leftrightarrow -2 \leq 2\text{sen}\left(3x + \frac{\pi}{3}\right) \leq 2$

$\Leftrightarrow -1 \leq 1 + 2\text{sen}\left(3x + \frac{\pi}{3}\right) \leq 3$

Temos que $D'_f = [-1,3]$.

-1 é então mínimo de f, logo os minimizantes são os valores de x tais que $f(x) = -1$.

$$1+2\text{sen}\left(3x+\frac{\pi}{3}\right)=-1 \Leftrightarrow \text{sen}\left(3x+\frac{\pi}{3}\right)=-1 \Leftrightarrow 3x+\frac{\pi}{3}=\frac{3\pi}{2}+2k\pi, k\in\mathbb{Z}$$

$$\Leftrightarrow 3x=\frac{3\pi}{2}-\frac{\pi}{3}+2k\pi, k\in\mathbb{Z}$$

$$\Leftrightarrow x=\frac{7\pi}{18}+\frac{2k\pi}{3}, k\in\mathbb{Z}$$

7.3. $g(x)=0$

Assim:

$$2-2\text{sen}^2x+\sqrt{3}\cos x=0 \Leftrightarrow 2(1-\text{sen}^2x)+\sqrt{3}\cos x=0$$

$$\Leftrightarrow 2\cos^2x+\sqrt{3}\cos x=0$$

$$\Leftrightarrow \cos x(2\cos x+\sqrt{3})=0$$

$$\Leftrightarrow \cos x=0 \ \vee \ 2\cos x+\sqrt{3}=0$$

$$\Leftrightarrow x=\frac{\pi}{2}+k\pi, k\in\mathbb{Z} \ \vee \ \cos x=-\frac{\sqrt{3}}{2}$$

$$\Leftrightarrow x=\frac{\pi}{2}+k\pi \ \vee \ x=\pm\frac{5\pi}{6}+2k\pi, k\in\mathbb{Z}$$

No intervalo $]-\pi,\pi[$, os zeros de g são $-\frac{\pi}{2}, \frac{\pi}{2}, \frac{5\pi}{6}$ e $-\frac{5\pi}{6}$.

8. $2\text{sen}\theta=1-k^2 \Leftrightarrow \text{sen}\theta=\frac{1-k^2}{2}$

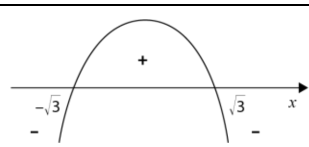
Como $\theta\in\left]\pi,\frac{3\pi}{2}\right[$, então $-1\leq\text{sen}\theta<0$. Assim:

$$-1\leq\frac{1-k^2}{2}<0 \Leftrightarrow -2\leq 1-k^2<0 \Leftrightarrow 1-k^2\geq -2 \ \wedge \ 1-k^2<0$$

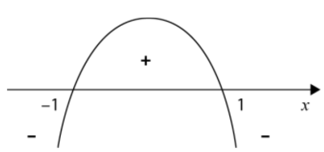
$$\Leftrightarrow 3-k^2\geq 0 \ \wedge \ 1-k^2<0$$

Cálculos auxiliares

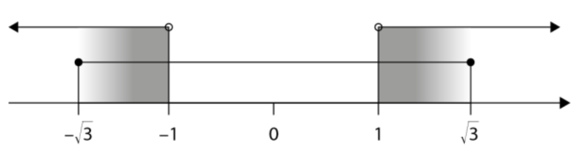
- $3-k^2=0 \Leftrightarrow k^2=3 \Leftrightarrow k=\pm\sqrt{3}$



- $1-k^2=0 \Leftrightarrow k^2=1 \Leftrightarrow k=\pm 1$



$$\Leftrightarrow -\sqrt{3}\leq k\leq\sqrt{3} \ \wedge \ (k<-1 \ \vee \ k>1)$$



Logo, $k\in\left[-\sqrt{3},-1\right[\cup\left]1,\sqrt{3}\right]$.