Unidade 7: Números complexos

1.1. $(1+2i)-(4-4i)=1+2i-4+4i=-3+6i$

1.2. $2\times 3i^{15}-3i=2\times 3i^{3}-3i==2\times 3(-i)-3i=-9i$ $\quad |15=4\times 3+3$

1.3. $(5-3i)\times\left(\frac{1}{2}+i\right)=5\times\frac{1}{2}+5i-\frac{3}{2}i-3i^2=\frac{5}{2}+3+5i-\frac{3}{2}i=\frac{11}{2}+\frac{7}{2}i$

1.4. $\left(1+i\sqrt{2}\right)^2+\left(\sqrt{2}-i\right)^2=1+2\sqrt{2}i+2i^2+2-2\sqrt{2}i+i^2=3-2-1=0$

1.5. $\left(5i^{17}-2i^{14}\right)(1+i)=\left(5i^{1}-2i^{2}\right)(1+i)=(5i+2)(1+i)=$ $\quad |17=4\times 4+1$ $\quad |14=4\times 3+2$

$=5i+5i^2+2+2i=-5+2+7i=-3+7i$

1.6. $\left(1-i\sqrt{3}\right)^2-\sqrt{3}\,i^{51}=1-2\sqrt{3}i+3i^2-\sqrt{3}i^3=1-2\sqrt{3}i-3+\sqrt{3}i=-2-\sqrt{3}i$ $\quad |51=4\times 12+3$

1.7. $i^{4n+1}=i^{4n}\times i=\left(i^4\right)^n\times i=1^n\times i=i\ ,\ n\in\mathbb{N}$

1.8. $2i^{4n+2}+3i^{4n+3}=2i^{4n}\times i^2+3i^{4n}\times i^3=2\left(i^4\right)^n\times(-1)+3\left(i^4\right)^n\times(-i)=-2\times 1^n-3\times 1^n\times i=-2-3i$

2.1. $\frac{2-i^{17}}{(3-4i)(1-i)}=\frac{2-i^{1}}{3-3i-4i-4}=\frac{(2-i)(-1+7i)}{(-1-7i)(-1+7i)}=$

$=\frac{-2+14i+i+7}{(-1)^2-(7i)^2}=\frac{5+15i}{1+49}$

$=\frac{5+15i}{50}=\frac{1}{10}+\frac{3}{10}i$

2.2. $\frac{1}{2+i}-\frac{3}{1-2i}=\frac{1\times(2-i)}{(2+i)(2-i)}-\frac{3(1+2i)}{(1-2i)(1+2i)}=$

$=\frac{2-i}{4-i^2}-\frac{3+6i}{1-4i^2}=\frac{2-i}{5}-\frac{3+6i}{5}$

$=\frac{2-i-3-6i}{5}=\frac{-1-7i}{5}=-\frac{1}{5}-\frac{7}{5}i$

2.3. $\frac{(1-i)^2}{(1+i)^2}-\frac{1}{i}+\frac{20-15i}{1+2i}=$

$=\frac{1-2i+i^2}{1+2i+i^2}-\frac{1\times(-i)}{i\times(-i)}+\frac{(20-15i)(1-2i)}{(1+2i)(1-2i)}=$

$=\frac{1-2i-1}{1+2i-1}-\frac{-i}{-i^2}+\frac{20-40i-15i+30i^2}{1^2-4i^2}=$

$=\frac{-2i}{2i}-\frac{-i}{1}+\frac{20-55i-30}{1+4}=-1+i+\frac{-10-55i}{5}=$

$=-1+i-2-11i=-3-10i$

2.4. $$\frac{\left(1-i^3\right)^3}{1+i^{31}}+2-\frac{(2+2i)^4}{8i^{101}}=\frac{\left(1-(-i)\right)^3}{1+i^3}+2-\frac{\left[(2+2i)^2\right]^2}{8i^1}=$$

$$=\frac{(1+i)^3}{1-i}+2-\frac{\left(4+8i+4i^2\right)^2}{8i}=$$

$$=\frac{(1+i)^2\times(1+i)}{1-i}+2-\frac{(4+8i-4)^2}{8i}=$$

$$=\frac{\left(1+2i+i^2\right)\times(1+i)}{1-i}+2-\frac{(8i)^2}{8i}=$$

$$=\frac{(1+2i-1)\times(1+i)}{1-i}+2-8i=$$

$$=\frac{2i\times(1+i)}{1-i}+2-8i=\frac{(-2+2i)(1+i)}{(1-i)(1+i)}+2-8i=$$

$$=\frac{-2-2i+2i+2i^2}{1^2-i^2}+2-8i=\frac{-2-2}{1+1}+2-8i=$$

$$=\frac{-4}{2}+2-8i=-2+2-8i=-8i$$

2.5. $$\left(\frac{\sqrt{2}-\sqrt{2}\,i}{\sqrt{2}+\sqrt{2}\,i}\right)^2+\left(\frac{\sqrt{3}+\sqrt{3}\,i}{\sqrt{3}-\sqrt{3}\,i}\right)^2=$$

$$=\frac{2-2\sqrt{2}\sqrt{2}\,i+2i^2}{2+2\sqrt{2}\sqrt{2}\,i+2i^2}+\frac{3-2\sqrt{3}\sqrt{3}\,i+3i^2}{3+2\sqrt{3}\sqrt{3}\,i+3i^2}=$$

$$=\frac{2-2\times2i-2}{2+2\times2i-2}+\frac{3-2\times3i-3}{3+2\times3i-3}=\frac{-4i}{4i}+\frac{-6i}{6i}=-1-1=-2$$

3.1. $$(2-i)\,z=5i\Leftrightarrow z=\frac{5i}{2-i}\Leftrightarrow z=\frac{5i(2+i)}{(2-i)(2+i)}\Leftrightarrow$$

$$\Leftrightarrow z=\frac{10i-5}{4-i^2}\Leftrightarrow z=\frac{-5+10i}{5}\Leftrightarrow z=-1+2i$$

$S=\{-1+2i\}$

3.2. $$z^2-6z+13=0\Leftrightarrow z=\frac{6\pm\sqrt{36-52}}{2}\Leftrightarrow z=\frac{6\pm\sqrt{-16}}{2}$$

$$\Leftrightarrow z=\frac{6\pm4i}{2}\Leftrightarrow z=3-2i\vee z=3+2i$$

$S=\{3-2i\,,\,3+2i\}$

3.3. $$z^2-10zi-26=0\Leftrightarrow z=\frac{10i\pm\sqrt{(-10i)^2-4\times(-26)}}{2}$$

$$\Leftrightarrow z=\frac{10i\pm\sqrt{-100+104}}{2}\Leftrightarrow z=\frac{10i\pm2}{2}\Leftrightarrow z=1+5i\vee z=-1+5i$$

$S=\{1+5i\,,\,-1+5i\}$

3.4. $z^3 + z = 0 \Leftrightarrow z\left(z^2+1\right) = 0 \Leftrightarrow z = 0 \vee z^2 + 1 = 0 \Leftrightarrow$

$\Leftrightarrow z = 0 \vee z^2 = -1 \Leftrightarrow z = 0 \vee z = -\mathrm{i} \vee z = \mathrm{i}$

$S = \{0\,,\,-\mathrm{i}\,,\,\mathrm{i}\}$

3.5. $z^3 - 4z^2 + 5z = 0 \Leftrightarrow z\left(z^2 - 4z + 5\right) = 0 \Leftrightarrow$

$\Leftrightarrow z = 0 \vee z^2 - 4z + 5 = 0 \Leftrightarrow z = 0 \vee z = \dfrac{4 \pm \sqrt{16 - 4\times 5}}{2} \Leftrightarrow$

$\Leftrightarrow z = 0 \vee z = \dfrac{4 \pm \sqrt{-4}}{2} \Leftrightarrow z = 0 \vee z = \dfrac{4 \pm 2\mathrm{i}}{2} \Leftrightarrow$

$\Leftrightarrow z = 0 \vee z = 2 - \mathrm{i} \vee z = 2 + \mathrm{i}$

$S = \{0\,,\,2-\mathrm{i}\,,\,2+\mathrm{i}\}$

3.6. $2\mathrm{i}\,z = 1 - \bar{z}$

Fazendo $z = x + y\mathrm{i}$, $x,\ y \in \mathbb{R}$, temos:

$2\mathrm{i}z = 1 - \bar{z} \Leftrightarrow 2\mathrm{i}\ (x + y\mathrm{i}) = 1 - (x - y\mathrm{i}) \Leftrightarrow$

$\Leftrightarrow 2x\mathrm{i} - 2y = 1 - x + y\mathrm{i} \Leftrightarrow -2y + 2x\mathrm{i} = (1 - x) + y\mathrm{i} \Leftrightarrow$

$\Leftrightarrow \begin{cases} -2x = 1 - x \\ 2x = y \end{cases} \Leftrightarrow \begin{cases} -2\times 2x = 1 - x \\ 2x = y \end{cases} \Leftrightarrow$

$\Leftrightarrow \begin{cases} -3x = 1 \\ 2x = y \end{cases} \Leftrightarrow \begin{cases} x = -\dfrac{1}{3} \\ y = -\dfrac{2}{3} \end{cases}$

Portanto, $2\mathrm{i}z = 1 - \bar{z} \Leftrightarrow z = -\dfrac{1}{3} - \dfrac{2}{3}\mathrm{i}$.

$S = \left\{-\dfrac{1}{3} - \dfrac{2}{3}\mathrm{i}\right\}$

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4. $P(z) = z^3 - z + 6$

	1	0	−1	6
−2		−2	4	−6
	1	−2	3	0

$P(z) = (z + 2)\ \left(z^2 - 2z + 3\right)$

$z^2 - z + 3 = 0 \Leftrightarrow z = \dfrac{2 \pm \sqrt{4 - 12}}{2} \Leftrightarrow z = \dfrac{2 \pm \sqrt{-8}}{2} \Leftrightarrow$

$\Leftrightarrow z = \dfrac{2 \pm \mathrm{i}\sqrt{8}}{2} \Leftrightarrow z = \dfrac{2 \pm 2\sqrt{2}\,\mathrm{i}}{2} \Leftrightarrow z = 1 - \sqrt{2}\,\mathrm{i} \vee z = 1 + \sqrt{2}\,\mathrm{i}$

$-2\,,\,1 - \sqrt{2}\,\mathrm{i}\,,\,1 + \sqrt{2}\,\mathrm{i}$ são os zeros do polinómio.

5.1. $z=\dfrac{a+\mathrm{i}}{1+\mathrm{i}}=\dfrac{(a+\mathrm{i})(1-\mathrm{i})}{(1+\mathrm{i})(1-\mathrm{i})}=\dfrac{a-a\mathrm{i}+\mathrm{i}+1}{1^2-\mathrm{i}^2}=$

$=\dfrac{(a+1)+(1-a)\mathrm{i}}{2}=\dfrac{a+1}{2}+\dfrac{1-a}{2}\,\mathrm{i}$

z é imaginário puro $\Leftrightarrow \operatorname{Re}(z)=0\wedge \operatorname{Im}(z)\neq 0 \Leftrightarrow$

$\Leftrightarrow \dfrac{a+1}{2}=0\wedge\dfrac{1-a}{2}\neq 0\Leftrightarrow a=-1$

5.2. $z=\dfrac{2+a\mathrm{i}}{1-2\mathrm{i}}+\mathrm{i}=\dfrac{(2+a\mathrm{i})(1+2\mathrm{i})}{(1-2\mathrm{i})(1+2\mathrm{i})}+\mathrm{i}=\dfrac{2+4\mathrm{i}+a\mathrm{i}-2a}{1^2-4\mathrm{i}^2}+\mathrm{i}=$

$=\dfrac{2-2a+4\mathrm{i}+a\mathrm{i}+5\mathrm{i}}{5}=\dfrac{2-2a}{5}+\dfrac{9+a}{5}\mathrm{i}$

z é imaginário puro $\Leftrightarrow \dfrac{2-2a}{5}=0\wedge\dfrac{9+a}{5}\neq 0\Leftrightarrow 2-2a=0\wedge 9+a\neq 0\Leftrightarrow a=1$

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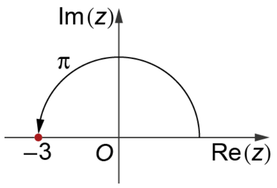
6.1. $z_1=-3$

$|z_1|=3$

$\operatorname{Arg}(z_1)=\pi$

$z_1=3e^{\mathrm{i}\pi}$

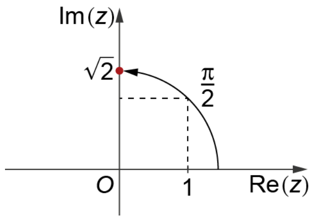


6.2. $z_2=\sqrt{2}\,\mathrm{i}$

$|z_2|=\sqrt{2}$

$\operatorname{Arg}(z_2)=\dfrac{\pi}{2}$

$z_2=\sqrt{2}e^{\mathrm{i}\frac{\pi}{2}}$

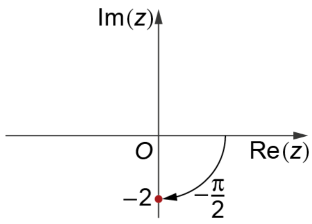


6.3. $z_3=-2\mathrm{i}$

$|z_3|=2$

$\operatorname{Arg}(z_3)=-\dfrac{\pi}{2}$

$z_3=2e^{-\mathrm{i}\frac{\pi}{2}}$

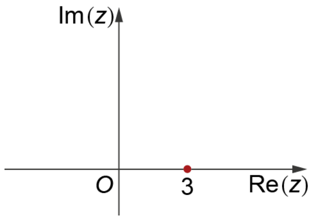


6.4. $z_4=3$

$|z_4|=3$

$\operatorname{Arg}(z_4)=0$

$z_3=e^{\mathrm{i}\times 0}$

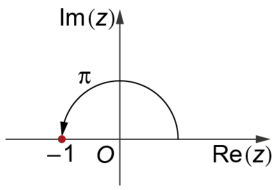


6.5. $z_5 = -1$

$|z_5| = 1$

$\text{Arg}(z_5) = \pi$

$z_3 = e^{i\pi}$



7.1. $z_1 = 1 + i$

$|z_1| = \sqrt{1+1} = \sqrt{2}$

Seja θ_1 um argumento de z_1

$$\begin{cases} \tan\,\theta_1 = \frac{1}{1} = 1 \\ \theta_1 \in 1.^\circ\ Q \end{cases} \Rightarrow \frac{\pi}{4} \text{ é um argumento de } z_1$$

$z_1 = \sqrt{2}e^{i\frac{\pi}{4}}$

7.2. $z_2 = -1 + \sqrt{3}\,i$

$|z_2| = \sqrt{1+3} = 2$

Seja θ_2 um argumento de z_2

$$\begin{cases} \tan\,\theta_2 = \frac{\sqrt{3}}{-1} = -\sqrt{3} \\ \theta_2 \in 2.^\circ\ Q \end{cases} \Rightarrow \pi - \frac{\pi}{3} = \frac{2\pi}{3} \text{ é um argumento de } z_2$$

$z_2 = 2e^{i\frac{2\pi}{3}}$

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7.3. $z_3 = -2 - 2i$

$|z_3| = \sqrt{4+4} = 2\sqrt{2}$

Seja θ_3 um argumento de z_3

$$\begin{cases} \tan\theta_3 = \frac{-2}{-2} = 1 \\ \theta_3 \in 3.^\circ\ Q \end{cases} \Rightarrow \pi + \frac{\pi}{4} = \frac{5\pi}{4} \text{ é um argumento de } z_3$$

$z_3 = 2\sqrt{2}e^{i\frac{5\pi}{4}}$

7.4. $z_4 = 3\sqrt{3} - 3i$

$|z_4| = \sqrt{\left(3\sqrt{3}\right)^2 + (-3)^2} = \sqrt{27+9} = 6$

Seja θ_4 um argumento de z_4

$$\begin{cases} \tan\theta_4 = \frac{-3}{3\sqrt{3}} = -\frac{\sqrt{3}}{3} \\ \theta_4 \in 4.^\circ\ Q \end{cases} \Rightarrow -\frac{\pi}{6} \text{ é um argumento de } z_4$$

$z_4 = 6\,e^{-i\frac{\pi}{6}}$

7.5. $z_5 = -\frac{3}{2} - \frac{3\sqrt{3}}{2}\text{i}$

$$|z_5| = \sqrt{\left(-\frac{3}{2}\right)^2 + \left(-\frac{3\sqrt{3}}{2}\right)^2} = \sqrt{\frac{9}{4} + \frac{27}{4}} = \sqrt{\frac{36}{4}} = 3$$

Seja θ_5 um argumento de z_5.

$$\begin{cases} \tan\,\theta_5 = \dfrac{-\frac{3\sqrt{3}}{2}}{-\frac{3}{2}} = \sqrt{3} \\ \theta_5 \in 3.^\circ\ \text{Q} \end{cases} \Rightarrow \pi + \frac{\pi}{3} = \frac{4\pi}{3} \text{ é um argumento de } z_5$$

$$z_5 = 3\,e^{\text{i}\frac{4\pi}{3}} = 3\,e^{\text{i}\left(\frac{4\pi}{3} - 2\pi\right)} = 3\,e^{-\text{i}\frac{2\pi}{3}}$$

7.6. $z_6 = -4\sqrt{3} + 4\text{i}$

$$|z_6| = \sqrt{\left(-4\sqrt{3}\right)^2 + 4^2} = \sqrt{48 + 16} = 8$$

Seja θ_6 um argumento de z_6

$$\begin{cases} \tan\,\theta_6 = \dfrac{4}{-4\sqrt{3}} = -\dfrac{\sqrt{3}}{3} \\ \theta_6 \in 2.^\circ\ \text{Q} \end{cases} \Rightarrow \pi - \frac{\pi}{6} = \frac{5\pi}{6} \text{ é um argumento de } z_6$$

$$z_6 = 8e^{\text{i}\frac{5\pi}{6}}$$

7.7. $z_7 = 3\sqrt{2} - 3\sqrt{2}\,\text{i}$

$$|z_7| = \sqrt{\left(3\sqrt{2}\right)^2 + \left(-3\sqrt{2}\right)^2} = \sqrt{18 + 18} = 6$$

Seja θ_7 um argumento de z_7

$$\begin{cases} \tan\,\theta_7 = \dfrac{-3\sqrt{2}}{3\sqrt{2}} = -1 \\ \theta_7 \in 4.^\circ\ \text{Q} \end{cases} \Rightarrow -\frac{\pi}{4} \text{ é um argumento de } z_7$$

$$z_7 = 6e^{-\text{i}\frac{\pi}{4}}$$

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8.1. $z_1 = 2\sqrt{2}\,e^{\text{i}\frac{\pi}{4}} = 2\sqrt{2}\left(\cos\frac{\pi}{4} + \text{i}\sin\frac{\pi}{4}\right) = 2\sqrt{2}\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\text{i}\right) = 2 + 2\text{i}$

8.2. $z_2 = 4e^{\text{i}\frac{2\pi}{3}} = 4\left(\cos\frac{2\pi}{3} + \text{i}\sin\frac{2\pi}{3}\right) = 4\left(-\cos\frac{\pi}{3} + \text{i}\sin\frac{\pi}{3}\right) =$

$$= 4\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}\,\text{i}\right) = -2 + 2\sqrt{3}\,\text{i}$$

8.3. $z_3 = 2\,e^{i\frac{7\pi}{6}} = 2\left(\cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6}\right) = 2\left(-\cos\frac{\pi}{6} - i\sin\frac{\pi}{6}\right) =$

$= 2\left(-\frac{\sqrt{3}}{2} - \frac{1}{2}\,i\right) = -\sqrt{3} - i$

8.4. $z_4 = 3\,e^{-i\frac{\pi}{6}} = 3\left(\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right) =$

$= 3\left(\cos\frac{\pi}{6} - i\sin\frac{\pi}{6}\right) = 3\left(\frac{\sqrt{3}}{2} - \frac{1}{2}\,i\right) = \frac{3\sqrt{3}}{2} - \frac{3}{2}\,i$

8.5. $z_5 = 3\,e^{i\times 0} = 3(\cos 0 + i\sin 0) = 3(1 + 0i) = 3$

8.6. $z_6 = 2\,e^{i\frac{\pi}{2}} = 2\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right) = 2(0 + i) = 2i$

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9. $z_1 = 4e^{i\frac{\pi}{3}}$; $z_2 = 2\,e^{i\frac{\pi}{6}}$

9.1. $\left(\frac{z_1}{z_2}\right)^3 = \left(\frac{4\,e^{i\frac{\pi}{3}}}{2\,e^{i\frac{\pi}{6}}}\right)^3 = \left(\frac{4}{2}\,e^{i\left(\frac{\pi}{3}-\frac{\pi}{6}\right)}\right)^3 =$

$= \left(2\,e^{i\frac{\pi}{6}}\right)^3 = 2^3 e^{i\frac{3\pi}{6}} = 8e^{i\frac{\pi}{2}}$

$= 8\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right) = 8(0 + i) = 8i$

9.2. $\frac{(z_1)^2 \times (z_2)^3}{64} = \frac{\left(4e^{i\frac{\pi}{3}}\right)^2 \times \left(2\,e^{i\frac{\pi}{6}}\right)^3}{64} =$

$= \frac{4^2 e^{i\frac{2\pi}{3}} \times 2^3 e^{i\frac{3\pi}{6}}}{64} = \frac{\left(4^2 \times 2^3\right)\,e^{i\left(\frac{2\pi}{3}+\frac{3\pi}{6}\right)}}{64} = \frac{128e^{i\frac{7\pi}{6}}}{64} =$

$= 2\left(\cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6}\right) = 2\left(-\cos\frac{\pi}{6} - i\sin\frac{\pi}{6}\right) = 2\left(-\frac{\sqrt{3}}{2} - \frac{1}{2}\,i\right) = -\sqrt{3} - i$

10. $z = \frac{\left(1+\sqrt{3}\,i\right)^3 - \left(1 + e^{i\left(-\frac{\pi}{2}\right)}\right)^5}{e^{i\frac{\pi}{3}} + i^6} =$

$= \frac{\left(2e^{i\frac{\pi}{3}}\right)^3 - \left(1 + (-i)\right)^5}{\cos\frac{\pi}{3} + i\sin\frac{\pi}{3} + i^{2+4}} =$

$u = 1 + \sqrt{3}\,i\,;\ \theta = \operatorname{Arg}(u)$

$|u| = \sqrt{1^2 + \left(\sqrt{3}\right)^2} = 2$

$\begin{cases} \tan\theta = \frac{\sqrt{3}}{1} = \sqrt{3} \\ \theta \in 1.^{\circ}Q \end{cases} \Leftrightarrow \theta = \frac{\pi}{3}$

$1 + \sqrt{3}\,i = 2e^{i\frac{\pi}{3}}$

$$= \frac{2^3 e^{i\frac{3\pi}{3}} - (1-i)^5}{\frac{1}{2} + \frac{\sqrt{3}}{2}i + i^2} = \frac{8e^{i\pi} - \left(\sqrt{2}e^{i\left(-\frac{\pi}{4}\right)}\right)^5}{\frac{1}{2} + \frac{\sqrt{3}}{2}i - 1} =$$

$$= \frac{-8 - (\sqrt{2})^5 e^{i\left(-\frac{5\pi}{4}\right)}}{-\frac{1}{2} + \frac{\sqrt{3}}{2}i} = \frac{-8 - 4\sqrt{2}\left(\cos\left(-\frac{5\pi}{4}\right) + i\sin\left(-\frac{5\pi}{4}\right)\right)}{\frac{1}{2}\left(-1 + \sqrt{3}i\right)} =$$

$v = -1 + \sqrt{3}i;\ \theta = \mathrm{Arg}(v)$

$|v| = \sqrt{(-1)^2 + (\sqrt{3})^2} = 2$

$\begin{cases} \tan\theta = \frac{\sqrt{3}}{-1} = -\sqrt{3} \\ \theta \in 2.^{\circ}Q \end{cases} \Leftrightarrow \theta = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$

$-1 + \sqrt{3}i = 2e^{i\frac{2\pi}{3}}$

$$= \frac{-8 - 4\sqrt{2}\left(\cos\frac{5\pi}{4} - i\sin\frac{5\pi}{4}\right)}{\frac{1}{2} \times 2e^{i\frac{2\pi}{3}}} = \frac{-8 - 4\sqrt{2}\left(-\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right)}{e^{i\frac{2\pi}{3}}} =$$

$$= \frac{-8 + 4 - 4i}{e^{i\frac{2\pi}{3}}} = \frac{-4 - 4i}{e^{i\frac{2\pi}{3}}} = \frac{4\sqrt{2}e^{i\left(-\frac{3\pi}{4}\right)}}{e^{i\frac{2\pi}{3}}} =$$

$w = -1 - i;\ \theta = \mathrm{Arg}(w)$

$|w| = \sqrt{(-1)^2 + (-1)^2} = \sqrt{2}$

$\begin{cases} \tan\theta = \frac{-1}{-1} = 1 \\ \theta \in 3.^{\circ}Q \end{cases} \Leftrightarrow \theta = -\pi + \frac{\pi}{4} = -\frac{3\pi}{4}$

$-1 - i = \sqrt{2}e^{i\left(-\frac{3\pi}{4}\right)}$

$$= 4\sqrt{2}e^{i\left(-\frac{3\pi}{4} - \frac{2\pi}{3}\right)} = 4\sqrt{2}e^{i\left(-\frac{9\pi}{4} - \frac{8\pi}{3}\right)} = 4\sqrt{2}e^{i\left(-\frac{17\pi}{12}\right)} =$$

$$= 4\sqrt{2}e^{i\left(-\frac{17\pi}{12} + 2\pi\right)} = 4\sqrt{2}e^{i\frac{7\pi}{12}}$$

Temos, portanto, $|z| = 4\sqrt{2}$ e $\mathrm{Arg}(z) = \frac{7\pi}{12}$

11.1. $\cos\frac{\pi}{7} + i\sin\frac{\pi}{7} = e^{i\frac{\pi}{7}}$

$\cos\frac{\pi}{7} - i\sin\frac{\pi}{7}$ é o conjugado de $\cos\frac{\pi}{7} + i\sin\frac{\pi}{7}$. Logo $\cos\frac{\pi}{7} - i\sin\frac{\pi}{7} = e^{i\left(-\frac{\pi}{7}\right)}$

11.2 $$z = \frac{20\,i^{14} + \left(4 - 2\sqrt{3}i\right)^2 - \left(2e^{i\frac{\pi}{15}}\right)^5}{\cos\frac{\pi}{5} - i\sin\frac{\pi}{5}} = \frac{20\,i^{3\times4+2} + 16 - 16\sqrt{3}i + 12i^2 - 2^5 e^{i\frac{5\pi}{15}}}{\overline{\cos\frac{\pi}{5} + i\sin\frac{\pi}{5}}} =$$

$$= \frac{20\,i^2 + 16 - 16\sqrt{3}i - 12 - 32e^{i\frac{\pi}{3}}}{\overline{e^{i\frac{\pi}{5}}}} = \frac{-20 + 4 - 16\sqrt{3}i - 32\left(\cos\frac{\pi}{3} - i\sin\frac{\pi}{3}\right)}{\overline{e^{i\frac{\pi}{5}}}} =$$

$$= \frac{-16 - 16\sqrt{3}i - 32\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)}{e^{i\left(-\frac{\pi}{5}\right)}} = \frac{-16 - 16\sqrt{3}i - 16 - 16\sqrt{3}i}{e^{i\left(-\frac{\pi}{5}\right)}} =$$

$$= \frac{32\left(-1 - \sqrt{3}i\right)}{e^{i\left(-\frac{\pi}{5}\right)}} = \frac{32 \times 2e^{i\left(-\frac{2\pi}{3}\right)}}{e^{i\left(-\frac{\pi}{5}\right)}} = \frac{64}{1}e^{i\left(-\frac{2\pi}{3} + \frac{\pi}{5}\right)} =$$

$$= 64e^{i\left(-\frac{10\pi}{15} + \frac{3\pi}{15}\right)} = 64e^{i\left(-\frac{7\pi}{15}\right)}$$

$v = -1 - \sqrt{3}i;\ \theta = \mathrm{Arg}(v)$

$|v| = \sqrt{(-1)^2 + (-\sqrt{3})^2} = 2$

$\begin{cases} \tan\theta = \frac{-\sqrt{3}}{-1} = \sqrt{3} \\ \theta \in 3.^{\circ}Q \end{cases} \Leftrightarrow \theta = -\pi + \frac{\pi}{3} = -\frac{2\pi}{3}$

$-1 - \sqrt{3}i = 2e^{i\left(-\frac{2\pi}{3}\right)}$

12.1. $z=\cos\alpha+\mathrm{i}\sin\alpha=e^{\mathrm{i}\alpha}$

$$z^k+z^{-k}=\left(e^{\mathrm{i}\alpha}\right)^k+\left(e^{\mathrm{i}\alpha}\right)^{-k}=$$
$$=e^{\mathrm{i}k\alpha}+e^{\mathrm{i}(-k\alpha)}=$$
$$=\cos(k\alpha)+\mathrm{i}\sin(k\alpha)+\cos(-k\alpha)+\mathrm{i}\sin(-k\alpha)=$$
$$=\cos(k\alpha)+\mathrm{i}\sin(k\alpha)+\cos(k\alpha)-\mathrm{i}\sin(k\alpha)=$$
$$=\cos(k\alpha)+\cos(k\alpha)=$$
$$=2\cos(k\alpha)$$

12.2. $w=\sqrt{3}z-\mathrm{i}z=z\left(\sqrt{3}-\mathrm{i}\right)$

Seja $u=\sqrt{3}-\mathrm{i}$

$\|u\|=\sqrt{3+1}=2$

Se $\theta=\operatorname{Arg}u$, $\tan\theta=-\dfrac{1}{\sqrt{3}}=-\dfrac{\sqrt{3}}{3}$

Como $\theta\in 4.^{\text{o}}$ quadrante, vem $\theta=-\dfrac{\pi}{6}$

Portanto $u=2e^{\mathrm{i}\left(-\frac{\pi}{6}\right)}$

$$w=\sqrt{3}z-\mathrm{i}z=z\left(\sqrt{3}-\mathrm{i}\right)=e^{\mathrm{i}\alpha}\times 2e^{\mathrm{i}\left(-\frac{\pi}{6}\right)}=2e^{\mathrm{i}\left(\alpha-\frac{\pi}{6}\right)}$$

O número complexo w é um número imaginário puro se e só se

$$\alpha-\frac{\pi}{6}=\frac{\pi}{2}+k\pi,\ k\in\mathbb{Z}\Leftrightarrow\alpha=\frac{\pi}{2}+\frac{\pi}{6}+k\pi,\ k\in\mathbb{Z}\Leftrightarrow\alpha=\frac{2\pi}{3}+k\pi,\ k\in\mathbb{Z}$$

Logo, os valores de α pertencentes ao intervalo $]-\pi,\pi]$, para os quais o número complexo w é um número imaginário puro são $-\dfrac{\pi}{3}$ (para $k=-1$) e $\dfrac{2\pi}{3}$ (para $k=0$)

13.
$$z=\frac{\mathrm{i}^{11}(1-\mathrm{i})^4}{2-2\operatorname{cis}\left(-\frac{\pi}{3}\right)}=\frac{\mathrm{i}^{2\times4+3}\left[\sqrt{2}\operatorname{cis}\left(-\frac{\pi}{4}\right)\right]^4}{2-2\left[\cos\left(-\frac{\pi}{3}\right)+\mathrm{i}\sin\left(-\frac{\pi}{3}\right)\right]}=$$
$$=\frac{\mathrm{i}^3\left(\sqrt{2}\right)^4\operatorname{cis}(-\pi)}{2-2\left(\frac{1}{2}-\frac{\sqrt{3}}{2}\mathrm{i}\right)}=\frac{-\mathrm{i}\times4\times(-1)}{2-1+\sqrt{3}\,\mathrm{i}}=$$
$$=\frac{4\mathrm{i}}{1+\sqrt{3}\,\mathrm{i}}=\frac{4\operatorname{cis}\frac{\pi}{2}}{2\operatorname{cis}\frac{\pi}{3}}=2\operatorname{cis}\left(\frac{\pi}{2}-\frac{\pi}{3}\right)=2\operatorname{cis}\frac{\pi}{6}$$

$v=1+\sqrt{3}\,\mathrm{i}=|v|\operatorname{cis}\alpha$

$|v|=\sqrt{1+3}=2$

$\begin{cases}\tan\alpha=\dfrac{\sqrt{3}}{1}=\sqrt{3}\\ \left(1,\sqrt{3}\right)\in 1.^{\text{o}}Q\end{cases}\Rightarrow$

$\Rightarrow\dfrac{\pi}{3}$ é um argumento de v

$v=1+\sqrt{3}\,\mathrm{i}=|v|\operatorname{cis}\alpha$

$|v|=\sqrt{1+3}=2$

$\begin{cases}\tan\alpha=\dfrac{\sqrt{3}}{1}=\sqrt{3}\\ \left(1,\sqrt{3}\right)\in 1.^{\text{o}}Q\end{cases}\Rightarrow$

$\Rightarrow\dfrac{\pi}{3}$ é um argumento de v

$z^n = \left(2\operatorname{cis}\frac{\pi}{6}\right)^n = 2^n \operatorname{cis}\frac{n\pi}{6}$

$z^n \in \mathbb{R}^- \Leftrightarrow \frac{n\pi}{6} = \pi + 2k\pi\,,\, k\in\mathbb{Z} \Leftrightarrow n = 6+12k\,,\, k\in\mathbb{Z}$

O menor valor de $n\in\mathbb{N}$ para o qual z^n é um número real negativo é $n=6$ e obtém-se para $k=0$.

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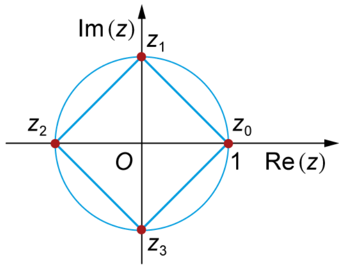
14.1. $\sqrt[4]{1} = \sqrt[4]{e^{i\times 0}} = \sqrt[4]{1}\, e^{i\left(\frac{0}{4}+\frac{2k\pi}{4}\right)} = e^{i\left(\frac{k\pi}{2}\right)}\,,\, k=0\,,1\,,2\,,3$

Se $k=0$, $z_0 = e^{i\times 0} = 1$.

Se $k=1$, $z_1 = e^{i\frac{\pi}{2}} = i$.

Se $k=2$, $z_2 = e^{i\pi} = -1$.

Se $k=3$, $z_2 = e^{i\frac{3\pi}{2}} = -i$.



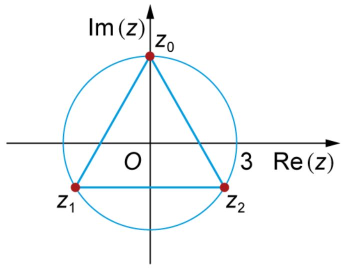
14.2.

$\sqrt[3]{-27i} = \sqrt[3]{27e^{i\frac{3\pi}{2}}} = \sqrt[3]{27}\, e^{i\left(\frac{\frac{3\pi}{2}}{3}+\frac{2k\pi}{3}\right)} = 3e^{i\left(\frac{\pi}{2}+\frac{2k\pi}{3}\right)} = 3e^{i\left(\frac{3\pi+4k\pi}{6}\right)}\,,\, k=0\,,1\,,2$

Se $k=0$, $z_0 = 3e^{i\left(\frac{3\pi+4k\pi}{6}\right)} = 3e^{i\frac{\pi}{2}}$.

Se $k=1$, $z_1 = 3e^{i\left(\frac{3\pi+4k\pi}{6}\right)} = 3e^{i\frac{7\pi}{6}} = 3e^{-i\frac{5\pi}{6}}$.

Se $k=2$, $z_2 = 3e^{i\left(\frac{3\pi+4k\pi}{6}\right)} = 3e^{i\frac{11\pi}{6}} = 3e^{-i\frac{\pi}{6}}$.



15. O polígono é um heptágono. Logo, a equação tem sete soluções (C ou D)

O número $-i$ é uma das soluções

$(-i)^7 = -i^{4+3} = -\left(i^4\times i^3\right) = -1\times(-i) = i = i$

$(-i)^7 + i = i + i = 2i$; $-i$ não é solução da equação $z^7 + i = 0$

$(-i)^7 - i = i - i = 0$; $-i$ é solução da equação $z^7 - i = 0$

Resposta: (D)

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16.1. $z^6 - 64 = 0 \Leftrightarrow z^6 = 64\, e^{i\times 0} \Leftrightarrow z = \sqrt[6]{64\, e^{i\times 0}} \Leftrightarrow$

$\Leftrightarrow z = \sqrt[6]{64}\, e^{i\left(\frac{0+2k\pi}{6}\right)}\,,\, k = 0,\ 1,\ 2,\ 3,\ 4,\ 5$

$\Leftrightarrow z = \sqrt[6]{64}\, e^{i\frac{k\pi}{3}}\,,\, k = 0,\ 1,\ 2,\ 3,\ 4,\ 5$

$\Leftrightarrow z = 2\, e^{i\times 0} \vee z = 2\, e^{i\frac{\pi}{3}} \vee z = 2e^{i\frac{2\pi}{3}} \vee z = 2\, e^{i\times\pi} \vee z = 2e^{i\frac{4\pi}{3}} \vee z = 2\, e^{i\frac{5\pi}{3}}$

$S = \left\{2\,,\, 2e^{i\frac{\pi}{3}}\,,\, 2e^{i\frac{2\pi}{3}}\,,\, -2\,,\, 2\, e^{i\frac{4\pi}{3}}\,,\, 2\, e^{i\frac{5\pi}{3}}\right\}$

16.2. $z^4 + z = 0 \Leftrightarrow z\left(z^3 + 1\right) = 0 \Leftrightarrow z = 0 \vee z^3 + 1 = 0 \Leftrightarrow$

$\Leftrightarrow z = 0 \vee z^3 = -1 \Leftrightarrow z = 0 \vee z^3 = e^{i \times \pi} \Leftrightarrow$

$\Leftrightarrow z = 0 \vee z = \sqrt[3]{e^{i \times \pi}} \Leftrightarrow$

$\Leftrightarrow z = 0 \vee z = e^{i\left(\frac{\pi + 2k\pi}{3}\right)}, \; k = 0, \, 1, \, 2$

$\Leftrightarrow z = 0 \vee z = e^{i\frac{\pi}{3}} \vee z = e^{i \times \pi} \vee z = e^{i\frac{5\pi}{3}}$

$S = \left\{ 0 \, , \, e^{i\frac{\pi}{3}} \, , \, -1 \, , \, e^{i\frac{5\pi}{3}} \right\}$

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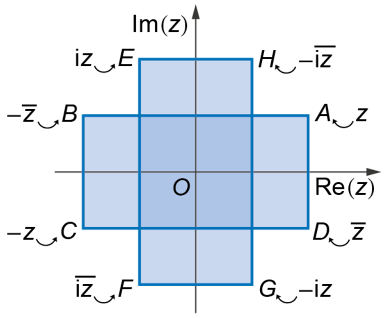
17.

Por exemplo:

$z \curvearrowright A$	$\bar{z} \curvearrowright D$
$-z \curvearrowright C$	$-\bar{z} \curvearrowright B$
$iz \curvearrowright E$	$\overline{iz} \curvearrowright F$
$-iz \curvearrowright G$	$\overline{-iz} \curvearrowright H$



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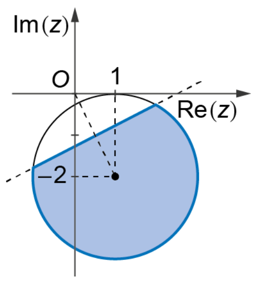
18. $|z| \geq |z - 1 + 2i| \wedge |z - 1 + 2i| \leq 2 \Leftrightarrow$

$\Leftrightarrow |z - (0 + 0i)| \geq |z - (1 - 2i)| \wedge |z - (1 - 2i)| \leq 2$

$Q(0\,,0) \qquad A(1\,,-2) \qquad A(1\,,-2)$

$|z - (0 + 0i)| \geq |z - (1 - 2i)| \rightarrow$ Semiplano definido pela mediatriz do segmento de reta [*OA*] e pelo ponto *A*

$|z - (1 - 2i)| \leq 2 \rightarrow$ Círculo de centro em $A(1\,,-2)$ e raio 2



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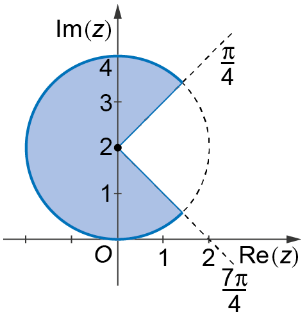
19. $\frac{\pi}{4} \leq \arg(z - 2i) \leq \frac{7\pi}{4} \wedge |z - 2i| \leq 2 \Leftrightarrow$

$\Leftrightarrow \frac{\pi}{4} \leq \arg(z - (0 + 2i)) \leq \frac{7\pi}{4} \wedge |z - (0 + 2i)| \leq 2$

$A(0\,,2) \qquad\qquad A(0\,,2)$

$r = 2$

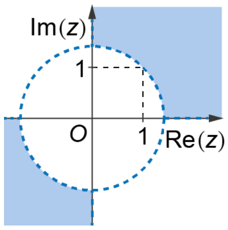


20. $\text{Re}(z) \times \text{Im}(z) > 0 \wedge |z| > |1+\text{i}| \Leftrightarrow$

- $\text{Re}(z) > 0 \times \text{Im}(z) > 0 \Leftrightarrow x \times y > 0 \Leftrightarrow \Leftrightarrow x \times y > 0 \Leftrightarrow$

  $\Leftrightarrow \underbrace{(x > 0 \wedge y > 0)}_{1.\text{º}Q} \vee \underbrace{(x < 0 \wedge y < 0)}_{3\text{º}Q}$

- $|z| > |1+\text{i}| \Leftrightarrow |z - 0| > \sqrt{1+1} \Leftrightarrow |z-0| > \sqrt{2}$



21.1. $|z+3| + |z-3| = 10$ define, no plano complexo, o conjunto de pontos cuja soma das distâncias a dois pontos F_1 e F_2,afixos de $z_1 = -3$ e $z_2 = 3$, respetivamente, é constante e igual a 10. Como $\overline{F_1F_2} = |-3-3| = 6 < 10$, este conjunto de pontos é uma elipse de focos F_1 e F_2.

21.2. Eixo maior: $2a = 10 \Leftrightarrow a = 5$

Eixo menor: $2b$

Distância focal: $2c = 6 \Leftrightarrow c = 3$

$a > b$

$a^2 = b^2 + c^2$

$5^2 = b^2 + 3^2 \Leftrightarrow = 16 \Leftrightarrow b = 4$

Vértices: $(-5, 0), (5, 0)$, $(0, -4)$ e $(0, 4)$

Focos: $(-3, 0)$ e $(3, 0)$

Os números complexos cujos afixos são os vértices e os focos são: -5, 5, -4i, 4i, -3 e 3

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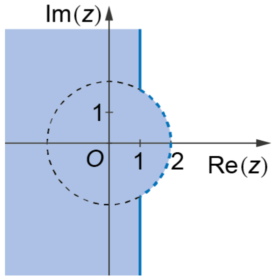
22.1. $\text{Re}(z+2) \leq 3 \vee |z| < 2$

- $\text{Re}(z+2) \leq 3 \Leftrightarrow \text{Re}(x + y\text{i} + 2) \leq 3 \Leftrightarrow$ $\quad | \; z = x + y\text{i}$

  $\Leftrightarrow \text{Re}((x+2) + y\text{i}) \leq 3 \Leftrightarrow$

  $\Leftrightarrow x + 2 \leq 3 \Leftrightarrow x \leq 1$



- $|z| < 2 \Leftrightarrow |z - (0 + 0\,\mathrm{i})| < 2$

Reunião do círculo aberto de centro na origem e raio 2, com o semiplano definido num referencial o.n. xOy pela condição $x \leq 1$.

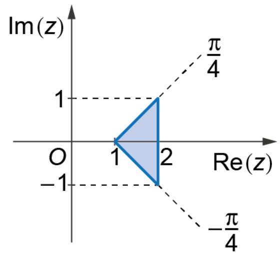
22.2. $-\frac{\pi}{4} \leq \operatorname{Arg}(z-1) \leq \frac{\pi}{4} \vee \operatorname{Im}(\mathrm{i}z - \mathrm{i}) \leq 1$

- $-\frac{\pi}{4} \leq \operatorname{Arg}(z-1) \leq \frac{\pi}{4} \Leftrightarrow -\frac{\pi}{4} \leq \operatorname{Arg}(z - (1 + 0\,\mathrm{i})) \leq \frac{\pi}{4}$
- $\operatorname{Im}(\mathrm{i}z - \mathrm{i}) \leq 1 \Leftrightarrow \quad | \; z = x + y\,\mathrm{i}$

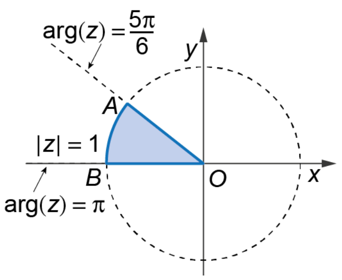
$\Leftrightarrow \operatorname{Im}(\mathrm{i}(x + y\,\mathrm{i}) - \mathrm{i}) \leq 1 \Leftrightarrow$

$\Leftrightarrow \operatorname{Im}(x\,\mathrm{i} - y - \mathrm{i}) \leq 1 \Leftrightarrow$

$\Leftrightarrow \operatorname{Im}(-y + (x-1)\mathrm{i}) \leq 1 \Leftrightarrow x - 1 \leq 1 \Leftrightarrow x \leq 2$



23.1.



Por exemplo: $|z| \leq 1 \vee \frac{5\pi}{6} \leq \operatorname{Arg}(z) \leq \pi$

23.2. $\sqrt{2}\,e^{\mathrm{i}\frac{5\pi}{4}} = \sqrt{2}\left(\cos\frac{5\pi}{4} + \mathrm{i}\sin\frac{5\pi}{4}\right) =$

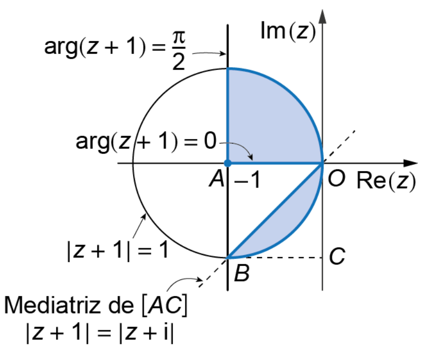
$= \sqrt{2}\,e^{\mathrm{i}\frac{5\pi}{4}} = \sqrt{2}\left(\cos\frac{5\pi}{4} + \mathrm{i}\sin\frac{5\pi}{4}\right)$

$= \sqrt{2}\left(-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\,\mathrm{i}\right) = -1 - \mathrm{i}$

$A(-1, 0) \leftarrow -1$; $B(-1, -1)$; $C(0, -1) \leftarrow -\mathrm{i}$

Por exemplo:

$|z+1| \leq 1 \wedge \left(0 \leq \operatorname{Arg}(z+1) \leq \frac{\pi}{2} \vee |z+1| \geq |z+\mathrm{i}|\right)$

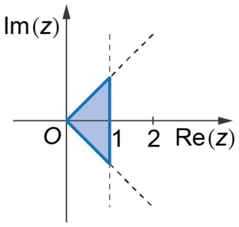


24. $-\frac{\pi}{2} \leq \operatorname{Arg}\left(\frac{z}{\bar{z}}\right) \leq \frac{\pi}{2} \wedge \left|\frac{z}{z-2}\right| < 1$

- $-\frac{\pi}{2} \leq \operatorname{Arg}\left(\frac{z}{\bar{z}}\right) \leq \frac{\pi}{2} \Leftrightarrow -\frac{\pi}{2} \leq \operatorname{Arg} z - \operatorname{Arg}\bar{z} \leq \frac{\pi}{2} \Leftrightarrow -\frac{\pi}{2} \leq \operatorname{Arg} z - (-\operatorname{Arg} z) \leq \frac{\pi}{2} \Leftrightarrow$

$\Leftrightarrow -\frac{\pi}{2} \leq 2\operatorname{Arg} z \leq \frac{\pi}{2} \Leftrightarrow -\frac{\pi}{4} \leq \operatorname{Arg} z \leq \frac{\pi}{4}$

- $\left|\frac{z}{z-2}\right| < 1 \Leftrightarrow \frac{|z|}{|z-2|} < 1 \Leftrightarrow |z-0| < |z-2| \wedge z \neq 2$



25. $\frac{13\pi}{7}$ é um argumento de z.

$z = \rho e^{i\frac{13\pi}{7}}$

$-\bar{z} = -\rho e^{-i\frac{13\pi}{7}} = \rho e^{i\left(\pi - \frac{13\pi}{7}\right)} = \rho e^{-i\frac{6\pi}{7}}$

$= \rho e^{i\left(-\frac{6\pi}{7}+2\pi\right)} = \rho e^{i\frac{8\pi}{7}}$

Resposta: (A)

26. $z = 3e^{i\frac{\pi}{7}}$

$\frac{1}{z^2} = \frac{1}{\left(3e^{i\frac{\pi}{7}}\right)^2} = \frac{e^{i\times 0}}{9e^{i\frac{2\pi}{7}}} = \frac{1}{9}e^{-i\frac{2\pi}{7}}$

Resposta: (B)

27. $z = e^{i\theta}$, $0 < \theta < \frac{\pi}{4}$

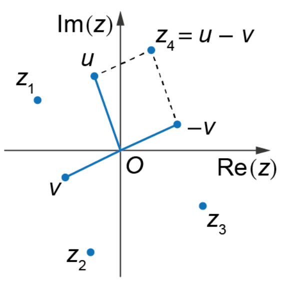
$iz^2 = i\left(e^{i\theta}\right)^2 = e^{i\frac{\pi}{2}} \times e^{i2\theta} = e^{i\left(\frac{\pi}{2}+2\theta\right)}$

$0 < \theta < \frac{\pi}{4} \Leftrightarrow 0 < 2\theta < \frac{\pi}{2} \Leftrightarrow \frac{\pi}{2} < \frac{\pi}{2} + 2\theta < \pi$

$\frac{\pi}{2} < \frac{3\pi}{4} < \pi$

Resposta: (A)

28.



$z_4 = u - v$

Resposta: (D)

29. $w=\rho e^{i\frac{3\pi}{4}}$ ou $w=\rho e^{i\frac{7\pi}{4}}$

$w\times(-w)=-w^2$

Se $w=\rho e^{i\frac{3\pi}{4}}$:

$$-w^2=-\rho^2 e^{i\left(2\times\frac{3\pi}{4}\right)}=-\rho^2 e^{i\frac{3\pi}{2}}=\rho^2 e^{i\frac{\pi}{2}}$$

Se $w=\rho e^{i\frac{7\pi}{4}}$:

$$-w^2=-\rho^2 e^{i\left(2\times\frac{7\pi}{4}\right)}=-\rho^2 e^{i\frac{7\pi}{2}}=-\rho^2 e^{i\frac{3\pi}{2}}=\rho^2 e^{i\frac{\pi}{2}}$$

Apenas z_2 tem argumento igual a $\frac{\pi}{2}$.

Resposta: (B)

30. $\overline{OP}=\overline{OQ}$ e $\overline{OP}\perp\overline{OQ}$

Então, $w=iz$.

$w^2=(iz)^2 \Leftrightarrow w^2=i^2z^2 \Leftrightarrow w^2=-z^2 \Leftrightarrow w^2+z^2=0$

Resposta: (B)

31. $z=\frac{i}{e^{i\alpha}}=\frac{e^{i\frac{\pi}{2}}}{e^{i\alpha}}=e^{i\left(\frac{\pi}{2}-\alpha\right)}$

$\bar{z}=e^{i\left(-\frac{\pi}{2}+\alpha\right)}=e^{i\left(2\pi-\frac{\pi}{2}+\alpha\right)}=e^{i\left(\frac{3\pi}{2}+\alpha\right)}$

Resposta: (D)

32. $z=-\sqrt{3}-i$

$|z|=\sqrt{3+1}=2$

Seja θ um argumento z.

$$\begin{cases}\tan\theta=\frac{-1}{-\sqrt{3}}=\frac{\sqrt{3}}{3}\\ \theta\in 3.^{\text{o}}\text{ Q}\end{cases}\Rightarrow \theta=\pi+\frac{\pi}{6}=\frac{7\pi}{6}$$

$z=2e^{i\frac{7\pi}{6}}$

180° ——— π rad

10° ——— x $\qquad x=\frac{10\pi}{180}\text{ rad}=\frac{\pi}{18}\text{ rad}$

B é o afixo do número complexo $2e^{i\left(\frac{7\pi}{6}+\frac{\pi}{18}\right)}$

$2e^{i\left(\frac{7\pi}{6}+\frac{\pi}{18}\right)}=2e^{i\frac{11\pi}{9}}$

Resposta: (C)

33. $z = \rho e^{i\frac{3\pi}{2}}$

$$\sqrt{z} = \sqrt{\rho}\, e^{i\left(\frac{\frac{3\pi}{2}+2k\pi}{2}\right)}, \; k = 0,\,1$$

$$= \sqrt{\rho}\, e^{i\frac{3\pi+4k\pi}{4}}, \; k = 0,\,1$$

Portanto as raízes quadradas de z são:

$\sqrt{\rho}\, e^{i\frac{3\pi}{4}}$ (k = 0) e $\sqrt{\rho}\, e^{i\frac{7\pi}{4}}$ (k = 1) cujos afixos pertencem à bissetriz dos quadrantes pares.

Resposta: (D)

34. $z = 2e^{i\theta}$

$$(iz)^5 = \left(2i\,e^{i\theta}\right)^5 = \left(2\,e^{i\frac{\pi}{2}} \times e^{i\theta}\right)^5 = \left(2e^{i\left(\frac{\pi}{2}+\theta\right)}\right)^5 = 2^5\, e^{i\left(\frac{5\pi}{2}+5\theta\right)}$$

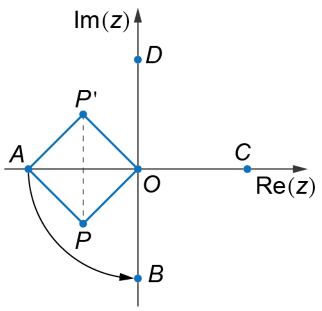
$$(iz)^5 \in \mathbb{R}^- \Leftrightarrow \frac{5\pi}{2} + 5\theta = \pi + 2k\pi,\; k \in \mathbb{Z} \Leftrightarrow$$

$$\Leftrightarrow 5\theta = -\frac{3\pi}{2} + 2k\pi,\; k \in \mathbb{Z} \Leftrightarrow \theta = -\frac{3\pi}{10} + \frac{2k\pi}{5},\; k \in \mathbb{Z}$$

Para k = 1 temos $\theta = -\frac{3\pi}{10} + \frac{2\pi}{5} = \frac{\pi}{10}$.

Resposta: (B)

35.



P é o afixo de w

P' é o afixo de $\overline{w}$

A é o afixo de $w + \overline{w}$

B é o afixo de $i \times \left(w + \overline{w}\right)$

Resposta: (B)

36. $\overline{BD}^2 = \overline{BA}^2 + \overline{AD}^2$

Como $\overline{BA}^2 = \overline{AD}^2 = 2$ (área do quadrado), temos:

$$\overline{BD}^2 = 2 + 2 \Leftrightarrow \overline{BD}^2 = 4 \overset{\overline{BD}>0}{\Leftrightarrow} \overline{BD} = 2$$

$$\overline{OD} = \frac{\overline{BD}}{2} = 1$$

Dado que reta *BD* faz com o eixo real um ângulo de amplitude $\frac{\pi}{6}$ radianos e $\overline{OD}=1$, o ponto *D* é o afixo do número complexo $e^{-i\frac{\pi}{6}}$ pelo que:

$$z=\left(e^{-i\frac{\pi}{6}}\right)^4=e^{-i\frac{4\pi}{6}}=e^{-i\frac{2\pi}{3}}=e^{i\left(-\frac{2\pi}{3}+2\pi\right)}=e^{i\frac{4\pi}{3}}$$

Resposta: (B)

37. $(z-1)^2=0\Leftrightarrow z=1$

A condição $(z-1)^2=0$ define, no plano complexo, o ponto de coordenadas (1 , 0).

Resposta: (D)

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38. $z^2=\rho e^{i\frac{\pi}{2}}$ ou $z^2=\rho e^{i\frac{3\pi}{2}}$

$z=\sqrt{\rho e^{i\frac{\pi}{2}}}$ ou $z=\sqrt{\rho e^{i\frac{3\pi}{2}}}$

$z=\sqrt{\rho}\,e^{i\left(\frac{\frac{\pi}{2}+2k\pi}{2}\right)}$ ou $z=\sqrt{\rho}\,e^{i\left(\frac{\frac{3\pi}{2}+2k\pi}{2}\right)}$, $k=0,1$

$z=\sqrt{\rho}\,e^{i\frac{\pi}{4}}\vee z=\sqrt{\rho}\,e^{i\frac{5\pi}{4}}\vee z=\sqrt{\rho}\,e^{i\frac{3\pi}{4}}\vee z=\sqrt{\rho}\,e^{i\frac{7\pi}{4}}$

Portanto, o afixo de *z* pertence à bissetriz dos quadrantes pares ou à bissetriz dos quadrantes ímpares, pelo que, sendo $z=x+yi$, temos que *y* = *x* ou *y* = – *x*, ou seja, $|\mathrm{Re}(z)|=|\mathrm{Im}(z)|$.

Resposta: (C)

39. $A:\ |z-i|\le|3+4i|\Leftrightarrow|z-i|\le\sqrt{9+16}\Leftrightarrow|z-i|\le 5$ (define um círculo)

$B:\ \left|\frac{1}{z-i}\right|<1\Leftrightarrow\frac{1}{|z-i|}<1\Leftrightarrow|z-i|>1\wedge z\ne i$ (define o exterior de um círculo)

$C:\ |z-1|=|z-2|$ (define uma reta)

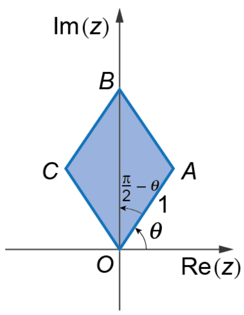
$D:\ \left|\frac{z-2}{z}\right|<1\Leftrightarrow\frac{|z-2|}{|z|}<1\Leftrightarrow|z-2|<|z|$ e $z\ne 0$ (define um semiplano)

Resposta: (D)

40.1. Se [*OABC*] é um losango, então [*OB*] é um eixo de simetria do polígono.

Portanto, como *A* é o afixo de $z=e^{i\theta}$:

- $\overline{OC}=\overline{OA}=1$
- $B\hat{O}C=A\hat{O}B=\frac{\pi}{2}-\theta$



Logo, sendo *w* o número complexo cujo afixo é *C*, temos $|w|=\overline{OC}=1$ e:

$\theta+2\times\left(\frac{\pi}{2}-\theta\right)=\theta+\pi-2\theta=\pi-\theta$; $\pi-\theta$ é um argumento de *w*

$$w=e^{i(\pi-\theta)}=e^{i\pi}\times e^{-i\theta}=-e^{-i\theta}=-\bar{z}$$

40.2. Seja u o número complexo cujo afixo é C.

Como $\overrightarrow{OA}+\overrightarrow{OC}=\overrightarrow{OB}$, temos:

$$u=\overline{z}+\overline{w}=e^{i\theta}-e^{-i\theta}=$$
$$=\cos\theta+i\sin\theta-(\cos(-\theta)+i\sin(-\theta))$$
$$=\cos\theta+i\sin\theta-(\cos\theta-i\sin\theta)$$
$$=\cos\theta+i\sin\theta-\cos\theta+i\sin\theta=2i\sin\theta$$

40.3. $A=\dfrac{\overline{OB}\times\overline{AC}}{2}=\dfrac{|u|\times|z-w|}{2}=\dfrac{2\sin\theta\times2\cos\theta}{2}=2\sin\theta\cos\theta=\sin(2\theta)$

dado que:

$|u|=|2i\sin\theta|=|2\sin\theta|=2\sin\theta$ porque $\sin\theta>0$.

$$|z-w|=\left|e^{i\theta}-\left(-e^{-i\theta}\right)\right|=|\cos\theta+i\sin\theta+\cos(-\theta)+i\sin(-\theta)|=$$
$$=|\cos\theta+i\sin\theta+\cos\theta-i\sin\theta|=$$
$$=|2\cos\theta|=2\cos\theta \text{ porque } \cos\theta>0$$

40.4. $(iz)^3=\left(ie^{i\theta}\right)^3=\left(e^{i\frac{\pi}{2}}\times e^{i\theta}\right)^3=\left[e^{i\left(\frac{\pi}{2}+\theta\right)}\right]^3=e^{i\left(\frac{3\pi}{2}+3\theta\right)}$

$(iz)^3\in\mathbb{R}\Leftrightarrow\dfrac{3\pi}{2}+3\theta=k\pi,\ k\in\mathbb{Z}\Leftrightarrow3\theta=-\dfrac{3\pi}{2}+k\pi,\ k\in\mathbb{Z}\Leftrightarrow\theta=-\dfrac{\pi}{2}+\dfrac{k\pi}{3},\ k\in\mathbb{Z}$

Como $\theta\in\left]0,\dfrac{\pi}{2}\right[$, $\theta=-\dfrac{\pi}{2}+2\times\dfrac{\pi}{3}=\dfrac{\pi}{6}$ (k = 2).

$A=\sin(2\theta)$

Se $\theta=\dfrac{\pi}{6}$, $A=\sin\left(2\times\dfrac{\pi}{6}\right)=\sin\dfrac{\pi}{3}=\dfrac{\sqrt{3}}{2}$

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41.1. $\overline{OA}=|2i|=2$

$|z|=2$ define a circunferência de centro na origem e raio 2.

41.2. $\overline{OB}=\overline{OA}=2$

Logo, $|w|=2$.

O triângulo [ABC] é equilátero. Assim, os arcos de circunferência AB, BC e CA são geometricamente iguais. Cada um deles tem amplitude igual a $\dfrac{2\pi}{3}$ radianos.

$$\text{Arg}(w)=\text{Arg}(2i)+\frac{2\pi}{3}=\frac{\pi}{2}+\frac{2\pi}{3}=\frac{7\pi}{6}$$

$$w=2e^{i\frac{7\pi}{6}}=2\left(\cos\frac{7\pi}{6}+i\sin\frac{7\pi}{6}\right)=2\left(-\frac{\sqrt{3}}{2}-\frac{1}{2}i\right)=-\sqrt{3}-i$$

42. $z_1 = \rho\, e^{i\frac{\pi}{6}}$ e $z_2 = 2\, e^{i\theta}$

42.1. $\left(2\sqrt{2}e^{i\frac{19\pi}{12}}\right)^2 z^3 = \dfrac{z_1}{|z_1|} \Leftrightarrow \left(2\sqrt{2}\right)^2 e^{i\frac{2\times 19\pi}{12}} \times z^3 = \dfrac{\rho\, e^{i\frac{\pi}{6}}}{\rho} \Leftrightarrow$

$\Leftrightarrow 8e^{i\frac{19\pi}{6}} \times z^3 = e^{i\frac{\pi}{6}} \Leftrightarrow z^3 = \dfrac{e^{i\frac{\pi}{6}}}{8e^{i\frac{19\pi}{6}}} \Leftrightarrow z^3 = \dfrac{1}{8}e^{i\left(\frac{\pi}{6}-\frac{19\pi}{6}\right)} \Leftrightarrow z^3 = \dfrac{1}{8}e^{-i3\pi} \Leftrightarrow$

$\Leftrightarrow z^3 = \dfrac{1}{8}e^{i\pi} \Leftrightarrow z = \sqrt[3]{\dfrac{1}{8}e^{i\pi}} \Leftrightarrow$

$\Leftrightarrow z = \sqrt[3]{\dfrac{1}{8}}e^{i\left(\frac{\pi+2k\pi}{3}\right)},\ k = 0,\ 1,\ 2 \Leftrightarrow z = \dfrac{1}{2}e^{i\left(\frac{\pi+2k\pi}{3}\right)},\ k = 0,\ 1,\ 2 \Leftrightarrow$

$\Leftrightarrow z = \dfrac{1}{2}e^{i\frac{\pi}{3}} \vee z = \dfrac{1}{2}e^{i\pi} \vee z = \dfrac{1}{2}e^{i\frac{5\pi}{3}}$

42.2. $z_1 = \rho e^{i\frac{\pi}{6}}$, $z_2 = 2\, e^{i\theta}$

Se z_1 e z_2 são raízes cúbicas do mesmo número complexo z, então $\theta - \dfrac{\pi}{6} = k \times \dfrac{2\pi}{3}$, $k \in \mathbb{Z}$ (a diferença dos argumentos é um múltiplo de $\dfrac{2\pi}{3}$)

$\theta - \dfrac{\pi}{6} = k \times \dfrac{2\pi}{3},\ k \in \mathbb{Z} \Leftrightarrow \theta = \dfrac{\pi}{6} + \dfrac{2k\pi}{3},\ k \in \mathbb{Z}$

Para $k = 0$, $\theta = \dfrac{\pi}{6}$.

Para $k = 1$, $\theta = \dfrac{5\pi}{6}$.

Para $k = 2$, $\theta = \dfrac{3\pi}{2}$

Como o afixo de z_2 pertence ao segundo quadrante, temos:

$z_2 = 2e^{i\frac{5\pi}{6}} = 2\left(\cos\dfrac{5\pi}{6} + i\sin\dfrac{5\pi}{6}\right) = 2\left(-\dfrac{\sqrt{3}}{2} + \dfrac{1}{2}i\right) = -\sqrt{3} + i$

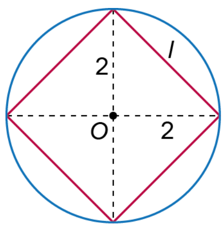
43. $z_1 = 16\, e^{i\theta}$

43.1. a) O polígono é um quadrado inscrito numa circunferência de raio $r = \sqrt[4]{16} = 2$.

Seja ℓ o lado desse quadrado

$l^2 = 2^2 + 2^2 \Leftrightarrow l^2 = 2 \times 2^2 \Leftrightarrow l = \sqrt{2 \times 2^2} \Leftrightarrow l = 2\sqrt{2}$

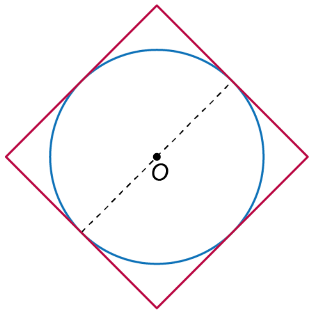
Perímetro = $4 \times 2\sqrt{2} = 8\sqrt{2}$



b) O centro da circunferência é a origem do referencial.
O diâmetro é igual ao lado do quadrado.

Logo, o raio é $\frac{2\sqrt{2}}{2} = \sqrt{2}$.

Uma condição que define a circunferência é $|z| = \sqrt{2}$.



43.2. $\sqrt[4]{z_1} = \sqrt[4]{16\,e^{i\frac{2\pi}{3}}} =$

$= \sqrt[4]{16}e^{i\left(\frac{\frac{2\pi}{3}+2k\pi}{4}\right)}, \; k = 0, 1, 2, 3$

$= 2\,e^{i\left(\frac{2\pi+6k\pi}{12}\right)}, \; k = 0, 1, 2, 3$

$= 2e^{i\left(\frac{\pi+3k\pi}{6}\right)}, \; k = 0, 1, 2, 3$

$k = 0 \Rightarrow z_0 = 2\,e^{i\frac{\pi}{6}}$

$k = 1 \Rightarrow z_1 = 2\,e^{i\frac{4\pi}{6}} = 2\,e^{i\frac{2\pi}{3}}$

$k = 2 \Rightarrow z_2 = 2\,e^{i\frac{7\pi}{6}}$

$k = 3 \Rightarrow z_3 = 2\,e^{i\frac{10\pi}{6}} = 2\,e^{i\frac{5\pi}{3}}$

$\left\{2\,e^{i\frac{\pi}{6}}, 2\,e^{i\frac{2\pi}{3}}, 2\,e^{i\frac{7\pi}{6}}, 2\,e^{i\frac{5\pi}{3}}\right\}$

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44. $z = \rho\,e^{i\frac{\pi}{4}}$

$z^3 = \left(\rho\,e^{i\frac{\pi}{4}}\right)^3 = \rho^3 e^{i\frac{3\pi}{4}}$

$\frac{3\pi}{4}$ é um argumento de z^3.

Resposta: (B)

45. $z = 2\,i\,e^{i\left(\theta-\frac{\pi}{12}\right)} = 2\,e^{i\frac{\pi}{2}} \times e^{i\left(\theta-\frac{\pi}{12}\right)} =$

$= 2\,e^{i\left(\frac{\pi}{2}+\theta-\frac{\pi}{12}\right)} = 2\,e^{i\left(\theta+\frac{5\pi}{12}\right)}$

$z \in \mathbb{R}^- \Leftrightarrow \theta + \frac{5\pi}{12} = \pi + 2k\pi, \; k \in \mathbb{Z}$

$\Leftrightarrow \theta = \pi - \frac{5\pi}{12} + 2k\pi, \; k \in \mathbb{Z}$

$\Leftrightarrow \theta = \frac{7\pi}{12} + 2k\pi, \; k \in \mathbb{Z}$

Para $k = 0$, temos $\theta = \frac{7\pi}{12}$.

Resposta: (A)

46. *P* é o afixo de *u*.

Como $OQ \perp \overline{OP}$ e $\overline{OQ} = 2\overline{OP}$, temos que *Q* é o afixo de $\mathrm{i} \times (2u) = 2\mathrm{i}u$.

Logo, $u \times v = 2\mathrm{i}u$, ou seja, $v = 2\mathrm{i}$.

Como $|2\mathrm{i}| = 2 = 2\overline{OP} = \overline{OQ}$, apenas *C* pode ser o afixo de *v*.

Resposta: (C)

47. $|w| < 1 \Rightarrow |w|^2 < |w| \Rightarrow |z| < |w|$ ((*A*) ou (*C*))

Se $z = w^2$, $\arg z = 2\arg w$ ((*A*) ou (*D*))

Resposta: (A)

48. $z = \rho\, \mathrm{e}^{\mathrm{i}\frac{5\pi}{4}},\ \rho > 0$

$$z^n = \left(\rho\, \mathrm{e}^{\mathrm{i}\frac{5\pi}{4}}\right)^n = \rho^n \mathrm{e}^{\mathrm{i}\frac{5n\pi}{4}},\ \rho^n > 0$$

$$z^n \in \mathbb{R}^+ \Rightarrow \frac{5n\pi}{4} = 2k\pi,\ k \in \mathbb{Z} \Leftrightarrow 5n = 8k,\ k \in \mathbb{Z} \Leftrightarrow n = \frac{8k}{5},\ k \in \mathbb{Z}$$

Para *k* = 5, temos *n* = 8.

Resposta: (D)

49. $\overline{OA} = \overline{OC}$ e $[OA] \perp [OC]$

Se *A* é o afixo de *z*, *C* é o afixo de i*z*.

Logo, como [*OABC*] é um quadrado e, portanto, é um paralelogramo, *B* é o afixo de $z + \mathrm{i}z = z(1+\mathrm{i})$.

Resposta: (A)

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50. $$z = \frac{\mathrm{i}}{\mathrm{e}^{\mathrm{i}\left(\frac{\pi}{8}-\theta\right)}} = \frac{\mathrm{e}^{\mathrm{i}\frac{\pi}{2}}}{\mathrm{e}^{\mathrm{i}\left(\frac{\pi}{8}-\theta\right)}} = \mathrm{e}^{\mathrm{i}\left(\frac{\pi}{2}-\frac{\pi}{8}+\theta\right)} = \mathrm{e}^{\mathrm{i}\left(\frac{3\pi}{8}+\theta\right)}$$

z é um imaginário puro $\Leftrightarrow$

$$\Leftrightarrow \frac{3\pi}{8} + \theta = \frac{\pi}{2} + k\pi,\ k \in \mathbb{Z} \Leftrightarrow \theta = \frac{\pi}{2} - \frac{3\pi}{8} + k\pi,\ k \in \mathbb{Z} \Leftrightarrow \theta = \frac{\pi}{8} + k\pi,\ k \in \mathbb{Z}$$

$$k = 0 \Rightarrow \theta = \frac{\pi}{8};\ k = 1 \Rightarrow \theta = \frac{9\pi}{8};\ k = 2 \Rightarrow \theta = \frac{17\pi}{8}$$

Resposta: (C)

51. $$\left|1 + \frac{1}{z}\right| \leq 1 \Leftrightarrow \left|\frac{z+1}{z}\right| \leq 1 \Leftrightarrow \frac{|z+1|}{|z|} \leq 1 \Leftrightarrow$$

$$\Leftrightarrow |z+1| \leq |z| \wedge z \neq 0 \Leftrightarrow |z-(-1)| \leq |z| \wedge z \neq 0$$

Sendo $A(-1, 0)$ e $O(0, 0)$, a condição define o semiplano definido pela mediatriz do segmento de reta [*OA*] e pelo ponto *A*.

Resposta: (C)

52. $w=1+2i$

52.1. $\left(\frac{w-1}{2}\right)^7\times\frac{10}{1+3i}=\left(\frac{1+2i-1}{2}\right)^7\times\frac{10(1-3i)}{(1+3i)(1-3i)}=i^7\times\frac{10(1-3i)}{1^2-(3i)^2}=i^3\times\frac{10(1-3i)}{10}=-i(1-3i)=-3-i$

52.2. $w^2=(1+2i)^2=1+4i+(2i)^2=1-4+4i=-3+4i$

$4\sqrt{2}e^{i\frac{3\pi}{4}}+1=4\sqrt{2}\left(\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}\right)+1=4\sqrt{2}\left(-\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i\right)+1=-4+4i+1=-3+4i$

Portanto, $w^2=4\sqrt{2}\,e^{i\frac{3\pi}{4}}+1$.

53.1. $z^3+1=0\Leftrightarrow z^3=-1\Leftrightarrow z^3=e^{i\pi}\Leftrightarrow$

$\Leftrightarrow z=\sqrt[3]{e^{i\pi}}\Leftrightarrow z=e^{i\left(\frac{\pi+2k\pi}{3}\right)}, k=0,1,2$

$\Leftrightarrow z=e^{i\frac{\pi}{3}}\vee z=e^{i\pi}\vee z=e^{i\frac{5\pi}{3}}$

$S=\left\{e^{i\frac{\pi}{3}}, e^{i\pi}, e^{i\frac{5\pi}{3}}\right\}$

53.2. a) As raízes cúbicas de – 1 são os zeros do polinómio z^3+1.
– 1 é a raiz cúbica real de – 1.
Usando a Regra de Ruffini, temos:

	1	0	0	1
–1		–1	1	–1
	1	–1	1	0

$z^3+1=(z+1)(z^2-z+1)$

Portanto as raízes cúbicas não reais de – 1 são as soluções da equação $z^2-z+1=0$.
Logo, se k é uma raiz cúbica não real de – 1, vem $k^2-k+1=0\Leftrightarrow k^2=k-1$.

b) $(1-k)^6=(k-1)^6=(k^2)^6=k^{12}=(k^3)^4=(-1)^4=1$

54. $P(z)=2z^4-5z^3+4z^2-5z+2$

54.1. $P(w)=2w^4-5w^3+4w^2-5w+2=0$

$P\left(\frac{1}{w}\right)=2\left(\frac{1}{w}\right)^4-5\left(\frac{1}{w}\right)^3+4\left(\frac{1}{w}\right)^2-5\times\frac{1}{w}+2=$

$=\frac{2}{w^4}-\underset{(w)}{\frac{5}{w^3}}+\underset{(w^2)}{\frac{4}{w^2}}-\underset{(w^3)}{\frac{5}{w}}+\underset{(w^4)}{2}=\frac{2-5w+4w^2-5w^3+2w^4}{w^4}=$

$=\frac{2w^4-5w^3+4w^2-5w+2}{w^4}=\frac{P(w)}{w^4}=\frac{0}{w^4}=0$ ($w\neq 0$ porque 0 não é um zero de $P(z)$).

Logo, se $P(w)=0$, então $P\left(\frac{1}{w}\right)=0$.

54.2. Se 2 é uma solução, então $\frac{1}{2}$ também é solução (**48.1.**).

Usando a regra de Ruffini:

	2	−5	4	−5	2
2		4	−2	4	−2
	2	−1	2	−1	0
$\frac{1}{2}$		1	0	1	
	2	0	2	0	

$$P(z)=0 \Leftrightarrow (z-2)\left(z-\frac{1}{2}\right)(2z^2+2)=0 \Leftrightarrow z-2=0 \vee z-\frac{1}{2}=0 \vee 2z^2+2=0 \Leftrightarrow z=2 \vee z=\frac{1}{2} \vee z^2=-1 \Leftrightarrow$$

$$\Leftrightarrow z=2 \vee z=\frac{1}{2} \vee z=\mathrm{i} \vee z=-\mathrm{i}$$

$$S=\left\{-\mathrm{i}\,,\,\mathrm{i}\,,\,\frac{1}{2}\,,\,2\right\}$$

55. $z=1+e^{\mathrm{i}2\alpha}$

55.1. $z=1+e^{\mathrm{i}2\alpha}=1+\cos(2\alpha)+\mathrm{i}\sin(2\alpha)=$

$$=1+\cos^2\alpha-\sin^2\alpha+2\mathrm{i}\sin\alpha\cos\alpha=$$

$$=\cos^2\alpha+\cos^2\alpha+2\mathrm{i}\sin\alpha\cos\alpha=$$

$$=2\cos\alpha\times e^{\mathrm{i}\alpha}$$

55.2. $z=2\cos\frac{8\pi}{7}\times e^{\mathrm{i}\frac{8\pi}{7}}=2\cos\left(\pi+\frac{\pi}{7}\right)\times e^{\mathrm{i}\left(\pi+\frac{\pi}{7}\right)}=$

$$=-2\cos\frac{\pi}{7}\times e^{\mathrm{i}\left(\pi+\frac{\pi}{7}\right)}=2\cos\frac{\pi}{7}\times e^{\mathrm{i}\frac{\pi}{7}}$$

$\frac{8\pi}{7}=\pi+\frac{\pi}{7}$

$\cos\frac{8\pi}{7}<0$

56. $z_1=e^{\mathrm{i}\frac{\pi}{4}}$

56.1. $\frac{5\mathrm{i}^{13}}{1+2z_1^2}+\mathrm{i}=\frac{5\mathrm{i}^{1+12}}{1+2\left(e^{\mathrm{i}\frac{\pi}{4}}\right)^2}+\mathrm{i}=\frac{5\mathrm{i}}{1+2\,e^{\mathrm{i}\frac{\pi}{2}}}+\mathrm{i}=$

$$=\frac{5\mathrm{i}}{1+2\mathrm{i}}+\mathrm{i}=\frac{5\mathrm{i}(1-2\mathrm{i})}{(1+2\mathrm{i})(1-2\mathrm{i})}+\mathrm{i}=\frac{5\mathrm{i}+10}{1+4}+\mathrm{i}=$$

$$=2+\mathrm{i}+\mathrm{i}=2+2\mathrm{i}=2\sqrt{2}e^{\mathrm{i}\frac{\pi}{4}}$$

$\rho=\sqrt{4+4}=2\sqrt{2}$

$\begin{cases}\tan\theta=1\\ \theta\in 1.^{\text{o}}\text{ Q}\end{cases}\Rightarrow\theta=\frac{\pi}{4}$

56.2. $w=\frac{(k+4\mathrm{i})e^{\mathrm{i}\frac{3\pi}{2}}}{k-\mathrm{i}}=\frac{(k+4\mathrm{i})\times(-\mathrm{i})}{k-\mathrm{i}}=$

$$=\frac{-k\mathrm{i}+4}{k-\mathrm{i}}=\frac{(4-k\mathrm{i})(k+\mathrm{i})}{(k-\mathrm{i})(k+\mathrm{i})}=$$

$$=\frac{4k+4\mathrm{i}-k^2\mathrm{i}+k}{k^2+1}=$$

$$=\frac{5k}{k^2+1}+\frac{4-k^2}{k^2+1}\mathrm{i}$$

$w\in\mathbb{R}\Leftrightarrow 4-k^2=0\Leftrightarrow k=2\vee k=-2$

56.3. $\left(\dfrac{\text{i}}{z_1}\right)^n = \left(\dfrac{e^{\text{i}\frac{\pi}{2}}}{e^{\text{i}\frac{\pi}{4}}}\right)^n = \left(e^{\text{i}\frac{\pi}{4}}\right)^n = e^{\text{i}\frac{n\pi}{4}}$

$$\left(\frac{\text{i}}{z_1}\right)^n = \text{i} \Leftrightarrow e^{\text{i}\frac{n\pi}{4}} = e^{\text{i}\frac{\pi}{2}} \Leftrightarrow$$

$$\Leftrightarrow \frac{n\pi}{4} = \underset{(2)}{\frac{\pi}{2}} + \underset{(4)}{2k\pi},\ k \in \mathbb{Z} \Leftrightarrow$$

$$\Leftrightarrow n = 2 + 8k,\ k \in \mathbb{Z}$$

O menor valor de n é 2.

57. $z_1 = e^{\text{i}\alpha}$; $z_2 = e^{\text{i}\frac{\pi}{3}}$

57.1. $w = 1 - \dfrac{\sqrt{3}\,\text{i}}{z_2} = 1 - \dfrac{\sqrt{3}\,e^{\text{i}\frac{\pi}{2}}}{e^{\text{i}\frac{\pi}{3}}} = 1 - \sqrt{3}\,e^{\text{i}\left(\frac{\pi}{2}-\frac{\pi}{3}\right)} =$

$$= 1 - \sqrt{3}e^{\text{i}\frac{\pi}{6}} = 1 - \sqrt{3}\left(\cos\frac{\pi}{6} + \text{i}\sin\frac{\pi}{6}\right) =$$

$$= 1 - \sqrt{3}\left(\frac{\sqrt{3}}{2} + \frac{1}{2}\text{i}\right) = 1 - \frac{3}{2} - \frac{\sqrt{3}}{2}\text{i} = -\frac{1}{2} - \frac{\sqrt{3}}{2}\text{i} = e^{\text{i}\frac{4\pi}{3}}$$

Cálculo auxiliar

$$-\frac{1}{2} - \frac{\sqrt{3}}{2}\text{i} = \rho\, e^{\text{i}\theta}$$

$$\rho = \sqrt{\frac{1}{4} + \frac{3}{4}} = 1$$

$$\begin{cases} \tan\theta = \dfrac{-\frac{\sqrt{3}}{2}}{-\frac{1}{2}} = \sqrt{3} \\ \theta \in 3.^{\text{o}}\ \text{Q} \end{cases} \Rightarrow \theta = \pi + \frac{\pi}{3} = \frac{4\pi}{3}$$

$$w^3 = \left(e^{\text{i}\frac{4\pi}{3}}\right)^3 = e^{\text{i}\left(3\times\frac{4\pi}{3}\right)} = e^{\text{i}4\pi} = e^{\text{i}\times 0} = 1$$

Logo, w é uma raiz cúbica de z_1.

57.2. $z_1 + z_2 = e^{\text{i}\alpha} + e^{\text{i}\frac{\pi}{3}} =$

$$= \cos\alpha + \text{i}\sin\alpha + \cos\frac{\pi}{3} + \text{i}\sin\frac{\pi}{3}$$

$$= \cos\alpha + \cos\frac{\pi}{3} + \text{i}\left(\sin\alpha + \sin\frac{\pi}{3}\right)$$

$$= \left(\cos\alpha + \frac{1}{2}\right) + \text{i}\left(\sin\alpha + \frac{\sqrt{3}}{2}\right)$$

$$z_1 + z_2 \in \mathbb{R} \Leftrightarrow \sin\alpha + \frac{\sqrt{3}}{2} = 0 \Leftrightarrow \sin\alpha = -\frac{\sqrt{3}}{2}$$

Como $\alpha \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$, vem $\alpha = -\frac{\pi}{3}$.

57.3. Seja $|z| = 1$. Então, $z = e^{i\alpha}$, $\alpha \in \mathbb{R}$.

$$\frac{1}{z} = \frac{1}{e^{i\alpha}} = \frac{e^{i\times 0}}{e^{i\alpha}} = e^{i(0-\alpha)} = e^{i(-\alpha)} = \bar{z}$$

58. $z_1 = e^{i\frac{\pi}{8}}$, $z_2 = \frac{\sqrt{3}}{2} + \frac{1}{2}i$, $z_3 = e^{i\alpha}$

58.1.
$$\frac{z_1^{12} + z_2}{(1-2i)^3 - 9e^{i\pi}} = \frac{\left(e^{i\frac{\pi}{8}}\right)^{12} + \frac{\sqrt{3}}{2} + \frac{1}{2}i}{(1-2i)^3 - 9e^{i\pi}} =$$

$$= \frac{e^{i\frac{12\pi}{8}} + \frac{\sqrt{3}}{2} + \frac{1}{2}i}{(1-4i-4)(1-2i)+9} = \frac{e^{i\frac{3\pi}{2}} + \frac{\sqrt{3}}{2} + \frac{1}{2}i}{(-3-4i)(1-2i)+9} =$$

$$= \frac{-i + \frac{\sqrt{3}}{2} + \frac{1}{2}i}{-3+6i-4i-8+9} = \frac{\frac{\sqrt{3}}{2} - \frac{1}{2}i}{-2+2i} =$$

$$= \frac{e^{i\left(-\frac{\pi}{6}\right)}}{2\sqrt{2}\, e^{i\frac{3\pi}{4}}} = \frac{1}{2\sqrt{2}} e^{i\left(-\frac{\pi}{6} - \frac{3\pi}{4}\right)} =$$

$$= \frac{\sqrt{2}}{2\times 2} e^{-i\frac{11\pi}{12}} = \frac{\sqrt{2}}{4} e^{-i\frac{11\pi}{12}}$$

Cálculos auxiliares

$\frac{\sqrt{3}}{2} - \frac{1}{2}i = \rho\, e^{i\theta}$

$\rho = \sqrt{\frac{3}{4} + \frac{1}{4}} = 1$

$\begin{cases} \tan\theta = -\frac{1}{\sqrt{3}} = -\frac{\sqrt{3}}{3} \\ \theta \in 4.^{\text{o}}\ \text{Q} \end{cases} \Rightarrow \theta = -\frac{\pi}{6}$

$-2 + 2i = r\, e^{i\alpha}$

$r = \sqrt{4+4} = 2\sqrt{2}$

$\begin{cases} \tan\alpha = -1 \\ \alpha \in 2.^{\text{o}}\ \text{Q} \end{cases} \Rightarrow \alpha = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$

58.2. $z_2 = \frac{\sqrt{3}}{2} + \frac{1}{2}i = \cos\frac{\pi}{6} + i\sin\frac{\pi}{6} = e^{i\frac{\pi}{6}}$

$$z_3^2 \times z_2 = \left(e^{i\alpha}\right)^2 \times e^{i\frac{\pi}{6}} = e^{i2\alpha} \times e^{i\frac{\pi}{6}} = = e^{i\left(2\alpha + \frac{\pi}{6}\right)}$$

O afixo de $z_3^2 \times z_2$ pertence à bissetriz do 2.º quadrante $\Leftrightarrow$

$$\Leftrightarrow 2\alpha + \frac{\pi}{6} = \frac{3\pi}{4} + 2k\pi,\ k \in \mathbb{Z} \Leftrightarrow 2\alpha = \frac{3\pi}{4} - \frac{\pi}{6} + 2k\pi,\ k \in \mathbb{Z} \Leftrightarrow$$

$$\Leftrightarrow 2\alpha = \frac{7\pi}{12} + 2k\pi,\ k \in \mathbb{Z} \Leftrightarrow \alpha = \frac{7\pi}{24} + k\pi,\ k \in \mathbb{Z}$$

Como $\alpha \in \left[0, \frac{\pi}{2}\right]$, temos $\alpha = \frac{7\pi}{24}$.

58.3.
$$|z_1 + z_2|^2 = \left|e^{i\frac{\pi}{8}} + e^{i\frac{\pi}{6}}\right|^2 = \left|\cos\frac{\pi}{8} + i\sin\frac{\pi}{8} + \cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right|^2$$

$$= \left|\left(\cos\frac{\pi}{8} + \cos\frac{\pi}{6}\right) + i\left(\sin\frac{\pi}{8} + \sin\frac{\pi}{6}\right)\right|^2 = \left(\sqrt{\left(\cos\frac{\pi}{8} + \cos\frac{\pi}{6}\right)^2 + \left(\sin\frac{\pi}{8} + \sin\frac{\pi}{6}\right)^2}\right)^2 =$$

$$= \cos^2\frac{\pi}{8} + 2\cos\frac{\pi}{8}\cos\frac{\pi}{6} + \cos^2\frac{\pi}{6} + \sin^2\frac{\pi}{8} + 2\sin\frac{\pi}{8}\sin\frac{\pi}{6} + \sin^2\frac{\pi}{6} =$$

$$= \left(\cos^2\frac{\pi}{8} + \sin^2\frac{\pi}{8}\right) + \left(\cos^2\frac{\pi}{6} + \sin^2\frac{\pi}{6}\right) + 2\left(\cos\frac{\pi}{8}\cos\frac{\pi}{6} + \sin\frac{\pi}{8}\sin\frac{\pi}{6}\right) =$$

$$= 1 + 1 + 2\cos\left(\frac{\pi}{6} - \frac{\pi}{8}\right) = 2 + 2\cos\left(\frac{\pi}{24}\right)$$

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59.1. $\dfrac{i^{8n-7}+\sqrt{2}\,e^{-i\frac{\pi}{4}}}{1+i\times\left(e^{i\frac{7\pi}{6}}\right)^3} = \dfrac{i^{8n}+i^{-7}+\sqrt{2}\left(\cos\frac{-\pi}{4}+i\sin\frac{-\pi}{4}\right)}{1+i\times e^{i\frac{3\times 7\pi}{6}}} =$

$$= \frac{\left(i^8\right)^n\times i^{-7+8}+\sqrt{2}\left(\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}i\right)}{1+i\,e^{i\frac{7\pi}{2}}} = \frac{1\times i+1-i}{1+i\times e^{i\frac{3\pi}{2}}} =$$

$$= \frac{1}{1+i\times(-i)} = \frac{1}{1+1} = \frac{1}{2}$$

59.2. $\dfrac{\sin\alpha+i\cos\alpha}{(\cos\alpha-i\sin\alpha)^3} = \dfrac{\cos\left(\frac{\pi}{2}-\alpha\right)+i\sin\left(\frac{\pi}{2}-\alpha\right)}{\left(\cos(-\alpha)+i\sin(-\alpha)\right)^3} =$

$$= \frac{e^{i\left(\frac{\pi}{2}-\alpha\right)}}{\left(e^{-i\alpha}\right)^3} = \frac{e^{i\left(\frac{\pi}{2}-\alpha\right)}}{e^{-i3\alpha}} = e^{i\left(\frac{\pi}{2}-\alpha+3\alpha\right)} = e^{i\left(\frac{\pi}{2}+2\alpha\right)}$$

60.1. a) $z_1 = 2\,e^{i\frac{\pi}{3}} = 2\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right) = 2\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right) = 1+\sqrt{3}i$

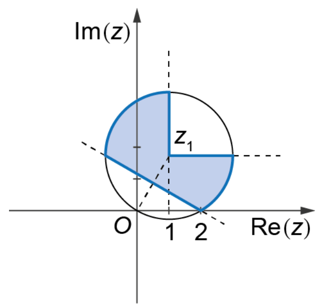
$$\frac{\left|z_1+\sqrt{3}-i\right|}{1+i} = \frac{\left|1+\sqrt{3}i+\sqrt{3}-i\right|}{1+i} = \frac{\left|\left(\sqrt{3}+1\right)+\left(\sqrt{3}-1\right)i\right|}{1+i} = \frac{\sqrt{\left(\sqrt{3}+1\right)^2+\left(\sqrt{3}-1\right)^2}}{1+i} =$$

$$= \frac{\sqrt{3+2\sqrt{3}+1+3-2\sqrt{3}+1}}{1+i} = \frac{\sqrt{8}\,(1-i)}{(1+i)(1-i)} = \frac{2\sqrt{2}\,(1-i)}{1^2-i^2} = \frac{2\sqrt{2}\,(1-i)}{2} = \sqrt{2}-\sqrt{2}i$$

b) $|z-z_1|\leq 2$ define o círculo de raio 2 cujo centro é o afixo de z_1.

$|z|\geq|z-z_1|$ define o semiplano limitado pela mediatriz do segmento cujos extremos são a origem do referencial e o afixo de z_1 ao qual pertence este último ponto.

$\frac{\pi}{2}\leq\arg(z-z_1)\leq 2\pi$ define a região do plano limitada pelas semirretas com origem no afixo de z_1 e que fazem com a parte positiva do eixo real ângulos de amplitudes $\frac{\pi}{2}$ rad e 2π rad, respetivamente (região a que pertence a origem do referencial).



Portanto, a região que se apresenta ao lado corresponde à condição

$$|z-z_1|\leq 2\wedge|z|\geq|z-z_1|\wedge\frac{\pi}{2}\leq\arg(z-z_1)\leq 2\pi$$

60.2. Se u e v são raízes cúbicas do mesmo número complexo vem $u^3=v^3$.

Se, por outro lado, $u=-v$, temos $(-v)^3=v^3\Leftrightarrow -v^3=v^3\overset{(v\neq 0)}{\Leftrightarrow}-1=1$, o que é falso.

Logo, se u e v são dois números complexos não nulos e simétricos, então não podem ser raízes cúbicas do mesmo número complexo.

61.1. $z_1=\frac{4}{-1-i\sqrt{3}}=\frac{4\left(-1+i\sqrt{3}\right)}{\left(-1-i\sqrt{3}\right)\left(-1+i\sqrt{3}\right)}=\frac{4\left(-1+i\sqrt{3}\right)}{1+3}=-1+i\sqrt{3}$

$z_2=2\,e^{i\frac{5\pi}{3}}=2\left(\cos\frac{5\pi}{3}+i\sin\frac{5\pi}{3}\right)=2\left(\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)=1-\sqrt{3}\,i$

$\frac{z_1+\bar{z}_2}{z_1^3}=\frac{\left(-1+i\sqrt{3}\right)+\left(1+\sqrt{3}\,i\right)}{\left(2\,e^{i\frac{2\pi}{3}}\right)^3}=\frac{2\sqrt{3}\,i}{2^3\,e^{i\left(3\times\frac{2\pi}{3}\right)}}=\frac{2\sqrt{3}\,i}{8}=$

$=\frac{\sqrt{3}}{4}i=\frac{\sqrt{3}}{4}e^{i\frac{\pi}{2}}$

Cálculos auxiliares

$z_1=-1+i\sqrt{3}$

$|z_1|=\sqrt{1+3}=2$

$\begin{cases}\tan(\arg z_1)=-\sqrt{3}\\ \left(-1,\sqrt{3}\right)\in 2.^{\text{o}}\,Q\end{cases}\Rightarrow \pi-\frac{\pi}{3}=\frac{2\pi}{3}$

é um argumento de z_1

$z_1=2\,e^{i\frac{2\pi}{3}}$

$e^{i2\pi}=\cos 2\pi+i\sin 2\pi=1+i\times 0=1$

61.2. $(z_1)^6=\left(2\,e^{i\frac{2\pi}{3}}\right)^6=2^6 e^{i\frac{12\pi}{3}}=64\,e^{i4\pi}=64$

$(z_2)^6=\left(2e^{i\frac{5\pi}{3}}\right)^6=2^6\,e^{i\frac{30\pi}{3}}=64\,e^{i10\pi}=64$

$z_1^6=z_2^6=64\in\mathbb{R}$

z_1 e z_2 são raízes de índice 6 de 64.

61.3. Atendendo a que:

- o centro da circunferência é o ponto $A\to\left(1,-\sqrt{3}\right)$, afixo de $1-i\sqrt{3}$;
- o raio da circunferência é $\overline{AC}=\sqrt{3}$;
- AB é uma reta paralela ao eixo real;
- $AC\perp AB$;

uma condição que defina a região colorida pode ser:

$$\left|z-\left(1-i\sqrt{3}\right)\right|\le\sqrt{3}\wedge\left(0\le \operatorname{Arg}\left(z-\left(1-i\sqrt{3}\right)\right)\le\frac{\pi}{2}\vee \operatorname{Re}(z)\le 0\right)$$

62.1. Designando por O a origem do referencial e atendendo a que:

- o ângulo que a semirreta $\dot{O}B$ faz com a parte positiva do eixo real é $\frac{5\pi}{6}$ rad;
- o ângulo que a semirreta $\dot{O}A$ faz com a parte positiva do eixo real é π rad;
- a circunferência tem centro em $A\to(-1,0)$ e raio 2;
- $-\frac{\pi}{2}\le \operatorname{Arg} z\le 0$ é uma condição que caracteriza os pontos do 4.º quadrante.

Uma condição que defina a região colorida pode ser:

$$|z-(-1)|\le 2\wedge\left(-\frac{\pi}{2}\le \operatorname{Arg} z\le 0\vee\frac{5\pi}{6}\le \operatorname{Arg} z\le\pi\right)$$

62.2. $\left(\sqrt{3}-\text{i}\right)z^3+16\,\text{e}^{\text{i}\frac{4\pi}{3}}=0 \Leftrightarrow 2\,\text{e}^{\text{i}\left(-\frac{\pi}{6}\right)}\times z^3=-16\,\text{e}^{\text{i}\frac{4\pi}{3}} \Leftrightarrow$

$\Leftrightarrow z^3=\dfrac{16\,\text{e}^{\text{i}\left(\pi+\frac{4\pi}{3}\right)}}{2\,\text{e}^{-\text{i}\frac{\pi}{6}}} \Leftrightarrow z^3=8\,\text{e}^{\text{i}\left(\pi+\frac{4\pi}{3}+\frac{\pi}{6}\right)} \Leftrightarrow$

$\Leftrightarrow z^3=8\,\text{e}^{\text{i}\frac{5\pi}{2}} \Leftrightarrow z=\sqrt[3]{8\,\text{e}^{\text{i}\frac{5\pi}{2}}} \Leftrightarrow$

$\Leftrightarrow z=\sqrt[3]{8}\,\text{e}^{\text{i}\left(\frac{\frac{5\pi}{2}+2k\pi}{3}\right)},\ k=0\,,1,\,2$

$\Leftrightarrow z=2\,\text{e}^{\text{i}\left(\frac{5\pi+4k\pi}{6}\right)},\ k=0\,,1\,,2$

$\Leftrightarrow z=2\,\text{e}^{\text{i}\frac{5\pi}{6}} \vee z=2\,\text{e}^{\text{i}\frac{3\pi}{2}} \vee z=2\text{e}^{\text{i}\frac{13\pi}{6}}$

Cálculos auxiliares

$u=\sqrt{3}-\text{i}$

$|u|=\sqrt{3+1}=2$

$\theta=\text{Arg}(u)$

$\begin{cases}\tan\theta=-\dfrac{1}{\sqrt{3}}=-\dfrac{\sqrt{3}}{3}\\ \left(\sqrt{3}\,,-1\right)\in 4.^\circ\text{Q}\end{cases}\Rightarrow\theta=-\dfrac{\pi}{6}$

$u=2\,\text{e}^{\text{i}\left(-\frac{\pi}{6}\right)}$

Como z_1 é solução da equação e $\text{Arg}(z_1)=\dfrac{\pi}{6}$, tem-se:

$z_1=2\,\text{e}^{\text{i}\frac{5\pi}{6}}=2\left(\cos\dfrac{5\pi}{6}+\text{i}\sin\dfrac{5\pi}{6}\right)=$

$2\left(-\dfrac{\sqrt{3}}{2}+\dfrac{1}{2}\,\text{i}\right)=-\sqrt{3}+\text{i}$

$z_1=-\sqrt{3}+\text{i}$

63. $z_1=(1+2\text{i})^2$ e $z_2=\dfrac{1-8\text{i}}{3-4\text{i}}$

63.1. $\dfrac{z^3-1}{z_1}=z_2 \Leftrightarrow \dfrac{z^3-1}{(1+2\text{i})^2}=\dfrac{1-8\text{i}}{3-4\text{i}} \Leftrightarrow$

$\Leftrightarrow \dfrac{z^3-1}{(1+4\text{i}-4)}=\dfrac{1-8\text{i}}{3-4\text{i}} \Leftrightarrow z^3-1=(-3+4\text{i})\times\dfrac{1-8\text{i}}{-(-3+4\text{i})} \Leftrightarrow$

$\Leftrightarrow z^3-1=-1+8\text{i} \Leftrightarrow z^3=8\text{i} \Leftrightarrow z^3=8\,\text{e}^{\text{i}\frac{\pi}{2}} \Leftrightarrow$

$\Leftrightarrow z=\sqrt[3]{8\,\text{e}^{\text{i}\frac{\pi}{2}}} \Leftrightarrow z=\sqrt[3]{8}\,\text{e}^{\text{i}\left(\frac{\frac{\pi}{2}+2k\pi}{3}\right)},\ k=0\,,1\,,2 \Leftrightarrow$

$\Leftrightarrow z=2\,\text{e}^{\text{i}\left(\frac{\pi+4k\pi}{6}\right)},\ k=0\,,1,\,2$

$\Leftrightarrow z=2\,\text{e}^{\text{i}\frac{\pi}{6}} \vee z=2\,\text{e}^{\text{i}\frac{5\pi}{6}} \vee z=2\,\text{e}^{\text{i}\frac{9\pi}{6}}$

$\Leftrightarrow z=2\left(\cos\dfrac{\pi}{6}+\text{i}\sin\dfrac{\pi}{6}\right)\vee z=2\left(\cos\dfrac{5\pi}{6}+\text{i}\sin\dfrac{5\pi}{6}\right)\vee z=2\,\text{e}^{\text{i}\frac{3\pi}{2}}$

$\Leftrightarrow z=2\left(\dfrac{\sqrt{3}}{2}+\dfrac{1}{2}\text{i}\right)\vee z=2\left(-\dfrac{\sqrt{3}}{2}+\dfrac{1}{2}\text{i}\right)\vee z=-2\text{i}$

$\Leftrightarrow z=\sqrt{3}+\text{i} \vee z=-\sqrt{3}+\text{i} \vee z=-2\text{i}$

$S=\left\{-2\text{i}\,,\ \sqrt{3}+\text{i}\,,-\sqrt{3}+\text{i}\right\}$

64.2. $\dfrac{1+e^{i(2\alpha)}}{i\sin\alpha-\cos\alpha}=\dfrac{1+\cos(2\alpha)+i\sin(2\alpha)}{-\cos\alpha+i\sin\alpha}=\dfrac{1+\cos^2\alpha-\sin^2\alpha+2i\sin\alpha\cos\alpha}{\cos(\pi-\alpha)+i\sin(\pi-\alpha)}=$

$$=\frac{\cos^2\alpha+\cos^2\alpha+2i\sin\alpha\cos\alpha}{e^{i(\pi-\alpha)}}=$$

$$=\frac{2\cos\alpha(\cos\alpha+i\sin\alpha)}{e^{i(\pi-\alpha)}}=\frac{2\cos\alpha\,e^{i\alpha}}{e^{i(\pi-\alpha)}}=$$

$$=2\cos\alpha e^{i(\alpha-\pi+\alpha)}=2\cos(-\pi+2\alpha)=$$

$$=2\cos\alpha\,e^{i(-\pi+2\alpha+2\pi)}=2\cos\alpha\,e^{i(\pi+2\alpha)}$$

65. $w=\dfrac{z}{z+i}$

Seja $z=x+yi,\ x,\ y\in\mathbb{R}$.

$$w=\frac{z}{z+i}=\frac{x+yi}{x+yi+i}=\frac{x+yi}{x+(y+1)i}=\frac{(x+yi)[x-(y+1)i]}{[x+(y+1)i][x-(y+1)i]}=$$

$$=\frac{x^2-x(y+1)i+xyi+y(y+1)}{x^2-(y+1)^2i^2}=\frac{x^2+y^2+y-xi}{x^2+(y+1)^2}=$$

$$=\frac{x^2+y^2+y-xi}{x^2+(y+1)^2}=$$

$$=\frac{x^2+y^2+y}{x^2+(y+1)^2}-\frac{x}{x^2+(y+1)^2}i$$

Se w é um número real, então $-\dfrac{x}{x^2+(y+1)^2}=0$ pelo que $x=0$.

Logo, z é um imaginário puro, pois $z=x+yi=0+yi=yi$ com $y\neq 0$.

65.1. $z_1=e^{i\frac{\pi}{2}}=i$

$$z_2=\frac{2}{i^3}=\frac{2}{-i}=\frac{2i}{-i\times i}=2i$$

$$(z_2-z_1)^{10}=(2i-i)^{10}=i^{10}=i^2=-1$$

10	4
2	2

65.2. $z^2\times z_1=\overline{z}\Leftrightarrow$ $\quad z=\rho\,e^{i\theta}$

$$\Leftrightarrow(\rho\,e^{i\theta})^2\times e^{i\frac{\pi}{2}}=\rho\,e^{i(-\theta)}\Leftrightarrow$$

$z=\rho\,e^{i\theta}$

$\overline{z}=\rho\,e^{i(-\theta)}$

$$\Leftrightarrow\rho^2e^{i(2\theta)}\times e^{i\frac{\pi}{2}}=\rho e^{i(-\theta)}\Leftrightarrow\rho^2e^{i\left(2\theta+\frac{\pi}{2}\right)}=\rho e^{i(-\theta)}\Leftrightarrow$$

$$\Leftrightarrow\begin{cases}\rho^2=\rho\\ 2\theta+\dfrac{\pi}{2}=-\theta+2k\pi,\ k\in\mathbb{Z}\end{cases}\Leftrightarrow$$

$$\Leftrightarrow \begin{cases} \rho(\rho-1)=0 \\ 3\theta = -\frac{\pi}{2}+2k\pi,\ k\in\mathbb{Z} \end{cases} \Leftrightarrow \begin{cases} \rho=0 \vee \rho=1 \\ \theta = -\frac{\pi}{6}+\frac{2k\pi}{3},\ k=0,1,2 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow z=0 \vee z=e^{i\left(-\frac{\pi}{6}\right)} \vee z=e^{i\left(-\frac{\pi}{6}+\frac{2\pi}{3}\right)} \vee z=e^{i\left(-\frac{\pi}{6}+\frac{4\pi}{3}\right)} \Leftrightarrow$$

$$\Leftrightarrow z=0 \vee z=e^{i\left(-\frac{\pi}{6}\right)} \vee z=e^{i\frac{\pi}{2}} \vee z=e^{i\frac{7\pi}{6}} \Leftrightarrow$$

$$\Leftrightarrow z=0 \vee z=\frac{\sqrt{3}}{2}-\frac{1}{2}i \vee z=i \vee z=-\frac{\sqrt{3}}{2}-\frac{1}{2}i$$

65.3. $\text{Im}(z)\geq 1 \Leftrightarrow \text{Im}(x+yi)\geq 1 \Leftrightarrow y\geq 1$ $\quad | \ z=x+yi$

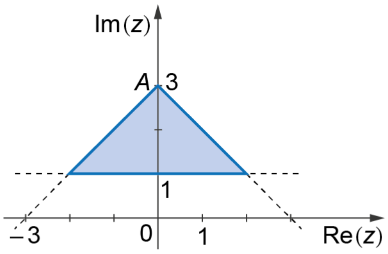
$3z_1 = 3i \ \to \ A(0,3)$

$\frac{5\pi}{4}\leq \arg(z-3z_1)\leq\frac{7\pi}{4}$ define a porção de plano limitado pelas semirretas de origem em A, afixo de $3z_1$ que fazem com a parte positiva do eixo real ângulos de $\frac{5\pi}{4}$ e $\frac{7\pi}{4}$ rad, respetivamente.

O conjunto definido por:

$$\text{Im}(z_1)\geq 1 \wedge \frac{5\pi}{4}\leq \arg(z-3z_1)\leq\frac{7\pi}{4}$$

é a interseção dos dois conjuntos anteriores referidos:



66. $z=\sqrt{2+\sqrt{3}}+\sqrt{2-\sqrt{3}}\ i$

66.1. $z^2=\left(\sqrt{2+\sqrt{3}}+\sqrt{2-\sqrt{3}}\ i\right)^2=$

$$=\left(2+\sqrt{3}+2\sqrt{2+\sqrt{3}}\sqrt{2-\sqrt{3}}\ i+\left(2-\sqrt{3}\right)i^2\right)=$$

$$=\left(2+\sqrt{3}+2\sqrt{\left(2+\sqrt{3}\right)\left(2-\sqrt{3}\right)}\ i-2+\sqrt{3}\right)=$$

$$=2\sqrt{3}+2\sqrt{4-3}\ i=2\sqrt{3}+2i$$

$$|z^2|=\sqrt{\left(2\sqrt{3}\right)^2+4}=\sqrt{16}=4$$

$$\begin{cases} \tan(\arg z^2)=\frac{2}{2\sqrt{3}}=\frac{\sqrt{3}}{3} \\ \arg z^2\in 1.^{\circ}\text{ Q} \end{cases} \Rightarrow \frac{\pi}{6} \text{ é um argumento de } z_2.$$

$$z^2=4\,e^{i\frac{\pi}{6}}$$

66.2. Se $z^2 = 4\,e^{i\frac{\pi}{6}}$:

$$z = \sqrt{4\,e^{i\frac{\pi}{6}}} = \sqrt{4}\,e^{i\left(\frac{\frac{\pi}{6}+2k\pi}{2}\right)}\,,\ k = 0\,,1$$

$$= 2\,e^{i\left(\frac{\pi+12k\pi}{12}\right)},\ k = 0\,,1$$

Para $k = 0$, $z = 2\,e^{i\frac{\pi}{12}}$

Para $k = 1$, $z = 2\,e^{i\frac{13\pi}{12}}$

Como o afixo de z pertence ao primeiro quadrante, temos $z = 2\,e^{i\frac{\pi}{12}}$, ou seja, $|z| = 2$ e $\operatorname{Arg}(z) = \frac{\pi}{12}$.

67.1. $$w = \frac{(2-i)^4 - 3i^{4n+3}}{1+2\,e^{i\frac{3\pi}{2}}} = \frac{\left((2-i)^2\right)^2 - 3i^{4n}\times i^3}{1-2i} =$$

$$= \frac{\left(4-4i+i^2\right)^2 - 3\left(i^4\right)^n\times(-i)}{1-2i} = \frac{(4-4i-1)^2 - 3\times 1^n\times(-i)}{1-2i} =$$

$$= \frac{(3-4i)^2+3i}{1-2i} = \frac{9-24i+16i^2+3i}{1-2i} = \frac{-7-21i}{1-2i} =$$

$$= \frac{(-7-21i)(1+2i)}{(1-2i)(1+2i)} = \frac{-7-21i-14i+42}{1-4i^2} = \frac{35-35i}{5} = 7-7i$$

$$|w| = \sqrt{7^2+7^2} = 7\sqrt{2}$$

$$\begin{cases}\tan(\arg\ w) = \dfrac{-7}{7} = -1 \\ \arg w \in 4.^\circ\ Q\end{cases} \Rightarrow -\frac{\pi}{4} \text{ é um argumento de } w$$

$$w = 7\sqrt{2}\,e^{i\left(-\frac{\pi}{4}\right)}$$

67.2. $z_1 = e^{i\alpha}$

$$u = z_1 - i\,\overline{z}_1 = e^{i\alpha} - i\,e^{i(-\alpha)} = e^{i\alpha} - e^{i\frac{\pi}{2}}\times e^{i(-\alpha)} =$$

$$= e^{i\alpha} - e^{i\left(\frac{\pi}{2}-\alpha\right)} =$$

$$= \cos\alpha + i\sin\alpha - \cos\left(\frac{\pi}{2}-\alpha\right) - i\sin\left(\frac{\pi}{2}-\alpha\right) =$$

$$= \cos\alpha + i\sin\alpha - \sin\alpha - i\cos\alpha =$$

$$= (\cos\alpha - \sin\alpha) + i(-\cos\alpha + \sin\alpha)$$

Como $\operatorname{Re}(u) = -\operatorname{Im}(u)$, o afixo, no plano complexo, do número complexo u pertence à bissetriz dos quadrantes pares (reta de equação $y = -x$).

68. $z_1 = e^{i\frac{\pi}{12}}$; $z_2 = 1-\sqrt{3}i$; $z_3 = 2\sqrt{2}e^{i\pi}$

68.1. $|z_2| = \sqrt{1+3} = 2$

Seja θ_2 um argumento de z_2

$$\begin{cases} \tan\theta_2 = -\sqrt{3} \\ \theta_2 \in 4.^\text{o}\ Q \end{cases} \Rightarrow \theta_2 = -\frac{\pi}{3}$$

$$z_2 = 2e^{i\left(-\frac{\pi}{3}\right)}$$

$$\frac{\overline{z_1}}{z_2} \times z_3 = \frac{e^{i\left(-\frac{\pi}{12}\right)}}{2e^{i\left(-\frac{\pi}{3}\right)}} \times 2\sqrt{2}\,e^{i\pi} = \frac{1}{2}e^{i\left(-\frac{\pi}{12}+\frac{\pi}{3}\right)} \times 2\sqrt{2}\,e^{i\pi} = \frac{1}{2} \times 2\sqrt{2}\,e^{i\left(-\frac{\pi}{12}+\frac{\pi}{3}+\pi\right)} =$$

$$= \sqrt{2}\,e^{i\left(\frac{5\pi}{4}\right)} = \sqrt{2}\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right) = \sqrt{2}\left(-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i\right) = -1-i$$

68.2. $z^2 = \bar{z} \times z_2 \Leftrightarrow \left(\rho\,e^{i\theta}\right)^2 = \rho\,e^{i(-\theta)} \times 2\,e^{i\left(-\frac{\pi}{3}\right)} \Leftrightarrow$ | $z = \rho\,e^{i\theta}$

$$\Leftrightarrow \rho^2\,e^{i(2\theta)} = 2\,\rho e^{i\left(-\theta-\frac{\pi}{3}\right)} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} \rho^2 = 2\rho \\ 2\theta = -\theta - \frac{\pi}{3} + 2k\pi,\ k \in \mathbb{Z} \end{cases} \Leftrightarrow \begin{cases} \rho(\rho-2) = 0 \\ 3\theta = \frac{-\pi}{3} + 2k\pi,\ k \in \mathbb{Z} \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} \rho = 0 \vee (\rho-2) = 0 \\ \theta = \frac{-\pi}{9} + \frac{2k\pi}{3},\ k \in \mathbb{Z} \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} \rho = 0 \vee (\rho-2) = 0 \\ \theta = \frac{-\pi}{9} + \frac{2k\pi}{3},\ k \in \mathbb{Z} \end{cases} \Leftrightarrow \begin{cases} \rho = 2 \\ \theta = \frac{-\pi}{9} + \frac{2k\pi}{3},\ k \in \mathbb{Z} \end{cases} \Leftrightarrow$$

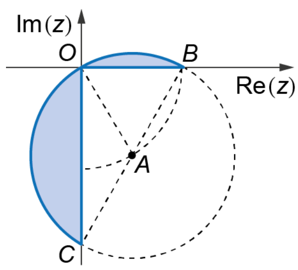
$$\Leftrightarrow z = 0 \vee z = e^{i\left(-\frac{\pi}{9}\right)} \vee z = 2\,e^{i\frac{5\pi}{9}} \vee z = 2e^{i\frac{11\pi}{9}}$$

$$S = \left\{0,\ e^{i\left(-\frac{\pi}{9}\right)},\ 2\,e^{i\frac{5\pi}{9}},\ 2e^{i\frac{11\pi}{9}}\right\}$$

68.3. $z_2 = 2\,e^{i\left(-\frac{\pi}{3}\right)} = 1-\sqrt{3}\,i \curvearrowright A$

$$|z - z_2| \le |z_2| \wedge 0 \le \text{Arg}(z) \le \frac{3\pi}{2} \Leftrightarrow$$

$$\Leftrightarrow \left|z - \left(1-\sqrt{3}i\right)\right| \le 2 \wedge 0 \le \arg(z) \le \frac{3\pi}{2}$$



[*BOA*] é um triângulo isósceles.

- $\overline{OB} = 2 \times |\text{abcissa de } a| = 2 \times 1 = 2$
- [*AOC*] é um triângulo isósceles
- $\overline{OC} = 2 \times |\text{ordenada de } A| = 2\sqrt{3}$

$$A_{[CBO]} = \frac{2 \times 2\sqrt{3}}{2} = 2\sqrt{3}$$

$$A_{\text{parte colorida}} = \frac{1}{2} \text{ área do círculo} - A_{[CBO]} = \frac{1}{2}\pi \times 2^2 - 2\sqrt{3} = 2\pi - 2\sqrt{3}$$

69. Seja r o raio da circunferência e seja θ um argumento de z. Então:

$$z = r\,e^{i\theta} \text{ e } w = r\,e^{i\left(\theta + \frac{2\pi}{3}\right)}$$

$$z + w = r\,e^{i\theta} + r\,e^{i\left(\theta + \frac{2\pi}{3}\right)} = r\left(e^{i\theta} + e^{i\theta} \times e^{i\frac{2\pi}{3}}\right) =$$

$$= r\,e^{i\theta}\left(1 + e^{i\frac{2\pi}{3}}\right) = r\,e^{i\theta}\left(1 + \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$$

$$= r\,e^{i\theta}\left(1 - \frac{1}{2} + \frac{\sqrt{3}}{2}i\right) = r\,e^{i\theta}\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) =$$

$$= r\,e^{i\theta}\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right) = r\,e^{i\theta} \times e^{i\frac{\pi}{3}} = r\,e^{i\left(\theta + \frac{\pi}{3}\right)}$$

Como $|z + w| = r$, o afixo de $z + w$ também pertence à circunferência de centro O e raio r.

70. • $u^3 = v^3 = z$

• $u = 2\,e^{i\theta}$ com $\theta \in \left]\pi, \frac{3\pi}{2}\right[$

• $w = -\bar{u} = -2\,e^{i(-\theta)} = 2\,e^{i(\pi - \theta)}$

$$u^3 = v^3 \Leftrightarrow \left(2\,e^{i\theta}\right)^3 = \left[2\,e^{i(\pi-\theta)}\right]^3 \Leftrightarrow$$

$$\Leftrightarrow 8\,e^{i(3\theta)} = 8\,e^{i(3\pi - 3\theta)} \Leftrightarrow 3\theta = 3\pi - 3\theta + 2k\pi,\ k \in \mathbb{Z} \Leftrightarrow$$

$$\Leftrightarrow 6\theta = 3\pi + 2k\pi,\ k \in \mathbb{Z} \Leftrightarrow \theta = \frac{\pi}{2} + \frac{k\pi}{3},\ k \in \mathbb{Z}$$

Como $\theta \in \left]\pi, \frac{3\pi}{2}\right[$, vem $\theta = \frac{\pi}{2} + \frac{2\pi}{3} = \frac{7\pi}{6}$.

$$z = u^3 = \left(2\cos\frac{7\pi}{6}\right)^3 = 8\,e^{i\left(\frac{3\times7\pi}{6}\right)} = 8\,e^{i\frac{7\pi}{2}} = 8e^{i\frac{3\pi}{2}} = -8i$$

71. $z_0 = 2i = 2\,e^{i\frac{\pi}{2}}$ é uma das raízes cúbicas de z.

71.1. As outras raízes são:

$$z_1 = z_0 \times e^{i\frac{2\pi}{3}} = 2\,e^{i\frac{\pi}{2}} \times e^{i\frac{2\pi}{3}} = 2\ e^{i\left(\frac{\pi}{2}+\frac{2\pi}{3}\right)} = 2\ e^{i\frac{7\pi}{6}} =$$

$$= 2\left(\cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6}\right) = 2\left(-\frac{\sqrt{3}}{2} - \frac{1}{2}i\right) = -\sqrt{3} - i$$

$$z_2 = z_1 \times e^{i\frac{2\pi}{3}} = 2\,e^{i\frac{7\pi}{6}}\,e^{i\frac{2\pi}{3}} = 2\,e^{i\left(\frac{7\pi}{6}+\frac{2\pi}{3}\right)} = 2e^{i\frac{11\pi}{6}} =$$

$$= 2\left(\cos\frac{11\pi}{6} + i\sin\frac{11\pi}{6}\right) = 2\left(\frac{\sqrt{3}}{2} - \frac{1}{2}i\right) = \sqrt{3} - i$$

71.2. a) Trata-se de um triângulo equilátero inscrito numa circunferência de raio 2.

360º : 3 = 120º

120º : 2 = 60º

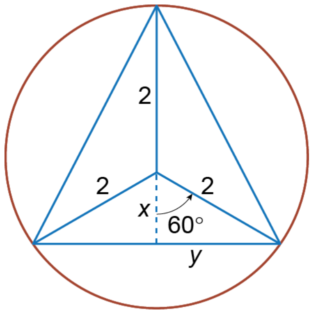
$\frac{x}{2} = \cos 60° \Leftrightarrow x = 2 \times \frac{1}{2} \Leftrightarrow x = 1$

$\frac{y}{2} = \sin 60° \Leftrightarrow y = 2 \times \frac{\sqrt{3}}{2} \Leftrightarrow y = \sqrt{3}$

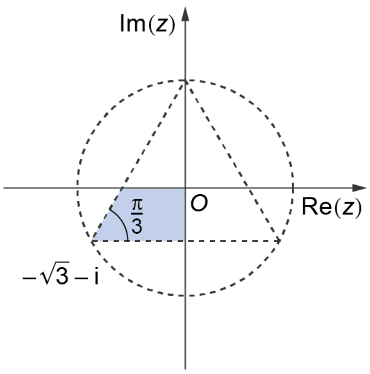
Base do triângulo: $2y = 2\sqrt{3}$

Altura do triângulo: $2 + x = 2 + 1 = 3$

Área do triângulo: $\frac{2\sqrt{3} \times 3}{2} = 3\sqrt{3}$ u.a.



b)



Por exemplo: $0 \le \text{Arg}\left(z + \sqrt{3} + i\right) \le \frac{\pi}{3} \wedge \text{Re}(z) < 0 \wedge \text{Im}(z) < 0$

72. $w_1 = -1 - i$; $w_2 = e^{i\alpha}$, $\alpha \in \mathbb{R}$

72.1. $w_1 = -1 - i = \sqrt{2}\ e^{i\frac{5\pi}{4}}$

$$\frac{w_1^{\,5}}{1 + w_1} = \frac{\left(\sqrt{2}\ e^{i\frac{5\pi}{4}}\right)^5}{1 + (-1 - i)} =$$

$$= \frac{\left(\sqrt{2}\right)^5 e^{i\frac{25\pi}{4}}}{-i} = \frac{\left(\sqrt{2}\right)^4 \sqrt{2}\ e^{i\frac{\pi}{4}}}{e^{i\left(-\frac{\pi}{2}\right)}} =$$

$$= 4\sqrt{2}\ e^{i\left(\frac{\pi}{4}+\frac{\pi}{2}\right)} = 4\sqrt{2}\ e^{i\frac{3\pi}{4}} =$$

$$= 4\sqrt{2}\left(-\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\,i\right) = -4 + 4i$$

$w_1 = -1 - i = |w_1| e^{i\theta_1}$

$|w_1| = \sqrt{1+1} = \sqrt{2}$

$\begin{cases} \tan\theta_1 = \frac{-1}{-1} = 1 \\ (-1, -1) \in 3.º\ Q \end{cases} \Rightarrow \frac{5\pi}{4}$ é um argumento de w_1

$w_1 = \sqrt{2}\ e^{i\frac{5\pi}{4}}$

$\frac{25\pi}{4} - 6\pi = \frac{\pi}{4}$

72.2. $\overline{w_1 w_2} = \sqrt{2} w_2 \Leftrightarrow \overline{\sqrt{2}\ e^{i\left(\frac{5\pi}{4}\right)} \times e^{i\alpha}} = \sqrt{2}\ e^{i\alpha} \Leftrightarrow$

$\Leftrightarrow \overline{\sqrt{2}\ e^{i\left(\alpha + \frac{5\pi}{4}\right)}} = \sqrt{2}\, e^{i\alpha} \Leftrightarrow$

$\Leftrightarrow \sqrt{2}\, e^{i\left(-\alpha - \frac{5\pi}{4}\right)} = \sqrt{2}\, e^{i\alpha} \Leftrightarrow$

$\Leftrightarrow \sqrt{2} = \sqrt{2} \wedge -\alpha - \frac{5\pi}{4} = \alpha + 2k\pi,\ k \in \mathbb{Z} \Leftrightarrow$

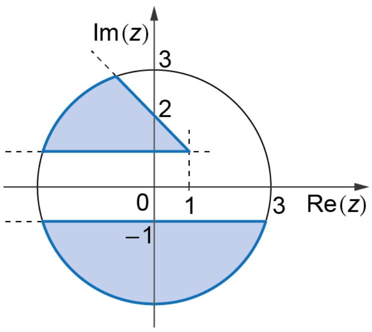
$\Leftrightarrow 2\alpha = -\frac{5\pi}{4} + 2k\pi,\ k \in \mathbb{Z} \Leftrightarrow \alpha = -\frac{5\pi}{8} + k\pi,\ k \in \mathbb{Z}$

72.3. $|z| \leq 3 \rightarrow$ Círculo fechado de centro na origem e raio 3

$\frac{3\pi}{4} \leq \operatorname{Arg}\,(z + w_1) \leq \pi \rightarrow$ Porção de plano limitada pelas semirretas com origem em $A(1, 1)$, afixo de $-w_1$, e que fazem com a parte positiva do eixo real ângulos de $\frac{3\pi}{4}$ e π radianos, respetivamente.

$\operatorname{Im} z \leq \operatorname{Im} w_1 \Leftrightarrow \operatorname{Im}(x + y\mathrm{i}) \leq \operatorname{Im}(-1 - \mathrm{i}) \Leftrightarrow y \leq -1$ $\quad | z = x + y\mathrm{i}$

Conjunto definido pela condição dada:



73. $z_1 = \mathrm{i}\ ;\ z_2 = e^{i\frac{5\pi}{6}}\ ;\ z_3 = e^{i\theta}$

$z_1^n = z_2^n = w$

73.1. $z_1 = \mathrm{i} = e^{i\frac{\pi}{2}}\ ;\ \ z_2 = e^{i\frac{5\pi}{6}}$

As imagens geométricas de z_1 e z_2 são vértices de um polígono regular de n lados, com centro na origem do referencial.

O ângulo ao centro definido por dois vértices consecutivos tem amplitude $\frac{2\pi}{n}$. Então

$\arg(z_2) - \arg(z_1) = k \times \frac{2\pi}{n},\ k \in \mathbb{Z}$

$\frac{5\pi}{6} - \frac{\pi}{2} = \frac{2k\pi}{n},\ k \in \mathbb{Z} \Leftrightarrow \frac{1}{3} = \frac{2k}{n},\ k \in \mathbb{Z} \Leftrightarrow n = 6k,\ k \in \mathbb{Z}$

O menor valor de $n \in \mathbb{N}$ é 6 (para k = 1).

73.2. a) O valor mínimo de P ocorre se n for 6. Neste caso, o polígono é um hexágono regular onde as imagens de z_1 e z_2 são vértices consecutivos

$$P = |z_1 - z_2| = \left|\mathrm{i} - e^{\mathrm{i}\frac{5\pi}{6}}\right| = \left|\mathrm{i} - \cos\frac{5\pi}{6} - \mathrm{i}\sin\frac{5\pi}{6}\right| = \left|\frac{\sqrt{3}}{2} + \mathrm{i} - \frac{1}{2}\mathrm{i}\right| = \left|\frac{\sqrt{3}}{2} + \frac{1}{2}\mathrm{i}\right| = \sqrt{\frac{3}{4} + \frac{1}{4}} = 1$$

b) Qualquer que seja o valor de n, P é um polígono de n lados inscrito numa circunferência de raio igual a $|z_1| = |z_2| = |\,\mathrm{i}\,| = 1$, esta circunferência tem perímetro igual a $2\pi \times 1 = 2\pi$.

Qualquer polígono inscrito na circunferência tem um perímetro interior ao perímetro desta. Logo, $P < 2\pi$, qualquer que seja $n \in \mathbb{N}$.

73.3. $$w = \frac{z_3}{z_1 - z_2} = \frac{z_3}{\frac{\sqrt{3}}{2} + \frac{1}{2}\mathrm{i}} = \frac{e^{\mathrm{i}\theta}}{\cos\frac{\pi}{6} + \mathrm{i}\sin\frac{\pi}{6}} = \frac{e^{\mathrm{i}\theta}}{e^{\mathrm{i}\frac{\pi}{6}}} = e^{\mathrm{i}\left(\theta - \frac{\pi}{6}\right)}$$

$$w \in \mathbb{R}^- \Leftrightarrow \theta - \frac{\pi}{6} = \pi + 2k\pi,\ k \in \mathbb{Z} \Leftrightarrow \theta = \frac{7\pi}{6} + 2k\pi,\ k \in \mathbb{Z}$$

Como $\theta \in \left]-\pi, \pi\right[$, $\theta = \frac{7\pi}{6} - 2\pi = -\frac{5\pi}{6}$.

73.4. $$z\bar{z} \geq 1 \Leftrightarrow |z|^2 \geq 1 \overset{|z|>0}{\Leftrightarrow} |z| \geq 1$$

$z + \bar{z} \geq 0 \Leftrightarrow$

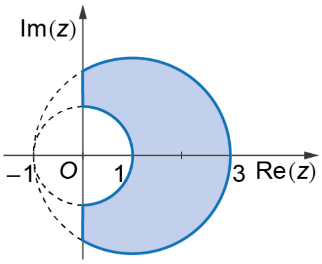
$\Leftrightarrow x + y\mathrm{i} + x - y\mathrm{i} \geq 0 \Leftrightarrow$ $\quad | \ z = x + y\mathrm{i}$

$\Leftrightarrow 2x \geq 0 \Leftrightarrow x \geq 0$

$z\bar{z} - 2(z + \bar{z}) \leq 3 \Leftrightarrow$ $\quad | \ z = x + y\mathrm{i}$

$\Leftrightarrow x^2 + y^2 - 2x \leq 3 \Leftrightarrow x^2 + y^2 - 2x \leq 3 \Leftrightarrow$

$\Leftrightarrow x^2 - 2x + 1 + y^2 - 1 \leq 3 \Leftrightarrow (x-1)^2 + y^2 \leq 4$ (círculo de centro em $A(1, 0)$ e raio 2)



74. Seja $w = \rho\, e^{\mathrm{i}\theta}$.

74.1. Se w e $\bar{w}$ são raízes de índice n de um mesmo número complexo z, então:

$w^n = \bar{w}^n = z$

$$w^n = \bar{w}^n \Leftrightarrow \left(\rho e^{\mathrm{i}\theta}\right)^n = \left(\rho e^{\mathrm{i}(-\theta)}\right)^n \Leftrightarrow \rho^n\, e^{\mathrm{i}(n\theta)} = \rho^n\, e^{\mathrm{i}(-n\theta)} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} \rho^n = \rho^n \\ n\theta = -n\theta + 2k\pi,\ k \in \mathbb{Z} \end{cases} \Leftrightarrow 2n\theta = 2k\pi,\ k \in \mathbb{Z} \Leftrightarrow n\theta = k\pi,\ k \in \mathbb{Z}$$

Como $z = w^n = \left(\rho\, e^{\mathrm{i}\theta}\right)^n = \rho^n e^{\mathrm{i}(n\theta)}$, se $n\theta = k\pi$ com $k \in \mathbb{Z}$, então $z = \rho^n\, e^{\mathrm{i}(k\pi)}$ é um número real.

74.2. Se w e $-w$ são raízes de índice n de um mesmo número complexo z, então:

$$w^n = (-w)^n = z$$

$$w^n = (-w)^n \Leftrightarrow \left(\rho\, e^{i\theta}\right)^n = \left(\rho\, e^{i(\pi+\theta)}\right)^n \Leftrightarrow$$

$$\Leftrightarrow \rho^n\, e^{i(n\theta)} = \rho^n\, e^{i(n\pi+n\theta)} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} \rho^n = \rho^n \\ n\theta = n\pi + n\theta + 2k\pi,\ k \in \mathbb{Z} \end{cases} \Rightarrow n\pi = 2k\pi,\ k \in \mathbb{Z} \Rightarrow n = 2k,\ k \in \mathbb{Z}$$, ou seja, n é um número par.

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1. $\arg(z) = \frac{6\pi}{5}$

$$\arg(\bar{z}) = -\frac{6\pi}{5} + 2k\pi,\ k \in \mathbb{Z}$$

$$\arg(-\bar{z}) = -\frac{6\pi}{5} + \pi + 2k\pi,\ k \in \mathbb{Z}$$

Para $k = 0$, $\arg(-\bar{z}) = -\frac{6\pi}{5} + \pi = -\frac{\pi}{5}$

Para $k = 1$, $\arg(-\bar{z}) = -\frac{6\pi}{5} + \pi + 2\pi = \frac{9\pi}{5}$

Resposta: (C)

2. $\arg(z) = \frac{\pi}{5}$

$$\arg(\bar{z}) = -\frac{\pi}{5} + 2k\pi$$

$$\arg(i\bar{z}) = -\frac{\pi}{5} + \frac{\pi}{2} + 2k\pi,\ k \in \mathbb{Z}$$

Para $k = 0$, $\arg(i\bar{z}) = -\frac{\pi}{5} + \frac{\pi}{2} = \frac{3\pi}{10}$

Resposta: (A)

3. $z = 2\ e^{i\frac{\pi}{5}}$

$$\arg(-z) = \frac{\pi}{5} + \pi + 2k\pi$$

$$\arg\left(\overline{-z}\right) = -\frac{\pi}{5} - \pi + 2k\pi,\ k \in \mathbb{Z}$$

Para $k = 0$, $\arg\left(\overline{-z}\right) = -\frac{\pi}{5} - \pi = -\frac{6\pi}{5}$

Para $k = 1$, $\arg\left(\overline{-z}\right) = -\frac{\pi}{5} - \pi + 2\pi = \frac{4\pi}{5}$

Resposta: (C)

4. $z = 2e^{i\frac{11\pi}{6}} - i$

$$z = 2\left(\cos\frac{11\pi}{6} + i\sin\frac{11\pi}{6}\right) - i =$$

$$= 2\left(\frac{\sqrt{3}}{2} - \frac{1}{2}i\right) - i = -\sqrt{3} - i - i = \sqrt{3} - 2i$$

$$|z| = \sqrt{3+4} = \sqrt{7}$$

Resposta: (B)

5. As n raízes de índice n de um número complexo z têm o mesmo módulo. Logo, $|u| = |v|$.

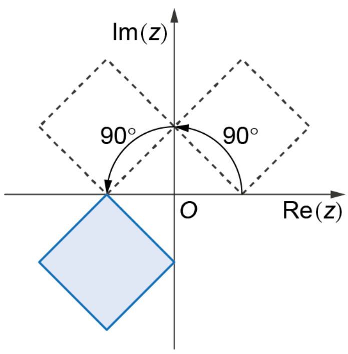
Resposta: (C)

6. $iz_1 \rightarrow$ rotação de centro na origem e amplitude 90°;

$\overline{iz_1} \rightarrow$ simetria relativamente ao eixo real.

Como os quatro vértices sofrem as mesmas transformações, o quadrado obtido é a imagem do quadrado dado por uma rotação de centro na origem e amplitude 90° seguida de uma simetria relativamente ao eixo real.

Resposta: (C)



7. $2e^{i\times 0} \curvearrowright A$

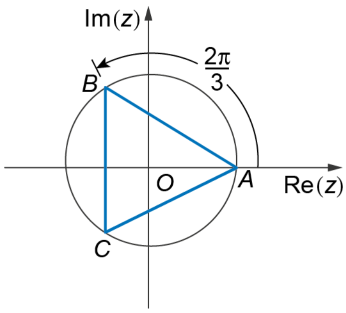
$2e^{i\left(0+\frac{2\pi}{3}\right)} \curvearrowright B$

$$2e^{i\frac{2\pi}{3}} = 2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right) =$$

$$= 2\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) = -1 + \sqrt{3}i$$

$-1 + \sqrt{3}i \curvearrowright B$

Resposta: (C)



8. A diferença dos argumentos de duas raízes de índice n "consecutivas" é $\frac{2\pi}{n}$ (os afixos das n raízes são os vértices de um polígono regular de n lados, de centro na origem do referencial).

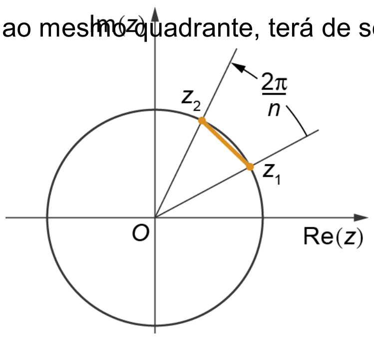
Então, se os afixos de z_1 e z_2 pertencem ao mesmo quadrante, terá de ser:

$$\frac{2\pi}{n} < \frac{\pi}{2} \Leftrightarrow$$

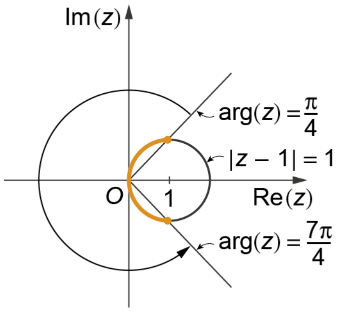
$$\Leftrightarrow \frac{2}{n} < \frac{1}{2} \Leftrightarrow n > 4$$

Logo, o valor de n pode ser 5.

Resposta: (D)



9. $\left\{ z \in \mathbb{C} : \ |z-1| = 1 \wedge \frac{\pi}{4} \leq \arg(z) \leq \frac{7\pi}{4} \right\}$



Resposta: (A)

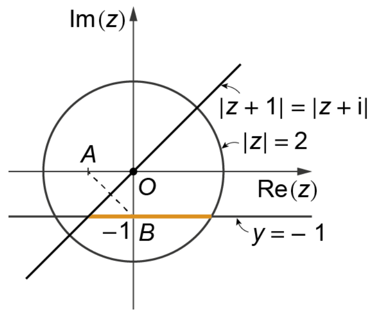
10. $\left\{ z \in \mathbb{C} : |z| \leq 2 \vee \operatorname{Re}(i z) = 1 \vee |z+1| \geq |z+i| \right\}$

$\operatorname{Re}(i z) = 1 \Leftrightarrow \operatorname{Re}(i(x+yi)) = 1 \Leftrightarrow \operatorname{Re}(x i - y) = 1 \Leftrightarrow -y = 1 \Leftrightarrow y = -1$

$|z+1| \geq |z+i| \Leftrightarrow |z-(-1+0i)| \geq |z-(0-i)|$

$A(-1\ ,\ 0)$ e $B(0\ ,\ -1)$

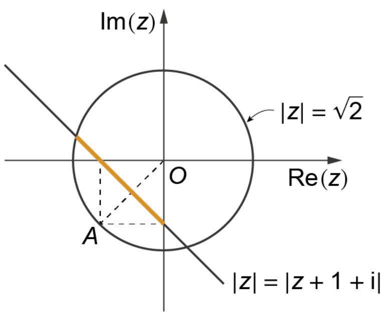
Resposta: (A)



11. $\left\{ z \in \mathbb{C} : |z| < \sqrt{2} \ \wedge |z| = |z+1+i| \right\}$

$|z| = |z+1+i| \Leftrightarrow |z-(0+0i)| = |z-(-1-i)|$

$O(0\ ,\ 0)$ e $A(-1\ ,\ -1)$



Resposta: (D)

12. $z = (1+i)(k+2i),\ k \in \mathbb{R}$

12.1. $z = k + 2i + ki + 2i^2 \Leftrightarrow z = (k-2) + (2+k)i$

z é imaginário puro $\Leftrightarrow \operatorname{Re}(z) = 0 \wedge \operatorname{Im}(z) \neq 0 \Leftrightarrow k - 2 = 0 \wedge 2 + k \neq 0 \Leftrightarrow k = 2$

12.2. $z = \bar{z} \Leftrightarrow (k-2) + (2+k)i = (k-2) - (2+k)i \Leftrightarrow$

$$\Leftrightarrow \begin{cases} k-2 = k-2 \\ 2+k = -(2+k) \end{cases} \Leftrightarrow \begin{cases} 0 = 0 \\ 2k = -4 \end{cases} \Leftrightarrow k = -2$$

13. $P(z) = z^3 + 8$

13.1. $P(-2)=(-2)^3+8=-8+8=0$

$P(z)=(z+2)(z^2-2z+4)$

	1	0	0	8
−2		−2	4	−8
	1	−2	4	0

$z^2-2z+4=0 \Leftrightarrow z=\dfrac{2\pm\sqrt{4-16}}{2} \Leftrightarrow$

$\Leftrightarrow z=\dfrac{2\pm\sqrt{-12}}{2} \Leftrightarrow z=\dfrac{2\pm i\sqrt{12}}{2} \Leftrightarrow$

$\Leftrightarrow z=\dfrac{2\pm 2i\sqrt{3}}{2} \Leftrightarrow z=1\pm\sqrt{3}i$

$\{-2\ ,\ 1-\sqrt{3}\,i\ ,\ 1+\sqrt{3}\,i\}$

13.2. Seja $z=a+bi\ (a,\ b\in\mathbb{R})$ um número complexo.

$z=\bar{z}\Rightarrow a+bi=a-bi\Rightarrow b=-b\Rightarrow 2b=0\Rightarrow b=0\Rightarrow z\in\mathbb{R}$

14.

14.1. $z^3-4z^2+7z=0\Leftrightarrow z(z^2-4z+7)=0\Leftrightarrow$

$\Leftrightarrow z=0\vee z^2-4z+7=0\Leftrightarrow z=0\vee z=\dfrac{4\pm\sqrt{16-28}}{2}\Leftrightarrow$

$\Leftrightarrow z=0\vee z=\dfrac{4\pm\sqrt{212}}{2}\Leftrightarrow z=0\vee z=\dfrac{4\pm i\times 2\sqrt{3}}{2}\Leftrightarrow z=0\vee z=2-\sqrt{3}i\vee z=2+\sqrt{3}i$

$S=\{0,\ 2-\sqrt{3}\,i,\ 2+\sqrt{3}\,i\}$

14.2. Seja $z=a+bi;\ a,\ b\in\mathbb{R}$

$iz+\dfrac{\bar{z}}{i}=i(a+bi)+\dfrac{(a-bi)}{i}=ai-b+\dfrac{(a-bi)\times(-i)}{i\times(-i)}=\ ai-b+\dfrac{-ai-b}{1}=ai-b-ai-b=$

$=-2b\in\mathbb{R}$

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15.

15.1. $(-1+i)^7=\left(\sqrt{2}\ e^{i\frac{3\pi}{4}}\right)^7=$

$=\left(\sqrt{2}\right)^7\ e^{i\frac{21\pi}{4}}=$

$=\sqrt{2}\left(\sqrt{2}\right)^6\ e^{i\left(\frac{21\pi}{4}-4\pi\right)}=$

$=\sqrt{2}\times 2^3 e^{i\frac{5\pi}{4}}=$

$=8\sqrt{2}\left(\cos\dfrac{5\pi}{4}+i\sin\dfrac{5\pi}{4}\right)=$

$=8\sqrt{2}\left(-\dfrac{\sqrt{2}}{2}-\dfrac{\sqrt{2}}{2}i\right)=8(-1-i)=-8(1+i)$

Cálculos auxiliares

$w=-1+i$

$|w|=\sqrt{1+1}=\sqrt{2}$

$\operatorname{Arg} z=\theta$

$\begin{cases}\tan\theta=-1\\ \theta\in 2.^{\circ}Q\end{cases}\Rightarrow\theta=\pi-\dfrac{\pi}{4}=\dfrac{3\pi}{4}$

$w=\sqrt{2}e^{i\frac{3\pi}{4}}$

15.2. $w=\left(\dfrac{-1+i}{1+i\sqrt{3}}\right)^{10}=$

$$=\left(\dfrac{\sqrt{2}e^{i\frac{3\pi}{4}}}{2e^{i\frac{\pi}{3}}}\right)^{10}=$$

$$=\left(\dfrac{\sqrt{2}}{2}e^{i\left(\frac{3\pi}{4}-\frac{\pi}{3}\right)}\right)^{10}=\left(\dfrac{\sqrt{2}}{2}e^{i\frac{5\pi}{12}}\right)^{10}=$$

$$=\left(\dfrac{\sqrt{2}}{2}\right)^{10}e^{i\frac{50\pi}{12}}=\dfrac{2^5}{2^{10}}e^{i\left(\frac{25\pi}{6}-4\pi\right)}=\dfrac{1}{32}e^{i\frac{\pi}{6}}$$

$|w|=\dfrac{1}{32}$; $\text{Arg}\,w=\dfrac{\pi}{6}$

Cálculos auxiliares

$w=-1+i=\sqrt{2}\ e^{i\frac{3\pi}{4}}$

$u=1+i\sqrt{3}$

$|u|=\sqrt{1+3}=2$

$\text{Arg}\,u=\alpha$

$\begin{cases}\tan\alpha=\dfrac{\sqrt{3}}{1}=\sqrt{3}\\ \alpha\in 1.^\circ\ Q\end{cases}\Rightarrow\alpha=\dfrac{\pi}{3}$

$u=2e^{i\frac{\pi}{3}}$

15.3. $z^3+4\sqrt{2}=4\sqrt{2}i\Leftrightarrow$

$$\Leftrightarrow z^3=-4\sqrt{2}+4\sqrt{2}\,i\Leftrightarrow$$

$$\Leftrightarrow z=\sqrt[3]{-4\sqrt{2}+4\sqrt{2}\ i}\Leftrightarrow$$

$$\Leftrightarrow z=\sqrt[3]{8e^{i\frac{3\pi}{4}}}\Leftrightarrow$$

$$\Leftrightarrow z=\sqrt[3]{8}\ e^{i\left(\frac{\frac{3\pi}{4}+2k\pi}{3}\right)},\ k=0,1,2\Leftrightarrow$$

$$\Leftrightarrow z=2\ e^{i\left(\frac{3\pi+8k\pi}{12}\right)},\ k=0,1,2\Leftrightarrow$$

$$\Leftrightarrow z=2\ e^{i\frac{3\pi}{12}}\vee z=2\ e^{i\frac{11\pi}{12}}\vee z=2\ e^{i\frac{19\pi}{12}}\Leftrightarrow$$

$$\Leftrightarrow z=2e^{i\frac{\pi}{4}}\vee z=2\ e^{i\frac{11\pi}{12}}\vee z=2\ e^{i\frac{19\pi}{12}}$$

Cálculos auxiliares

$u=-4\sqrt{2}+4\sqrt{2}i$

$|u|=\sqrt{16\times2+16\times2}=8$

$\text{Arg}\,u=\theta$

$\begin{cases}\tan\theta=-1\\ \theta\in 2.^\circ Q\end{cases}\Rightarrow\theta=\pi-\dfrac{\pi}{4}\Rightarrow\theta=\dfrac{3\pi}{4}$

16.

16.1. $z_1=\dfrac{2}{1-i^{25}}+e^{i\pi}=\dfrac{2}{1-i}-1=$

$$=\dfrac{2(1+i)}{(1-i)(1+i)}-1=\dfrac{2+2i}{1-i^2}-1=\dfrac{2+2i}{2}-1$$

$$=1+i-1=i$$

z_1 é um imaginário puro porque $\text{Re}(z_1)=0$.

16.2. $z^2-8z+18=0\Leftrightarrow z=\dfrac{8\pm\sqrt{64-72}}{2}\Leftrightarrow$

$$\Leftrightarrow z=\dfrac{8\pm\sqrt{-8}}{2}\Leftrightarrow z=\dfrac{8\pm i\sqrt{8}}{2}\Leftrightarrow$$

$$\Leftrightarrow z=\dfrac{8\pm2\sqrt{2}\,i}{2}\Leftrightarrow z=4\pm\sqrt{2}\,i\Leftrightarrow$$

$$\Leftrightarrow z=4-\sqrt{2}\,i\vee z=4+\sqrt{2}\,i$$

17. $w=\frac{1}{2}+\frac{\sqrt{3}}{2}\mathrm{i}$

17.1. $|w|=\sqrt{\frac{1}{4}+\frac{3}{4}}=1$

$$\begin{cases}\tan\theta=\dfrac{\frac{\sqrt{3}}{2}}{\frac{1}{2}}=\sqrt{3}\\ \theta\in 1.^{\text{o}}\ \text{Q}\end{cases}\Rightarrow \theta=\frac{\pi}{3}$$

$w=e^{\mathrm{i}\frac{\pi}{3}}$

$w^{12}=\left(e^{\mathrm{i}\frac{\pi}{3}}\right)^{12}=e^{\mathrm{i}\frac{12\pi}{3}}=e^{\mathrm{i}(4\pi)}=e^{\mathrm{i}\times 0}=\cos 0+\mathrm{i}\sin 0=1+0\mathrm{i}=1$

17.2. $2z^2+3\mathrm{i}\,z+2=0 \Leftrightarrow z=\frac{-3\mathrm{i}\pm\sqrt{(3\mathrm{i})^2-4\times2\times2}}{2\times2}\Leftrightarrow$

$\Leftrightarrow z=\frac{-3\mathrm{i}\pm\sqrt{-9-16}}{4}\Leftrightarrow z=\frac{-3\mathrm{i}\pm\sqrt{-25}}{4}\Leftrightarrow z=\frac{-3\mathrm{i}\pm5\mathrm{i}}{4}\Leftrightarrow$

$\Leftrightarrow z=\frac{-3\mathrm{i}-5\mathrm{i}}{4}\vee z=\frac{-3\mathrm{i}+5\mathrm{i}}{4}\Leftrightarrow$

$\Leftrightarrow z=-2\mathrm{i}\vee z=\frac{1}{2}\mathrm{i}$

18. $z^2=e^{\mathrm{i}\frac{\pi}{3}}\times\bar{z}\Leftrightarrow$ $\quad | \; z=\rho\,e^{\mathrm{i}\theta}$

$\Leftrightarrow\left(\rho e^{\mathrm{i}\theta}\right)^2=e^{\mathrm{i}\frac{\pi}{3}}\times\rho e^{\mathrm{i}(-\theta)}\Leftrightarrow\rho^2 e^{\mathrm{i}(2\theta)}=\rho e^{\mathrm{i}\left(\frac{\pi}{3}-\theta\right)}\Leftrightarrow$

$\Leftrightarrow\begin{cases}\rho^2=\rho\\ 2\theta=\frac{\pi}{3}-\theta+2k\pi,\ k\in\mathbb{Z}\end{cases}\Leftrightarrow\begin{cases}\rho^2-\rho=0\\ 3\theta=\frac{\pi}{3}+2k\pi,\ k\in\mathbb{Z}\end{cases}\Leftrightarrow$

$\Leftrightarrow\begin{cases}\rho(\rho-1)=0\\ \theta=\frac{\pi}{9}+\frac{2k\pi}{3},\ k\in\mathbb{Z}\end{cases}\Leftrightarrow\begin{cases}\rho=0\vee\rho=1\\ \theta=\frac{\pi}{9}+\frac{2k\pi}{3},\ k\in\mathbb{Z}\end{cases}$

$\Leftrightarrow z=0\vee z=e^{\mathrm{i}\frac{\pi}{9}}\vee z=e^{\mathrm{i}\frac{7\pi}{9}}\vee z=e^{\mathrm{i}\frac{13\pi}{9}}$

19. $z=\left(\frac{1}{2}+\frac{\sqrt{3}}{2}\mathrm{i}\right)^n=\left(\cos\frac{\pi}{3}+\mathrm{i}\sin\frac{\pi}{3}\right)^n=\left(e^{\mathrm{i}\frac{\pi}{3}}\right)^n=e^{\mathrm{i}\frac{n\pi}{3}}$

z é um imaginário puro $\Leftrightarrow$

$\Leftrightarrow\frac{n\pi}{3}=\frac{\pi}{2}+k\pi,\ k\in\mathbb{Z}\Leftrightarrow\frac{n}{3}=\frac{1}{2}+k,\ k\in\mathbb{Z}\Leftrightarrow n=\frac{3}{2}+3k,\ k\in\mathbb{Z}$

Dado que $\frac{3}{2}+3k$, não é um número natural, qualquer que seja $k\in\mathbb{Z}$, podemos concluir que não existe $n\in\mathbb{N}$ para o qual z é um imaginário puro.

20.

20.1. $z_1 = \frac{2i}{1-i^{25}} - e^{i\frac{\pi}{2}} = \frac{2i}{1-i} - i = \frac{2i(1+i)}{(1-i)(1+i)} - i =$

$= \frac{2i-2}{1-i^2} - i = \frac{2i-2}{2} - i = i - 1 - i = -1 \in \mathbb{R}$

20.2. $z^2 - 8z + 25 = 0 \Leftrightarrow z = \frac{8 \pm \sqrt{64-100}}{2} \Leftrightarrow$

$\Leftrightarrow z = \frac{8 \pm 6i}{2} \Leftrightarrow z = 4 - 3i \vee z = 4 + 3i$

20.3. $z = \left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right)^n = \left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)^n = \left(e^{i\frac{\pi}{6}}\right)^n = e^{i\frac{n\pi}{6}}$

z é imaginário puro $\Leftrightarrow$

$\Leftrightarrow \frac{n\pi}{6} = \frac{\pi}{2} + k\pi,\ k \in \mathbb{Z}$

$\Leftrightarrow n = 3 + 6k,\ k \in \mathbb{Z}$

Logo, para $k \in \mathbb{N}$, $n = 3 + 6k \in \mathbb{N}$ e, para estes valores de n, z é imaginário puro.

21. $w = 16\ e^{i\frac{4\pi}{3}}$ e $z_1 = \sqrt{3} - i$

21.1. $z_1 = \sqrt{3} - i$

$|z_1| = \sqrt{3+1} = 2$

$\begin{cases} \tan\theta_1 = \frac{-1}{\sqrt{3}} = -\frac{\sqrt{3}}{3} \\ \theta_1 \in 4.^{\text{o}}\ Q \end{cases} \Rightarrow \theta_1 = -\frac{\pi}{6}$

$z_1 = 2\,e^{i\left(-\frac{\pi}{6}\right)}$

$z_1^4 = \left[2\,e^{i\left(-\frac{\pi}{6}\right)}\right]^4 = 2^4\ e^{i\left(-\frac{4\pi}{6}\right)} = 16\ e^{i\left(-\frac{2\pi}{3}\right)} =$

$= 16\,e^{i\left(-\frac{2\pi}{3}+2\pi\right)} = 16\,e^{i\frac{4\pi}{3}} = w$

Como $z_1^4 = w$, z_1 é uma raiz quarta de w.

As restantes raízes têm o mesmo módulo e os argumentos em progressão aritmética de razão $\frac{2\pi}{4} = \frac{\pi}{2}$.

$z_1 = 2\,e^{i\left(-\frac{\pi}{6}\right)}$

$z_2 = 2\,e^{i\left(-\frac{\pi}{6}+\frac{\pi}{2}\right)} = 2\,e^{i\frac{\pi}{3}}$

$z_3 = 2\,e^{i\left(\frac{\pi}{3}+\frac{\pi}{2}\right)} = 2\,e^{i\frac{5\pi}{6}}$

$z_4 = 2\,e^{i\left(\frac{5\pi}{6}+\frac{\pi}{2}\right)} = 2\,e^{i\frac{4\pi}{3}}$

21.2. O polígono é um quadrado inscrito numa circunferência de raio igual a $|z_1| = 2$.

$l^2 = 2^2 + 2^2$

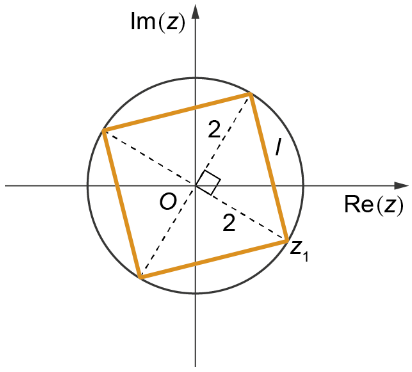
$l^2 = 2 \times 2^2$

$l = 2\sqrt{2}$

Perímetro $= 4l =$

$= 4 \times 2\sqrt{2}$

$= 8\sqrt{2}$



22. $z_1 = 2 + \mathrm{i}$

22.1. $u = \left(z_1 - \mathrm{i}\,\overline{z_1}\right)^6 = \left(2 + \mathrm{i} - \mathrm{i}(2 - \mathrm{i})\right)^6 = \left(2 + \mathrm{i} - 2\mathrm{i} + \mathrm{i}^2\right)^6 = \left(2 + \mathrm{i} - 2\mathrm{i} - 1\right)^6 = (1 - \mathrm{i})^6$

Seja $v = 1 - \mathrm{i}$.

$|v| = \sqrt{1+1} = \sqrt{2}$

$$\begin{cases} \tan\theta = \dfrac{-1}{1} = -1 \\ \theta \in 4.^\circ\text{Q} \end{cases} \Rightarrow \theta = -\frac{\pi}{4}$$

$v = \sqrt{2}\,\mathrm{e}^{\mathrm{i}\left(-\frac{\pi}{4}\right)}$

$$u = (1-\mathrm{i})^6 = \left(\sqrt{2}\,\mathrm{e}^{\mathrm{i}\left(-\frac{\pi}{4}\right)}\right)^6 = \left(\sqrt{2}\right)^6 \mathrm{e}^{\mathrm{i}\frac{6\pi}{4}} = 8\,\mathrm{e}^{\mathrm{i}\left(-\frac{3\pi}{2}\right)} = 8\mathrm{e}^{\mathrm{i}\frac{\pi}{2}} = 8\mathrm{i}$$

u é um imaginário puro porque $\operatorname{Re}(u) = 0$.

22.2. $z_1 = 2 + \mathrm{i}$

$z_2 = \mathrm{e}^{\mathrm{i}\alpha}$

$$\frac{z_1}{\overline{z_2}} = \frac{2+\mathrm{i}}{\overline{\mathrm{e}^{\mathrm{i}\alpha}}} = \frac{\sqrt{5}\,\mathrm{e}^{\mathrm{i}\theta}}{\mathrm{e}^{\mathrm{i}(-\alpha)}} = \sqrt{5}\;\;\mathrm{e}^{\mathrm{i}(\theta+\alpha)}$$

$0 < \theta < \frac{\pi}{4}$

$\pi < \alpha < \frac{5\pi}{4}$

$\pi < \theta + \alpha < \frac{6\pi}{4}$

$\pi < \theta + \alpha < \frac{3\pi}{2}$

$z_1 = 2 + \mathrm{i}$

$|z_1| = \sqrt{4+1} = \sqrt{5}$

$\operatorname{Arg}(z_1) = \theta$

$(2\,,\,1) \in 1.^\circ$quadrante $\Rightarrow$

$\Rightarrow 0 < \theta < \frac{\pi}{2}$

Como $\tan\theta = \frac{1}{2} < 1$

$0 < \theta < \frac{\pi}{4}$

Como $\pi < \theta + \alpha < \frac{3\pi}{2}$, o afixo de $\frac{z_1}{\overline{z_2}}$ situa-se no 3.º quadrante.

23. $w = 2\operatorname{cis}\left(\frac{4\pi}{3}\right)$

23.1. $z^2 - \overline{z \times w} = 0 \Leftrightarrow z^2 = \overline{z \times w} \Leftrightarrow$

$$\Leftrightarrow \left(\rho\, e^{i\theta}\right)^2 = \overline{\rho\, e^{i\theta} \times 2e^{i\frac{4\pi}{3}}} \Leftrightarrow \rho^2\, e^{i(2\theta)} = \overline{2\rho\, e^{i\left(\theta+\frac{4\pi}{3}\right)}} \Leftrightarrow$$

$$\Leftrightarrow \rho^2\, e^{i(2\theta)} = 2\rho\, e^{i\left(-\theta-\frac{4\pi}{3}\right)} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} \rho^2 = 2\rho \\ 2\theta = -\theta - \frac{4\pi}{3} + 2k\pi,\ k \in \mathbb{Z} \end{cases} \Leftrightarrow \begin{cases} \rho^2 - 2\rho = 0 \\ 3\theta = -\frac{4\pi}{3} + 2k\pi,\ k \in \mathbb{Z} \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} \rho(\rho-2) = 0 \\ \theta = -\frac{4\pi}{9} + \frac{2k\pi}{3},\ k \in \mathbb{Z} \end{cases} \Leftrightarrow \Leftrightarrow \begin{cases} \rho = 0 \ \vee\ \rho = 2 \\ \theta = \frac{-4\pi + 6k\pi}{9},\ k \in \mathbb{Z} \end{cases} \Leftrightarrow$$

$$\Leftrightarrow z = 0 \vee z = 2\, e^{i\left(-\frac{4\pi}{9}\right)} \vee z = 2\, e^{i\frac{2\pi}{9}} \vee z = 2\, e^{i\frac{8\pi}{9}}$$

23.2. $u = \left(\frac{w}{2}+1\right)^3 = \left(\frac{2e^{i\frac{4\pi}{3}}}{2}+1\right)^3 = \left(e^{i\frac{4\pi}{3}}+1\right)^3 =$

$$= \left(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3} + 1\right)^3 =$$

$$= \left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i + 1\right)^3 = \left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)^3 = \left(e^{i\left(-\frac{\pi}{3}\right)}\right)^3 = e^{i\left(\frac{-3\pi}{3}\right)} =$$

$$= e^{i(-\pi)} = e^{i\pi}$$

$|u| = 1$ e $\operatorname{Arg} u = \pi$

Cálculos auxiliares

$z = \frac{1}{2} - \frac{\sqrt{3}}{2}i$

$|z| = \sqrt{\frac{1}{4} + \frac{3}{4}} = 1$

$\theta = \operatorname{Arg} z$

$\begin{cases} \tan\theta = -\sqrt{3} \\ \theta \in 4.°Q \end{cases} \Rightarrow \theta = -\frac{\pi}{3}$

$z = e^{i\left(-\frac{\pi}{3}\right)}$

24. $z_1 = a - i$; $z_2 = 2 - 2\sqrt{3}\,i$; $z_3 = \frac{8}{-1+\sqrt{3}\,i}$

24.1. $z_1^2 = (a-i)^2 = a^2 - 2ai + i^2 = \left(a^2-1\right) - 2ai$

z_1 é raiz quadrada de um imaginário puro $\Leftrightarrow$

$\Leftrightarrow$ z_1^2 é imaginário puro $\Leftrightarrow$

$\Leftrightarrow a^2 - 1 = 0 \Leftrightarrow a = 1 \vee a = -1$

24.2.

- $z_4 = 1 - i$

  $|z_4| = \sqrt{1+1} = \sqrt{2}$

  $\begin{cases} \tan a(\arg z_4) = -1 \\ \arg z_4 \in 4.°Q \end{cases} \Rightarrow -\frac{\pi}{4}$ é um argumento de z_4

  $z_4 = \sqrt{2}\, e^{i\left(-\frac{\pi}{4}\right)}$

- $z_2 = 2 - 2\sqrt{3}i$

$|z_2| = \sqrt{4+4\times 3} = 4$

$$\begin{cases} \tan(\arg\ z_2) = \dfrac{-2\sqrt{3}}{2} = -\sqrt{3} \\ \arg(z_2) \in 4.^\circ\ Q \end{cases} \Rightarrow -\dfrac{\pi}{3} \text{ é um argumento de } z_2$$

$z_2 = 4\ e^{i\left(-\frac{\pi}{3}\right)}$

- $z_3 = \dfrac{8}{-1+\sqrt{3}\,i} = \dfrac{8\left(-1-\sqrt{3}\,i\right)}{\left(-1+\sqrt{3}i\right)\left(-1-\sqrt{3}\,i\right)} = \dfrac{8\left(-1-\sqrt{3}\,i\right)}{(-1)^2 - 3i^2} =$

$= \dfrac{8\left(-1-\sqrt{3}\,i\right)}{4} = 2\left(-1-\sqrt{3}\,i\right) = -2-2\sqrt{3}\,i$

$|z_3| = \sqrt{4+4\times 3} = 4$

$$\begin{cases} \tan(\arg\ z_3) = \dfrac{-2\sqrt{3}}{-2} = \sqrt{3} \\ \arg(z_3) \in 3.^\circ Q \end{cases} \Rightarrow \pi + \dfrac{\pi}{3} = \dfrac{4\pi}{3} \text{ é um argumento de } z_3$$

$z_3 = 4\ e^{i\frac{4\pi}{3}}$

- $\dfrac{z_2^2 + (1-i)^8}{z_3^3} = \dfrac{\left(4\,e^{i\left(-\frac{\pi}{3}\right)}\right)^2 + \left(\sqrt{2}\,e^{i\left(-\frac{\pi}{4}\right)}\right)^8}{\left(4\ e^{i\left(-\frac{4\pi}{3}\right)}\right)^3} =$

$= \dfrac{4^2\ e^{i\left(-\frac{2\pi}{3}\right)} + \left(\sqrt{2}\right)^8\ e^{i(-2\pi)}}{4^3\ e^{i(-\pi)}} =$

$= \dfrac{16\ e^{i\left(-\frac{-2\pi}{3}\right)} + 16\,e^{i\times 0}}{64\,e^{i\times 0}} =$

$= \dfrac{16\left(\ e^{i\left(-\frac{2\pi}{3}\right)} + e^{i\times 0}\right)}{64} = \dfrac{\cos\frac{-2\pi}{3} + i\sin\frac{-2\pi}{3} + 1}{4} =$

$= \dfrac{-\frac{1}{2} - \frac{\sqrt{3}}{2}i + 1}{4} = \dfrac{\frac{1}{2} - \frac{\sqrt{3}}{2}i}{4} =$

$= \dfrac{1}{8} - \dfrac{\sqrt{3}}{8}i = \dfrac{1}{4}\ e^{i\left(-\frac{\pi}{3}\right)}$

$v = \dfrac{1}{8} - \dfrac{\sqrt{3}}{8}i$

$|v| = \sqrt{\dfrac{1}{64} + \dfrac{3}{64}} = \sqrt{\dfrac{4}{64}} = \dfrac{1}{4}$

Seja θ um argumento de v

$$\begin{cases} \tan\theta = -\sqrt{3} \\ \arg\ v \in 4.^\circ\ Q \end{cases} \Rightarrow -\dfrac{\pi}{3} \text{ é um argumento de } v$$

24.3. $z_2^6 = \left(4\ e^{i\left(-\frac{\pi}{3}\right)}\right)^6 = 4^6\ e^{i(-2\pi)} = 4096\,e^{i\times 0} = 4096$

$z_3^6 = \left(4\ e^{i\frac{4\pi}{3}}\right)^6 = 4^6\ e^{i(8\pi)} = 4096\,e^{i\times 0} = 4096$

$z_2^6 = z_3^6 = 4096 \Rightarrow z_2$ e z_3 são raízes de índice 6 de 4096

$4096 \in \mathbb{R}$

24.4. $z_2^n + \overline{z}_2^n = \left(4\ e^{i\left(-\frac{\pi}{3}\right)}\right)^n + \left(4e^{i\frac{\pi}{3}}\right)^n = 4^n\ e^{i\left(-\frac{n\pi}{3}\right)} + 4^n\ e^{i\frac{n\pi}{3}} =$

$= 4^n\left(\cos\frac{-n\pi}{3} + i\sin\frac{-n\pi}{3} + \cos\frac{n\pi}{3} + i\sin\frac{n\pi}{3}\right) =$

$= \left(2^2\right)^n\left(\cos\frac{n\pi}{3} - i\sin\frac{n\pi}{3} + \cos\frac{n\pi}{3} + i\sin\frac{n\pi}{3}\right) =$

$= 2^{2n} \times 2\cos\frac{n\pi}{3} = 2^{2n+1}\cos\frac{n\pi}{3}$

24.5. $\overline{z} \times z_2 - z^2 = 0 \Leftrightarrow$

$\Leftrightarrow z^2 = \overline{z} \times z_2 \Leftrightarrow$

| $z = \rho\, e^{i\theta}$

$\Leftrightarrow \left(\rho\ e^{i\theta}\right)^2 = \rho\ e^{i(-\theta)} \times 4\ e^{i\left(-\frac{\pi}{3}\right)} \Leftrightarrow$

$\Leftrightarrow \rho^2\ e^{i(2\theta)} = 4\rho\ e^{i\left(-\theta-\frac{\pi}{3}\right)} \Leftrightarrow$

$\Leftrightarrow \begin{cases} \rho^2 = 4\rho \\ 2\theta = -\theta - \frac{\pi}{3} + 2k\pi,\ k \in \mathbb{Z} \end{cases} \Leftrightarrow \begin{cases} \rho(\rho - 4) = 0 \\ 3\theta = -\frac{\pi}{3} + 2k\pi,\ k \in \mathbb{Z} \end{cases} \Leftrightarrow \begin{cases} \rho = 0 \vee \rho = 4 \\ \theta = -\frac{\pi}{9} + \frac{2k\pi}{3},\ k \in \mathbb{Z} \end{cases} \Leftrightarrow$

$\Leftrightarrow z = 0 \vee z = 4\ e^{i\left(-\frac{\pi}{9}\right)} \vee z = 4\ e^{i\frac{5\pi}{9}} \vee z = 4\ e^{i\frac{11\pi}{9}}$

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25.

25.1. $z_0 = 1 + \sqrt{3}i$

Centro da circunferência: (0 , 0)

Raio da circunferência: $\overline{OA} = |z_0| = \sqrt{1+3} = 2$

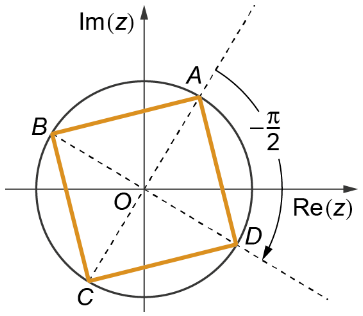
Condição pedida: $|z| = 2$

| $z_0 = 1 + \sqrt{3}\,i = \rho\, e^{i\theta}$
| $\rho = 2$
| $\tan\theta = \sqrt{3}$ e $\theta \in 1.^\text{o}$Q
| $\text{Arg}\, z_0 = \frac{\pi}{3}$

25.2. $z_0 = 1 + \sqrt{3}\,i = 2e^{i\frac{\pi}{3}}$

$A \curvearrowleft 2e^{i\frac{\pi}{3}}$

$D \curvearrowleft 2\ e^{i\left(\frac{\pi}{3}-\frac{\pi}{2}\right)} = 2\ e^{i\left(-\frac{\pi}{6}\right)}$



25.3. $w \curvearrowright P$

$iw \curvearrowright P'$ → transformado de P na rotação de centro O e amplitude 90°.

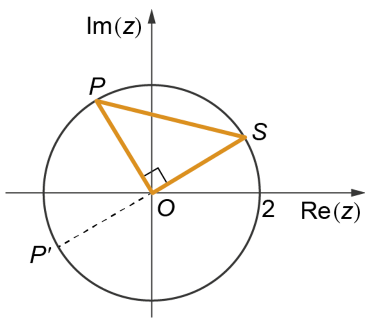
$-iw \to S$ → imagem de P' na rotação de 180° em torno da origem.

$\overline{OP} = \overline{OS} = 2$

$\overline{PS}^2 = \overline{OP}^2 + \overline{OS}^2$

$\overline{PS}^2 = 4 + 4 \Leftrightarrow \overline{PS} = \sqrt{8} \Leftrightarrow \overline{PS} = 2\sqrt{2}$

$\text{Perimetro}_{[OSP]} = 2 + 2 + 2\sqrt{2} = 4 + 2\sqrt{2}$



26.

26.1. $A \curvearrowleft z_1 = 2\,e^{i\alpha}$

$A' \curvearrowleft i z_1 \to A'$ é a imagem de A na rotação de 90° com centro na origem

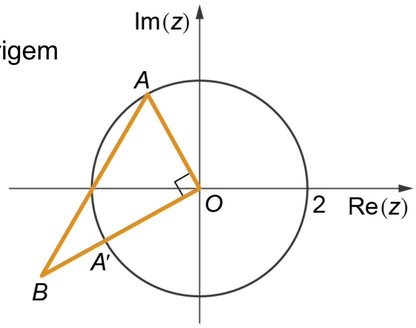
$B \leftarrow \sqrt{3}\,i\,z_1$

$\overline{OA} = 2;\ \overline{OB} = 2\sqrt{3};\ OA \perp OB$

$\overline{AB}^2 = \overline{OA}^2 + \overline{OB}^2$

$\overline{AB}^2 = 2^2 + \left(2\sqrt{3}\right)^2 \Leftrightarrow \overline{AB} = \sqrt{4 + 4\times 3} \Leftrightarrow \overline{AB} = 4$

$\text{Perimetro}_{[OAB]} = 2 + 2\sqrt{3} + 4 = 6 + 2\sqrt{3}$



26.2. $z = \sqrt{3} + i$

$z = 2\,e^{i\frac{\pi}{6}}$

$u = \left(\sqrt{3} + i\right)^{6n+3} = \left(2\,e^{i\frac{\pi}{6}}\right)^{6n+3} =$

$= 2^{6n+3}\,e^{i\left[\frac{(6n+3)\pi}{6}\right]}$

$= 2^{6n+3}\,e^{i\left(\frac{6n\pi+3\pi}{6}\right)} = 2^{6n+3}\,e^{i\left(n\pi+\frac{3\pi}{6}\right)}$

$= 2^{6n+3}\,e^{i\left(\frac{\pi}{2}+n\pi\right)}$

$|z| = \sqrt{3+1} = 2$

$\begin{cases} \tan(\operatorname{Arg} z) = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3} \\ \operatorname{Arg}(z) \in 1.^\circ\ Q \end{cases} \Rightarrow \operatorname{Arg} z = \frac{\pi}{6}$

$z = 2\,e^{i\frac{\pi}{6}}$

Dado que $\arg(u) = \frac{\pi}{2} + n\pi$, com $n \in \mathbb{N}$, u é um imaginário puro.

27. $z_1 = 16\,e^{i\frac{\pi}{4}}$

27.1. $-i \times z_1 = e^{i\left(-\frac{\pi}{2}\right)} \times 16\,e^{i\frac{\pi}{4}} = 16\,e^{i\left(-\frac{\pi}{4}\right)} = \overline{z_1}$

27.2. Trata-se da circunferência de centro na origem e raio r igual a $\sqrt[4]{|z_1|}$, ou seja, $r = \sqrt[4]{16} = 2$

Condição pedida: $|z| = 2$

28. $z_1 = 1 - i$

28.1. $u = \frac{z_1 + i^{35}}{6+3i} = \frac{1 - i + i^3}{6+3i} = \frac{1-i-i}{6+3i} = \frac{1-2i}{6+3i} =$

$= \frac{(1-2i)(6-3i)}{(6+3i)(6-3i)} = \frac{6 - 3i - 12i + 6i^2}{6^2 - (3i)^2} =$

$= \frac{-15i}{36+9} = -\frac{15}{45}i = -\frac{1}{3}i$

$-\frac{1}{3}i$ é imaginário puro porque $\operatorname{Re}(u) = 0 \wedge \operatorname{Im}(u) \neq 0$.

28.2. $e^{i\frac{\pi}{2}} - z_1 z^3 = 1 \Leftrightarrow z_1 z^3 = e^{i\frac{\pi}{2}} - 1$

$$\Leftrightarrow (1-i)z^3 = i - 1 \Leftrightarrow z^3 = \frac{-1+i}{1-i} \Leftrightarrow$$

$$\Leftrightarrow z^3 = -1 \Leftrightarrow z^3 = e^{i\pi} \Leftrightarrow z = \sqrt[3]{e^{i\pi}}$$

$$\Leftrightarrow z = \sqrt[3]{1}\ e^{i\left(\frac{\pi+2k\pi}{3}\right)},\ k = 0, 1, 2$$

$$\Leftrightarrow z = e^{i\frac{\pi}{3}} \vee z = e^{i\pi} \vee z = e^{i\frac{5\pi}{3}}$$

28.3. Círculo aberto de centro *C*, afixo de $z_1 = 1 - i$, e raio $= \overline{OC} = |z_1| = \sqrt{2}$:

$$|z - (1-i)| < \sqrt{2}$$

29.

29.1. A circunferência tem centro na origem e raio $\overline{OA} = |-3i| = 3$ e pode ser definida pela condição $|z| = 3$.

29.2. $A \leftarrow -3i = 3e^{i\frac{3\pi}{2}}$

$$B \leftarrow 3\ e^{i\left(\frac{3\pi}{2}+\frac{2\pi}{3}\right)} = 3\ e^{i\frac{13\pi}{6}} = 3\ e^{i\left(\frac{13\pi}{6}-2\pi\right)} = 3e^{i\frac{\pi}{6}}$$

$$= 3\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right) = 3\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right) = \frac{3\sqrt{3}}{2} + \frac{3}{2}i$$

29.3. $P \leftarrow z_1$

$P' \leftarrow iz_1$ P' é a imagem de P na rotação de 90° com centro na origem

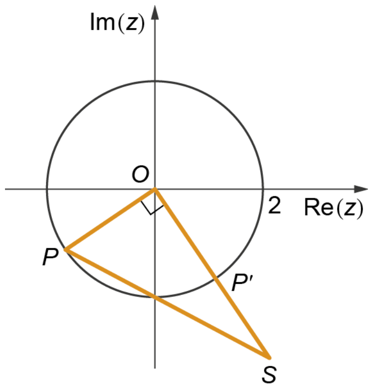
$S \leftarrow 2iz_1$

$\overline{OP} = \overline{OP'} = 3$

$\overline{OS} = 2\overline{OP'} = 6$

$\overline{OP} \perp \overline{OS}$

$$A_{[OPS]} = \frac{\overline{OP} \times \overline{OS}}{2} = \frac{3 \times 6}{2} = 9 \text{ u.a.}$$



30. $z_1 = e^{i\frac{\pi}{2}}$ e $z_2 = \frac{2}{i^{23}}$

30.1. $z_1 = i$; $z_2 = \frac{2}{i^{23}} = \frac{2}{i^3} = \frac{2}{-i} = \frac{2 \times i}{-i \times i} = \frac{2i}{1} = 2i$

$$(z_2 - z_1)^{10} = (2i - i)^{10} = i^{10} = i^2 = -1$$

30.2. $\sqrt[3]{z_1} = \sqrt[3]{1e^{i\frac{\pi}{2}}} = \sqrt[3]{1}\ e^{i\left(\frac{\frac{\pi}{2}+2k\pi}{3}\right)},\ k = 0, 1, 2$

$$= e^{i\left(\frac{\pi+4k\pi}{6}\right)},\ k = 0,\ 1,\ 2$$

$k=0 \Rightarrow w_0 = e^{i\frac{\pi}{6}} = \cos\frac{\pi}{6} + i\sin\frac{\pi}{6} = \frac{\sqrt{3}}{2} + \frac{1}{2}i$

$k=1 \Rightarrow w_1 = e^{i\frac{5\pi}{6}} = \cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6} = -\frac{\sqrt{3}}{2} + \frac{1}{2}i$

$k=2 \Rightarrow w_2 = e^{i\frac{3\pi}{2}} = -i$

30.3. A circunferência tem centro no afixo de $z_2 = 2i$ e tem raio igual a $|z_2| = |2i| = 2$ dado que passa na origem: $|z-2i| = 2$

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31. $z_1 = \rho e^{i\frac{2\pi}{3}}$

$z_2 = 3iz_1 = 3e^{i\frac{\pi}{2}} \times \rho e^{i\frac{2\pi}{3}} = 3\rho\, e^{i\left(\frac{\pi}{2}+\frac{2\pi}{3}\right)} = 3\rho\, e^{i\frac{7\pi}{6}}$

31.1. $\left(\frac{z_2}{z_1}\right)^4 - \left(\frac{z_1}{|z_1|}\right)^3 = \left(\frac{3iz_1}{z_1}\right)^4 - \left(\frac{\rho e^{i\frac{2\pi}{3}}}{\rho}\right)^3 =$

$= 3^4 i^4 - e^{i(2\pi)} = 81 \times 1 - e^{i\times 0}$

$= 81 - 1 = 80$

31.2. $z_1^3 = 8 \Leftrightarrow \left(\rho\, e^{i\frac{2\pi}{3}}\right)^3 = 8 \Leftrightarrow \rho^3\, e^{i(2\pi)} = 8 \Rightarrow$

$\Rightarrow \rho^3 = 8 \Leftrightarrow \rho = 2$

$z_1 = 2\, e^{i\frac{2\pi}{3}}$

a) $z_2 = 3\rho\, e^{i\frac{7\pi}{6}} =$ $\quad |\rho = 2$

$= 3 \times 2\left(\cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6}\right) =$

$= 6\left(-\frac{\sqrt{3}}{2} - \frac{1}{2}i\right) = -3\sqrt{3} - 3i$

b) $z_1 = 2\, e^{i\frac{2\pi}{3}} \to A$

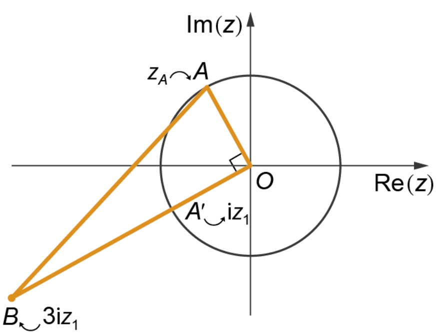
$z_2 = 3iz_1 \to B$

$\overline{OA} = 2$

$\overline{OB} = 3 \times 2 = 6$

$OA \perp OB$

$A_{[BOA]} = \frac{\overline{OA} \times \overline{OB}}{2} = \frac{2 \times 6}{2} = 6$



32. $z_1 = 1 - \text{i}$

32.1. $z \times z_1 = \text{i}^{11} + z_1 \times \overline{z_1} \Leftrightarrow z \times (1-\text{i}) = \text{i}^3 + (1-\text{i})(1+\text{i}) \Leftrightarrow z = \dfrac{-\text{i}+1-\text{i}^2}{1-\text{i}} \Leftrightarrow$

$\Leftrightarrow z = \dfrac{(2-\text{i})(1+\text{i})}{(1-\text{i})(1+\text{i})} \Leftrightarrow z = \dfrac{2+2\text{i}-\text{i}-\text{i}^2}{1^2-\text{i}^2} \Leftrightarrow z = \dfrac{3+\text{i}}{1+1} \Leftrightarrow z = \dfrac{3}{2}+\dfrac{1}{2}\text{i}$

32.2. $\dfrac{z_1^{\,2} + \left(\sqrt{2}\,\text{e}^{\text{i}\frac{5\pi}{12}}\right)^2}{\text{e}^{\text{i}\frac{3\pi}{2}}} = \dfrac{(1-\text{i})^2 + 2\,\text{e}^{\text{i}\left(\frac{2\times 5\pi}{12}\right)}}{-\text{i}} =$

$= \dfrac{1-2\text{i}+\text{i}^2+2\,\text{e}^{\text{i}\frac{5\pi}{6}}}{-\text{i}} = \dfrac{-2\text{i}+2\left(\cos\frac{5\pi}{6}+\text{i}\sin\frac{5\pi}{6}\right)}{-\text{i}} =$

$= \dfrac{-2\text{i}+2\left(-\frac{\sqrt{3}}{2}+\frac{1}{2}\text{i}\right)}{-\text{i}} = \dfrac{-2\text{i}-\sqrt{3}+\text{i}}{-\text{i}} = \dfrac{\left(-\sqrt{3}-\text{i}\right)\times\text{i}}{-\text{i}\times\text{i}} = \dfrac{-\sqrt{3}\,\text{i}-\text{i}^2}{1} = 1-\sqrt{3}\,\text{i}$

32.3. O polígono cujos vértices são os afixos das raízes quartas de w é um quadrado inscrito numa circunferência de raio r igual ao módulo das raízes, ou seja:

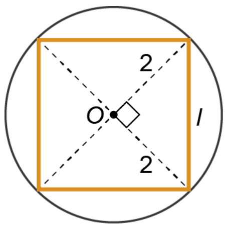
$r = \sqrt[4]{|w|}$

$w = 8(z_1)^2 = 8(1-\text{i})^2 = 8\left(1-2\text{i}+\text{i}^2\right) = -16\text{i}$

$|w| = |-16\text{i}| = 16$

$r = \sqrt[4]{16} = 2$

$A_{\text{quadrado}} = l^2 = 8$



$l^2 = 2^2 + 2^2$

$l^2 = 8$

33. $w_1 = \dfrac{2}{\sqrt{3}-\text{i}} + \text{i}^{43}$; $w_2 = \sqrt{2}\,\text{e}^{\text{i}\frac{\pi}{4}}$; $w_3 = 2 + y\,\text{i}$

33.1. $w_1 = \dfrac{2}{\sqrt{3}-\text{i}} + \text{i}^{43} = \dfrac{2\left(\sqrt{3}+\text{i}\right)}{\left(\sqrt{3}-\text{i}\right)\left(\sqrt{3}+\text{i}\right)} + \text{i}^3$

$= \dfrac{2\left(\sqrt{3}+\text{i}\right)}{3-\text{i}^2} - \text{i} = \dfrac{2\left(\sqrt{3}+\text{i}\right)}{4} - \text{i} =$

$= \dfrac{\sqrt{3}}{2} + \dfrac{1}{2}\text{i} - \text{i} = \dfrac{\sqrt{3}}{2} - \dfrac{1}{2}\text{i} =$

$= \text{e}^{\text{i}\left(-\frac{\pi}{6}\right)}$

$|w_1| = \sqrt{\dfrac{3}{4}+\dfrac{1}{4}} = 1$

$\begin{cases} \tan(\text{Arg } w_1) = -\dfrac{1}{\sqrt{3}} = -\dfrac{\sqrt{3}}{3} \Rightarrow \text{Arg} w_1 = -\dfrac{\pi}{6} \\ \text{Arg } w_1 \in 4.^\circ\text{Q} \end{cases}$

$w_1^{12} = \left(\text{e}^{\text{i}\left(-\frac{\pi}{6}\right)}\right)^{12} = \text{e}^{\text{i}\left(-\frac{12\pi}{6}\right)} = \text{e}^{\text{i}(-2\pi)} = \text{e}^{\text{i}\times 0} = 1 \in \mathbb{R}$

33.2. $w_2=\sqrt{2}\,e^{i\frac{\pi}{4}}=\sqrt{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)=\sqrt{2}\left(\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i\right)=1+i$

$$\frac{w_3}{w_2}=\frac{2+yi}{1+i}=\frac{(2+yi)(1-i)}{(1+i)(1-i)}=\frac{2-2i+yi-yi^2}{1-i^2}=\frac{2+y+(y-2)i}{1+1}=\frac{y+2}{2}+\frac{y-2}{2}i$$

$$w_2\times w_3=(1+i)(2+yi)=2+yi+2i+yi^2=(2-y)+(y+2)i$$

$$\frac{w_3}{w_2}\in\mathbb{R}\Rightarrow\frac{y-2}{2}=0\Rightarrow y=2\Rightarrow 2-y=0\Rightarrow$$

$\Rightarrow(2-y)+(y+2)i$ é imaginário puro $\Rightarrow w_2\times w_3$ é imaginário puro

33.3. $z^4+z\,e^{i\frac{\pi}{3}}=0\Leftrightarrow z\left(z^3+e^{i\frac{\pi}{3}}\right)=0\Leftrightarrow$

$$\Leftrightarrow z=0\vee z^3+e^{i\frac{\pi}{3}}=0\Leftrightarrow$$

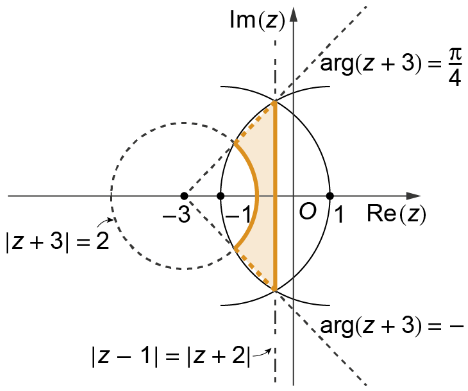
$$\Leftrightarrow z=0\vee z^3=-e^{i\frac{\pi}{3}}\Leftrightarrow z=0\vee z^3=e^{i\left(\frac{\pi}{3}+\pi\right)}\Leftrightarrow$$

$$\Leftrightarrow z=0\vee z=\sqrt[3]{e^{i\frac{4\pi}{3}}}\Leftrightarrow z=0\vee z=\sqrt[3]{1}\,e^{i\frac{\frac{4\pi}{3}+2k\pi}{3}},\ k=0,1,2\Leftrightarrow$$

$$\Leftrightarrow z=0\vee z=e^{i\frac{4\pi+6k\pi}{9}},\ k=0,1,2\Leftrightarrow$$

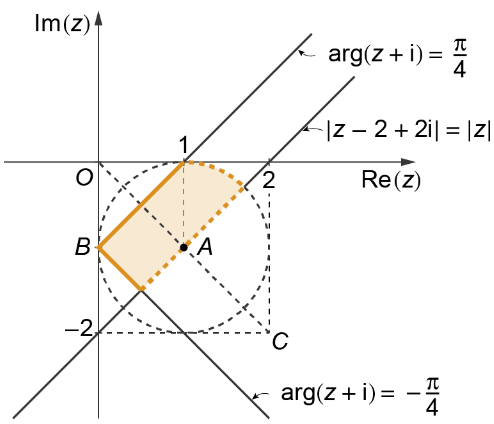
$$\Leftrightarrow z=0\vee z=e^{i\left(-\frac{2\pi}{9}\right)}\vee z=e^{i\frac{4\pi}{9}}\vee z=e^{i\frac{10\pi}{9}}$$

33.4. $|z-1|\geq|z+2|\wedge|z+3|\geq 2\wedge|\text{Arg}(z+3)|<\frac{\pi}{4}$



- $|z-1|\geq|z+2|\Leftrightarrow|z-(1+0i)|\geq|z-(-2+0i)|$
- $|z+3|\geq 2\Leftrightarrow|z-(-3+0i)|\geq 2$
- $|\text{Arg}(z+3)|<\frac{\pi}{4}\Leftrightarrow-\frac{\pi}{4}<\text{Arg}(z-(-3+0i))<\frac{\pi}{4}$

34. $|z-1+i|<1\wedge|\text{Arg}(z+i)|\leq\frac{\pi}{4}\wedge|z-2+2i|>|z|\Leftrightarrow$



- $|z-1+i|<1\Leftrightarrow|z-(1-i)|<1$

  $A(1\,,-1)$; $r=1$

- $|\text{Arg}(z+i)|\leq\frac{\pi}{4}\Leftrightarrow-\frac{\pi}{4}\leq\text{Arg}(z-(-i))\leq\frac{\pi}{4}$

  $B(0\,,\ -1)$

- $|z-2+2i|>|z|\Leftrightarrow|z-(2-2i)|>|z|$

  $C(2\,,-2)$

35.

35.1. $\mathrm{i}\left(z^3-4\right)=4\sqrt{3} \Leftrightarrow z^3-4=\dfrac{4\sqrt{3}}{\mathrm{i}} \Leftrightarrow z^3=4+\dfrac{4\sqrt{3}\times(-\mathrm{i})}{\mathrm{i}\times(-\mathrm{i})} \Leftrightarrow z^3=4-4\sqrt{3}\,\mathrm{i}$

Seja $u=4-4\sqrt{3}\,\mathrm{i}$.

$$|u|=\sqrt{4^2+\left(-4\sqrt{3}\right)^2}=\sqrt{16+48}=8$$

$$\begin{cases}\tan(\operatorname{Arg}u)=\dfrac{-4\sqrt{3}}{4}=-\sqrt{3}\\ \operatorname{Arg}u\in 4.^{\circ}\mathrm{Q}\end{cases}\Rightarrow \operatorname{Arg}u=-\dfrac{\pi}{3}$$

$u=8\,e^{\mathrm{i}\left(-\frac{\pi}{3}\right)}$

$$z^3=4-4\sqrt{3}\,\mathrm{i}\Leftrightarrow z^3=8\,e^{\mathrm{i}\left(-\frac{\pi}{3}\right)}\Leftrightarrow z=\sqrt[3]{8e^{\mathrm{i}\left(-\frac{\pi}{3}\right)}}\Leftrightarrow$$

$$\Leftrightarrow z=\sqrt[3]{8}\,e^{\mathrm{i}\left(\frac{-\frac{\pi}{3}+2k\pi}{3}\right)},\ k=0,\ 1,\ 2\Leftrightarrow$$

$$\Leftrightarrow z=2\,e^{\mathrm{i}\left(\frac{-\pi+6k\pi}{9}\right)},\ k=0,\ 1,\ 2\Leftrightarrow$$

$$\Leftrightarrow z=2\,e^{\mathrm{i}\left(-\frac{\pi}{9}\right)}\vee z=2\,e^{\mathrm{i}\frac{5\pi}{9}}\vee z=2\,e^{\mathrm{i}\frac{11\pi}{9}}$$

35.2. $z=e^{\mathrm{i}\theta}$

a) $|z+1|^2=\left|\,e^{\mathrm{i}\theta}+1\right|^2=|\cos\theta+\mathrm{i}\sin\theta+1|^2=$

$$=|(1+\cos\theta)+\mathrm{i}\sin\theta|^2=\left(\sqrt{(1+\cos\theta)^2+\sin^2\theta}\right)^2=$$

$$=1+2\cos\theta+\cos^2\theta+\sin^2\theta=1+2\cos\theta+1=$$

$$=2+2\cos\theta$$

b) $\tan\theta=\dfrac{5}{12},\ 0<\theta<\dfrac{\pi}{2}$

$$1+\tan^2\theta=\frac{1}{\cos^2\theta}$$

$$1+\left(\frac{5}{12}\right)^2=\frac{1}{\cos^2\theta}\Leftrightarrow\frac{1}{\cos^2\theta}=1+\frac{25}{144}\Leftrightarrow\frac{1}{\cos^2\theta}=\frac{169}{144}\Leftrightarrow\cos^2\theta=\frac{144}{169}$$

Como $0<\theta<\dfrac{\pi}{2}$, temos $\cos\theta=\sqrt{\dfrac{144}{169}}=\dfrac{12}{13}$

$$|z+1|^2=2+2\cos\theta=2+2\times\frac{12}{13}=\frac{50}{13}$$

Portanto, $|z+1|=\sqrt{\dfrac{50}{13}}$.

36. $z_1 = e^{i\frac{\pi}{3}}$; $z_1 = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3} = \frac{1}{2} + \frac{\sqrt{3}}{2}i$

- Circunferência com centro no afixo de z_1 e raio igual a $|z_1| = 1$: $\left|z - e^{i\frac{\pi}{3}}\right| = 1$
- Semirreta $\dot{O}B$: $\operatorname{Arg} z = 0$
- Semirreta $\dot{O}A$: $\operatorname{Arg} z = \frac{\pi}{3}$
- Reta AC : $\operatorname{Im}(z) = \frac{\sqrt{3}}{2}$ (reta horizontal que passa em A)

Condição pedida:

$$\left|z - e^{i\frac{\pi}{3}}\right| \le 1 \wedge 0 \le \operatorname{Arg}(z) \le \frac{\pi}{3} \wedge \operatorname{Im}(z) \le \frac{\sqrt{3}}{2}$$

37. $z_1 = 1 - i$

37.1. $$\frac{z_1^4 - 2i^{19}}{z_1(\overline{z}_1 - 3)} = \frac{(1-i)^4 - 2i^{16+3}}{(1-i)(1+i-3)} = \frac{(1-2i-1)^2 - 2i^3}{(1-i)(-2+i)} = \frac{(-2i)^2 + 2i}{-2+i+2i+1} = \frac{-4+2i}{-1+3i} =$$

$$= \frac{(-4+2i)(-1-3i)}{(-1+3i)(-1-3i)} = \frac{4+12i-2i+6}{1+9} = \frac{10+10i}{10} = 1+i$$

37.2. a) $z^3 - 4\sqrt{2}\, z_1 = 0 \Leftrightarrow z^3 = 4\sqrt{2}(1-i) \Leftrightarrow$

$\Leftrightarrow z^3 = 4\sqrt{2} \times \sqrt{2}\, e^{i\left(-\frac{\pi}{4}\right)} \Leftrightarrow z^3 = 8\, e^{i\left(-\frac{\pi}{4}\right)} \Leftrightarrow$

$\Leftrightarrow z = \sqrt[3]{8\, e^{i\left(-\frac{\pi}{4}\right)}} \Leftrightarrow z = \sqrt[3]{8}\, e^{i\left(\frac{-\frac{\pi}{4} + 2k\pi}{3}\right)}, \; k = 0,\ 1,\ 2 \Leftrightarrow$

$\Leftrightarrow z = 2\, e^{i\left(\frac{-\pi + 8k\pi}{12}\right)}, \; k = 0,\ 1,\ 2 \Leftrightarrow$

$\Leftrightarrow z = 2\, e^{i\left(-\frac{\pi}{12}\right)} \vee z = 2\, e^{i\frac{7\pi}{12}} \vee z = 2\, e^{i\frac{15\pi}{12}} \Leftrightarrow$

$\Leftrightarrow z = 2\, e^{i\left(-\frac{\pi}{12}\right)} \vee z = 2e^{i\frac{7\pi}{12}} \vee z = 2e^{i\frac{5\pi}{4}}$

$S = \left\{2\, e^{i\left(-\frac{\pi}{12}\right)},\ 2e^{i\frac{7\pi}{12}},\ 2e^{i\frac{5\pi}{4}}\right\}$

b) $z\overline{z} + \overline{z} = z_1 \Leftrightarrow$ $\quad |\, z = x + yi$

$\Leftrightarrow (x+yi)(x-yi) + (x-yi) = 1-i \Leftrightarrow x^2 + y^2 + x - yi = 1-i \Leftrightarrow$

$\Leftrightarrow \begin{cases} x^2 + y^2 + x = 1 \\ -y = -1 \end{cases} \Leftrightarrow \begin{cases} x^2 + 1 + x = 1 \\ y = 1 \end{cases} \Leftrightarrow \begin{cases} x(x+1) = 0 \\ y = 1 \end{cases} \Leftrightarrow$

$\Leftrightarrow \begin{cases} x = 0 \vee x + 1 = 0 \\ y = 1 \end{cases} \Leftrightarrow \begin{cases} x = 0 \\ y = 1 \end{cases} \vee \begin{cases} x = -1 \\ y = 1 \end{cases} \Leftrightarrow z = i \vee z = -1 + i$

$S = \{i, -1+i\}$

c) $|z| + z - 4 = 4\sqrt{2}e^{i\frac{\pi}{4}}$

Seja $z = z + y\text{i}$, $(x\,,\, y \in \mathbb{R})$.

$$|z| + z - 4 = 4\sqrt{2}e^{i\frac{\pi}{4}} \Leftrightarrow \sqrt{x^2+y^2} + x + y\text{i} - 4 = 4\sqrt{2}\left(\cos\frac{\pi}{4} + \text{i}\sin\frac{\pi}{4}\right) \Leftrightarrow$$

$$\Leftrightarrow \left(\sqrt{x^2+y^2} + x - 4\right) + y\text{i} = 4\sqrt{2}\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\text{i}\right) \Leftrightarrow \left(\sqrt{x^2+y^2} + x - 4\right) + y\text{i} = 4 + 4\text{i} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} \sqrt{x^2+y^2} + x - 4 = 4 \\ y = 4 \end{cases} \Leftrightarrow \begin{cases} \sqrt{x^2+16} = 8 - x \\ y = 4 \end{cases} \Rightarrow \begin{cases} x^2 + 16 = (8-x)^2 \\ y = 4 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} x^2 + 16 = 64 - 16x + x^2 \\ y = 4 \end{cases} \Leftrightarrow \begin{cases} 16x = 64 - 16 \\ y = 4 \end{cases} \Rightarrow \begin{cases} x = 3 \\ y = 4 \end{cases}$$

Vamos verificar se $x = 3$ é solução de $\sqrt{x^2+16} = 8 - x$:

$\sqrt{3^2+16} = 8 - 3 \Leftrightarrow \sqrt{25} = 5$ (proposição verdadeira)

Portanto, $|z| + z - 4 = 4\sqrt{2}\,e^{i\frac{\pi}{4}} \Leftrightarrow z = 3 + 4\text{i}$.