Lawrence W. Swienciki

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Lawrence W. Swienciki, Ph.D.



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Preface

Over the last twenty years, there has been increased interest in using multicultural themes throughout the mathematics curriculum. This worthy goal is not an easy one for the mathematics classroom teacher because information of this type requires research. It is challenging to bring this information to students in a way that is meaningful to the mathematics curriculum.

In 1972, I began publishing multicultural math puzzles to reinforce basic skills in elementary algebra. My experience has been that materials of this kind bring various cultures to the classroom and provide an interesting way to reinforce math skills.

Multicultural material suitable for more advanced classes such as geometry, intermediate algebra, analysis, or calculus hasn't been widely available. As I pondered the idea of bringing a multicultural dimension into these courses, it was very clear to me that teaching time was at a premium. With students being taken out of class for field trips, class activities, counseling, and so on, I realized that it was impractical to introduce multicultural themes as a distinct lesson in advanced classes. It seemed to me that the most practical way to deal with the proposition that multicultural material could be incorporated into all math classes was to develop a series of posters recognizing mathematical achievements of various cultures. In this way teachers could have interesting materials to decorate their rooms, to accomplish multicultural content objectives, and to focus on their own curriculum, without getting into topics that would be difficult to connect to their objectives. As a consequence, I developed and published the Math of Mexico poster in 1980. Since then, many more international mathematics posters have been added to the series, recognizing the mathematical achievements of Africans, Chinese, Japanese, Indians, Native Americans, and others. Thus, at the present time, the poster series has not only met the original goal of providing multicultural references to all levels of math students, but has been an effective method of introducing the global history of mathematics.

Through my research in developing the poster series, it became apparent that more topics could be developed for teachers to use with the curriculum. The poster format did not provide the space needed to present all of these topics. My aim is to add resource books to the international mathematics poster series to provide more depth of information for teachers to use with their students.

I wish to take this opportunity to thank all of my benefactors who contributed to my education. In particular, I wish to thank the Sisters of the Presentation at St. Patrick's Elementary School in San Jose, California, and the Jesuit fathers at Bellarmine College Preparatory and Santa Clara University.

Lawrence W. Swienciki, Ph.D. Santa Fe, New Mexico June 29, 2000



## Suggested Uses

When using multicultural material in a mathematics class, keep in mind your basic curriculum objectives. Multicultural themes can be a tool to motivate students, like the other tools you use to expand class activities such as math puzzles, math games, individual or group math projects, and student work at the chalkboard.

As teachers, we prepare students for the next mathematics class they take, while helping them to meet career challenges, and to appreciate the contributions of all cultures to our present-day knowledge. Multicultural content will enhance a mathematics class when it is presented in an interesting way for all students and especially when it reinforces the mathematics curriculum.

The Ambitious Horse: Ancient Chinese Mathematics Problems complements the international mathematics poster Math of China offered by Key Curriculum Press. This book focuses on ancient and medieval Chinese mathematics. There is a broad range of topics in this book that can serve a variety of needs for teachers and students. There are sections that deal with elementary level, secondary level, and college level mathematics. Topics in this book were chosen to make it possible for teachers and students to use Chinese mathematics that can be connected to modern curriculum. In fact, this volume is designed to be reproducible for instructional purposes.

#### Elementary and Middle School Level

At the elementary and middle school level, teachers can find topics on the base-10 number system, operations on the Chinese calculating board. For example, "Calculator Rod Addition" is a manipulative activity—very much like a game—that can increase understanding of place value. Also, the seven-piece puzzle is another manipulative activity that promotes basic skills in geometry. A teacher at the elementary school level will find the presentation in this book well-suited for the elementary school environment where curriculum is generally more integrated than secondary or college level curriculum. Topics presented in this book integrate mathematics, history, art, and culture.

#### High School Level

Several topics relate directly to the subjects of elementary algebra, geometry, and intermediate algebra. Teachers can find applications with traditional lessons or activities in such classes. Moreover, students have opportunities to solve problems with alternate methodologies. For example, the problem "The Ambitious Horse" provides a Chinese solution that is different from most Western explanations of problems on arithmetic series. That students can view multiple methods of solving a given problem is valuable in developing critical thinking—this experience encourages them to be more reflective.

#### College Level

There are problems involving the solution of polynomial equations of arbitrary degree from the area of algebra known as the theory of equations. Problems are developed in number theory related to operations with modular arithmetic. These and other topics can be especially valuable in a history of mathematics class or for students preparing to become teachers.

#### Mathematics Projects

Some topics in this book can be used for individual or group activity projects—in particular, the topics on extracting roots or solving polynomial equations. Another project taking the students outdoors involves the ancient Chinese mathematics of surveying.

#### Home School or Specialized Schools

Home schooling has become more popular in recent years. With this resource book, parents can provide their children a broad range of historical mathematics that is fundamentally sound. The material in this book offers opportunities for projects or activities.

Some Chinese-American students attend special classes on Chinese culture because they or their parents value the opportunity to learn about their cultural heritage. This book can be used by teachers offering such classes or by students studying at home, self-directed or assisted by parents or family.

#### Below.

The Great Wall of China is the most spectacular wall ever built. The central part stretches 1,500 miles across northern China from the Yellow Sea to the central Asian desert. This symbol of China was built by General Meng Tian on orders from the first emperor of China, Qin Shihuangdi, who unified China in 221 B.C. General Meng linked earlier walls together to make the Great Wall.

Chinese Writing

"Jing xi zi zhi"

The inventions of paper (second to first centuries B.C.), printing with wooden blocks (eighth century), and movable type (eleventh century) are accolades to China's writing tradition. This tradition, the longest and most continuous of any nation, defines Chinese culture more than any other characteristic.

Two anecdotes demonstrate how important the writing tradition is to the Chinese people. First, in ancient China to carelessly throw paper with writing into the street was considered disrespectful to the written word. Second, as late as the 1930s in Beijing,

> public trash containers bore the phrase *Jing xi zi zhi*—"Respect and save written paper."

Scientists have discovered Neolithic cultures in China dating before 6000 B.C., but the earliest written history begins with the late Shang dynasty (approximately 1250 to 1050 B.C.) with inscriptions on oracle bones and cast bronze artifacts.

The writing tradition of China is very different from that of other cultures, and it has enjoyed a

unity and continuity for thousands of years. Chinese graphs are composed of two parts: a signific or radical, indicating the general category to which the word belongs; and a phonetic, which gives its approximate sound. Only a small portion of these complex graphs were originally actual pictures, such as a horse or a flag.

In spite of the stability of Chinese writing, China's long history before unification in 221 B.C. was subject to intense wars that destroyed the written records of the defeated. Therefore much early evidence for the history of writing has been lost. Fortunately, new archaeological discoveries have provided historians and scientists with fresh material for understanding the history of the formation of numerals. Some discoveries even suggest that basic Chinese numerals appear scratched on pottery dating from 5000 B.C.

The concept of counting is elementary in any culture. Scientists have not provided a complete history of the development of numerals in China, but it is ancient and parallels the development of writing over thousands of years. There is also evidence that growth of the numeral system was subject to change and variations. For example, coins cast during the second century B.C. and used for hundreds of years bear numerals that are now discontinued in Chinese writing.

On the *Math of China* poster, the numerals displayed in the 40 boxes of the poster represent the traditional or literary method of writing Chinese numerals. However, four major types of numerals have developed in China: a traditional or literary form, a business or legal form, a calculating rod form, and an abbreviated form. The symbols in the center of the *Math of China* poster, the *Ba Gua*, are not numerals but, rather, mathematical permutations of the symbols for the Chinese concepts of *yin* and *yang*.



Above.

Shang dynasty characters were inscribed on divination bones and cast bronze artifacts as early as 1250–1050 в.с. yi er san si wu liu qi ba jiu shi bai qian wan

#### Above.

The romanized names for the Chinese numerals: (top to bottom) 1 through 10, 100, 1,000, and 10,000.

At right.

The three examples illustrate the property of place value. The Chinese use a special character for place value. Modern Hindu-Arabic notation uses position to show place value.

# Traditional Numerals

Chinese numerals, just as our own Hindu-Arabic symbols, are based on powers of 10. Although they can be written from left to right, the traditional form is to write them vertically from top to bottom. Some of the most commonly used numerals are illustrated in the student instruction charts on pages 5 and 6.

To write other numerals, such as multiples of 10, the symbols from 1 to 10 are employed with the idea of place value. For example, think of 60 as  $6 \times 10$ . In the Chinese system, put the character for 6 over the character for 10 to represent 60 as shown below.

To write the numeral for 600, think of 600 as  $6 \times 100$ . Similar to the previous example, write the 6 over the 100 symbol in the Chinese system as shown below.

Numerals such as 6,000 follow the same pattern. Think of 6,000 as  $6 \times 1,000$ . For Chinese numerals, write 6 over 1,000 as illustrated below.



To write numerals such as 250, think of 250 as  $(2 \times 100) + (5 \times 10)$ . In the Chinese system, start at the top with 2 over the 100. Continue under the 100 with the 5 over the 10. This is shown at the bottom of the page.

To write four-digit numerals such as 1,492, think of 1,492 as  $(1 \times 1,000) + (4 \times 100) + (9 \times 10) + 2$ . The Chinese representation follows the same pattern as the last example and is illustrated below.

If 1,066 is written as the sum of powers of 10, then  $1,066 = (1 \times 1,000) + (0 \times 100) +$  $(6 \times 10) + 6$ . However, the Chinese did not always have a zero in their system of mathematics. Think of 1,066 as  $(1 \times 1,000) +$  $(6 \times 10) + 6$ . To help the reader quickly understand that a multiple of 10 follows the 1,000, the symbol **\$\$** is placed between the 1,000 and the 6 symbol as illustrated below.





#### Above.

During the Ming dynasty (1368–1644), the Chinese character *ling* was adopted as zero in writing numerals in both traditional and accountant forms. The ancient meaning of *ling* is the small droplet of water left on a plant or some object after a rainstorm. It is speculated that the name was applied to the Sanskrit O introduced in the previous dynasty, the Song (960–1279).

#### At left.

The three examples illustrate more complex Chinese numerals. The example for 1,066 has a symbol that acts as a zero but is not a numeral. It can be understood as a Chinese punctuation symbol. The numeral O came from India. However, it was not introduced into Chinese mathematics until the middle of the thirteenth century A.D.







Below.

Ancient Chinese graph for long life (*shou*).













In 1662 the great emperor Kangxi ascended the throne of the Manchu, or Qing, dynasty (1644–1911). His reign brought military unity to China. He greatly favored the continuation of learning from the Ming dynasty. However, he allowed corrupt officials to remain in office and was indecisive in declaring his heir to the throne of China.

Legend has it that before his death, Kangxi designated his favorite son, the fourteenth, to be his heir. He wrote the will and named his fourteenth son to take his place upon his death. The will was hidden in a secret spot behind a royal tablet.

The emperor's secret was discovered by his fourth son, Yongzheng. He formed a conspiracy with his uncle to alter the will. A kung-fu master was employed to climb to the hidden location and change the will to read the fourth son instead of the fourteenth. This change was easily made because the will was not written with formal or legal numerals. The left column at right shows what Kangxi wrote: "Give reign to fourteenth prince." Notice the traditional numerals for fourteen are used. The ten numeral was easily altered by drawing a horizontal line on the top and adding a hook to the vertical line. This creates the graph yu meaning "to." The new form, shown in the right column, has the meaning "Give the throne to fourth prince."

#### Below.

Left column designates the fourteenth son as heir to the throne of China in traditional literary form of numerals. Right column names the fourth son in the same way.

傳傳 位位

When Kangxi died, Yongzheng the fourth son became emperor. Thus, China's history had been changed by the misuse of writing simple numeral characters.

Chinese Legal Numerals

If Kangxi had been careful in making his will, he would have used formal, or accountant, numerals. For legal documents, China had centuries before developed special characters to prevent such fraud. Shown in the caption at left is the legal method the emperor could have chosen. The left column designates the fourteenth prince as heir, while the right column designates the fourth prince.

At the bottom of the page, two examples of writing numerals are given in both traditional and accountant form. Compare these numerals to the example of 101 on page 6. Notice the *ling* zero symbol is used only once in 1,001. This is another indication that the symbol is not a true mathematical zero, but a punctuation symbol indicating that place value is being skipped between numerals.



#### Below.

Left column designates the fourteenth son as heir to the throne of China using formal, or accountant, numerals. Right column names the fourth son in the same manner.





For at least 2,000 years, Chinese mathematicians used an amazing system of calculating rods to solve a wide range of problems. Originally the rods were large, but a variety of types developed over the centuries. A classic set might consist of 271 small bamboo pieces about 4 cm long and 0.5 cm in diameter. Red pieces represented positive numbers, and black ones stood for negative numbers. The illustration depicts their use on a rectangular grid called a *counting board*.

Calculating rod numerals are based on the decimal system with a set of nine digits. Oracle bone inscriptions suggest that the Shang Chinese were the first known people to possess such a system in the ancient world. The nine digits are defined in Figure 1 on page 15. Mathematicians used two forms: one for odd powers of 10, another for even powers of 10. This system served as a check against error and made the numerals easier to read. Unlike the literary form for Chinese numerals, there are no special symbols for place value for calculating rod numerals. The rod numerals use position for place value and are read from left to right.

Below.

The abacus is generally associated with the mathematics of China. However, this device was not widely used before the fourteenth century. Calculating rods form the background for an understanding of ancient and other historical mathematics in China.



The numerals in Figure 2 posed no problem in writing on paper with rod numeral symbols, but when writing numerals such as 3,605 or 17,760 we do not know how to deal with the zero. On the counting board, there is no difficulty because a blank square is left for the zero. It wasn't until the eighth century A.D. that the O symbol from India was introduced into China. Ever since the Ming dynasty (1368–1644), the O has been utilized when writing out rod numerals on paper. In Figure 3, rod numerals for 3,605 and 17,760 are shown as they appeared in documents after the late Ming period.



Rod numerals are designed to be used for mathematics in the same way that the abacus or modern calculator is used to speed computation. Calculating rods are placed on a counting board from left to right. Operations such as addition, subtraction, multiplication, division, and finding square roots or even higher roots are made easier. Ancient Chinese mathematicians were able to solve many advanced algebraic problems with this tool.

The Chinese through the fourteenth century were ahead of most other peoples in mathematics. At that time no country had developed a convenient symbolic notation. Instead, problems were usually written out in words. The counting board with the rod numerals made it possible to solve advanced problems without the use of symbolic notation. However, in time this feature became a limitation to Chinese mathematics. China needed a symbolic notation to carry it beyond the glorious achievements of its past.

Rod Numeral Operations

Rod numeral operations on the counting board often involved more than one person. Numbers were called out by one individual, and a second person placed or moved the calculating pieces on the board. Because the number of operations left on the board at any one time was limited to the size of the board, steps were generally replaced through the solution of the problem. In our explanation of the calculating rods, the ordinary use of this device has a different style than it does on paper. It is suggested that you make a simple grid on paper. Using flat toothpicks for rod numerals, manipulate the pieces on the board for the various problems. In this way you sense the power of this calculating tool.

### Calculating Rod Addition

The examples shown in Figures 1 and 2 on page 18 demonstrate the operation of addition with the calculating rods. The first example, 345 + 271, shows the result, 616, on the third row of the counting board. The rod numerals are aligned in the same manner as in modern arithmetic. In the second example, 4,809 + 718 = 5,527. The 4,809 has a zero in the tens place. On the counting board, Figure 2 illustrates the rod numerals for 4,809 on the first line with a blank space where the zero occurs. Otherwise, the second example follows the same pattern as the first. Try this process with a column of numbers to be summed. But, instead of filling up the rows with symbols, ask someone to call out the numbers to you. In this way you will sense the speed of not writing down all the numerals. You can change the partial sums as the process continues in the same way it was done in ancient China.





Above.

The Song dynasty (960-1279) introduced monograms formed of rod numerals useful for table entries. The rod numeral for 4,716 is shown in monogram form.

### Calculating Rod Subtraction

The example of subtraction comes from the Song dynasty. The mathematician Qin Jiushao wrote the subtraction problem 1.470,000 - 64,464 = 1,405,536 in his book Shushu jiuzhang (Computational Techniques in Nine Chapters), published in 1247, as shown at the bottom of the page.

In Figure 3 on page 20, the same problem is represented on the counting board with the rod numerals. Notice in Qin Jiushao's writing of the results on paper, the ling symbol O was used. Furthermore, the book illustrates the fact that at various times and locations in China, some mathematicians used varied forms of rod numerals such as the crossed rods for the numeral 4. Try the problem on the counting board yourself. The problem requires you to borrow. What is the simplest method of borrowing using the rod numerals?

In Figure 4 on page 20, another example of a subtraction problem is displayed on a counting board for you to verify. In Figures 5 and 6, the subtraction problems are displayed on the counting board, but you will have to fill in the boxes below the boards in modern Hindu-Arabic notation.

I≣O≣IIII≡⊤ I≣⊤0000

 $T \times \parallel \parallel \perp \times$ 



Exercise 2 The problems below provide you with practice recognizing the rod numerals. Use Figure 1 on page 15 as a guide to complete the boxes. If the rod numerals are represented on a single row of the counting board, write the modern Hindu-Arabic numerals in the box to the right of the problem. For the Hindu-Arabic numerals given in the box, fill in the counting board row with rod numerals. a. ТГ = b. M Ш = c. 1,249 = d. 8,203 \_\_\_\_ e. Ξ f. 8,772 g. 42,365 = h. Ш i. 294,786 j. -Π Ш Ħ







#### Example 1

The sequence of three board patterns on page 26 illustrates the multiplication of  $49 \times 25$ . The first caption of the problem contains the commentary marks for the operations. The multiplier, 49, is placed at the top right on the board. At the very bottom of the board, the multiplicand, 25, is placed. Note that its last digit, 5, aligns with the first digit, 4, in the multiplier. This alignment is the usual way of starting the problem. The partial products and partial sums are done between the top and bottom rows.

In the second caption, the 4 in the multiplier and the entire multiplicand are more heavily framed for emphasis in the explanation. The 4 in the multiplier is applied to the 25 of the multiplicand. The first product,  $4 \times 2 = 8$ , is obtained. The 8 is written in the second row directly above the 2 because the place value position on the board makes the product really  $40 \times 20 = 800$ . In the diagram this is annotated at the left of the second row. In daily practice it is not necessary to think of the actual place value. The product is always written above the digit in the multiplicand. If the product has two digits, an additional column space to the left is used.

Next, multiply the 4 in the multiplier and the 5 in the multiplicand:  $4 \times 5 = 20$ , two digits. The zero, which is not marked on the board, aligns with the 5 in row three. The 2 must be written to the left of the zero position. The side note in the figure justifies this process since  $40 \times 5 = 200$ .

In the last caption, the pieces are moved to show the first partial sum in row two. The multiplicand is moved to the right one space. Notice that the multiplicand does not change symbols when its column position is moved. No notation change occurs because the operations are performed between the first and last rows.

In the last caption, the 9 in the multiplier is heavily framed with the multiplicand to indicate which numbers are used next.  $9 \times 2 = 18$ . Write 18 in row three above the 2 digit of the multiplicand. But 18 has two digits. Therefore, enter 8 above the 2 and 1 to the left. Next multiply 9 times 5. Enter 45 in row four. Add rows two, three, and four to obtain the final sum, 1,225.



Example 2

The second example of multiplication with calculating rods is  $736 \times 298$ . The process is shown in four captions beginning at the bottom of this page. The pattern is similar to the first example but involves larger numbers. The explanation is condensed.

The caption below shows the given problem with the multiplier at the top and the multiplicand at the bottom. Note the alignment of first and last digits.

In the second caption, at the top of page 28, the 7 is multiplied over the multiplicand. The three partial products are entered in rows two, three, and four. The three rows are added. The partial sum is entered in row five.

In the third caption, the multiplicand is shifted to the right one space. The pieces are moved, with the partial sum from the previous diagram entered in row two. Next, 3 in the multiplier operates on the multiplicand. The three partial products are entered and summed. Row five is the second partial sum. The latter sum is carried to row two of the last caption on page 28.

In the fourth caption, the multiplicand is moved one last time to the right. The 6 in the multiplier operates on the multiplicand. The three partial products are entered and summed to get the final answer in line five.



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Exercise 1

Each counting board below has a multiplication to be performed. Write down in modern numerical symbols what the problem is. Then use the calculating rods on a counting board to perform the multiplication. Check your answer by ordinary multiplication or with a calculator.



a. The product the board is set for you to solve is \_\_\_\_\_\_.



b. The product the board is set for you to solve is \_\_\_\_\_\_.



### Below.

Confucius or Kong Qiu (551–479 B.C.) was China's most influential teacher. He trained his students in the "Six Arts," one of which was mathematics. He considered his work a failure because he could not reform the politics of his time. But his teaching through the work of his students has had a monumental effect on Chinese life for many centuries.





Below.

Shou, symbol for longevity, from a nineteenth-century silk robe of a mandarin.



Exercises: Seven-Piece Puzzle

- 1. In the figure shown on page 30, the pieces are separated and rearranged to form the shaded figure of a person with a fan. Use the model you constructed to create this figure.
- 2. What is the area of the figure you constructed in the first question above? Express your answer in square units.
- 3. Arrange the Seven-Piece Puzzle to form the shaded figures of a kangaroo and a crane shown below. What is the area of the kangaroo figure in square units? What is the area of the crane figure in square units?
- 4. Write a statement about the area of the three figures you constructed from the Seven-Piece Puzzle. Write a hypothesis for a general case.







# Chinese Dissections

Dissection can be used to discover formulas for the areas of geometric figures such as parallelograms, triangles, trapezoids, and other polygons. For example, starting with the formula for the area of a rectangle as the product of its base and altitude, the formula for the area of a parallelogram follows easily by dissection.

At the left a parallelogram is depicted. To find a formula for its area,  $A_p$ , dissect the parallelogram along the segment representing its altitude to obtain a trapezoid and a right triangle. Can you state what the area of a parallelogram is? Use the diagrams at left to prove your formula.

From the parallelogram the area formula for a triangle easily follows. Dissect the parallelogram along one of its diagonals. Use the formula that you derived and refer to the diagram at left to prove that the area of a triangle is

$$A_t = \frac{1}{2}bh$$



R



Jnscribed Square Theorem

The *Jiuzhang suanshu (Nine Chapters of Mathematical Art)* is China's oldest mathematical text and it's the most important one from ancient times. This classic text was started in the Western Han dynasty and was completed in the early years of the Eastern Han dynasty (circa 50–100 A.D.), but it represents a mathematical tradition that echoes back to the period of the Zhou dynasty (founded during the eleventh century B.C.). It became the fundamental text on which subsequent Chinese mathematical Art comes the following geometrical theorem about the side of a square inscribed in any right triangle. Can you prove it?



The length of the side of a square inscribed in a right triangle is the quotient of the product and the sum of the legs.



Liu Hui's Solution

Liu Hui was one of the great mathematicians of ancient China. He lived during the Three Kingdoms period when China was ruled by three separate monarchies (220–280 A.D.). Little is known about Liu Hui's personal life, but he lived in the northern kingdom of Wei. From his *Commentary on the Nine Chapters of Mathematical Art* (263 A.D.), Liu Hui provides us with the following creative solution to the inscribed square problem on page 36.



Liu Hui's figure with additional modern notation

In the figure, right triangle ABC with inscribed square is given. Duplicate right triangle ABC and rotate it to form rectangle ACBE with base a and altitude b.

Duplicate two triangles of region 2 and one square of region 3. Arrange the copies to form rectangle *CDFG* with base a + b and altitude s.

We define A(R) as the area of region R. Give the reasons for the following algebraic steps.

- 1. A(ACBE) = ab
- 2.  $\land$  (CDFG) =  $\land$  (ACBE)

3. 
$$A(CDFG) = s(a + b)$$

4. 
$$s(a + b) = ab$$

5. 
$$s = \frac{ab}{a+b}$$
 Q.E.D.



From the Ancient Han Dynasty

Han Gaozu was the first emperor and founder of the Western Han dynasty (208 B.C.–8 A.D.). He reigned from 206 to 195 B.C. This period was impressive for the development of art, literature, and science as well as philosophy, political and social organization and the opening up of the Silk Road leading to commerce with the Roman Empire. From this time comes the *Jiuzhang suanshu (Nine Chapters of Mathematical Art)*, China's oldest mathematical text and its most important one from ancient times. From the *Nine Chapters of Mathematical Art* comes the following geometrical theorem about the diameter of a circle inscribed in any right triangle. Can you prove it?



Han Gaozu, based on a Ming painting

0

## Juscribed Circle Theorem

The diameter of a circle inscribed in a right triangle is the quotient of twice the product of the legs and the sum of the three sides.

Given: Right triangle *ABC* with inscribed circle *O* of diameter *d*; *AC* = *b*; *BC* = *a*; *AB* = *c* Prove:  $d = \frac{2ab}{a+b+c}$ 







China's oldest book, combining mathematics with astronomy, is the Zhoubi suanjing (Zhou Dynasty Computation with the Gnomon), finalized during the first century A.D. The two scholars Chen Zi and his student Rong Fang discuss the movements of the sun. Their conversation leads to the problem of calculating the height of the sun above the earth. Although the astronomy at that time was based on the assumption that the earth was flat, their solution demonstrates how ancient Chinese mathematicians used dissection to solve problems with right triangles. The problem, using modern algebraic notation, is defined below and on page 43 with the accompanying Figure 1. Can you derive this formula?

### The Problem

The sun at point *A* is directly over point *C* on the earth. Rays of light strike two vertical poles, or *gnomon*, of equal height h at locations *S* and *T*. They are a distance *d* apart. At the same time of day, the shadow lengths of the poles are recorded as  $s_1 = SG$ and  $s_2 = TB$ . Show that H = AC, the height of the sun above the earth, is given by the formula

# $H = \frac{hd}{(s_2 - s_1)} + h$

#### Below.

Ancient Chinese gnomon tower used to measure the shadows cast by the sun. This drawing is adapted from Joseph Needham's *Science and Civilisation in China*, vol. 3, plate XXXI.

The Ambitious Horse: Ancient Chinese Mathematics Problems







# The Sea Jsland

The Jiuzhang suanshu (Nine Chapters of Mathematical Art) is China's oldest book on mathematics. In 263 A.D., Liu Hui wrote a famous commentary on it in which he added nine problems on the topic of surveying. His methods became the basis of Chinese surveying for the next 1,000 years. So important were the nine problems considered that they were made into a separate book at the beginning of the Tang dynasty (618-907 A.D.) The book was given the title Haidao suanjing (Sea Island Canon of Mathematics) because the very first problem required a distant observer to determine the size and distance of an island from the seashore.

### The Problem

There is a sea island an unknown distance from the seashore. Someone wants to know how far away the island is and how high is its mountain peak. An observer erects two 30 ft poles 1,000 ft apart so that the mountain peak and the two poles are aligned. When the observer steps back 123 ft from the first pole, he sights the mountain peak from the ground; the top of the pole and the mountain peak align. When the observer steps back 127 ft from the second pole, he makes another sighting of the summit of the mountain so that the top of the second pole and the mountain peak align as viewed from the ground.

The solution begins on page 47.





Ancient Chinese Philosophy

Ancient Chinese teachers from the Zhou dynasty (1045–246 B.C.) said that there were two complementary principles in nature: the *Yin* and the *Yang*. Yin and Yang are usually understood as deriving out of the formless, unitary Dao (Tao), so it is the Dao that is the origin of all things. Yin and



Yang are "complementary," rather than "opposing," for there is Yin in Yang and Yang in Yin. It is the ethereal material energy or force (*qi*) that actually composes things in the universe, whereas Yin and Yang can be understood as the "principles" of things although "principle" was not a term used in the Zhou dynasty. The symbol for the yin-yang is shown in the figure at left.

## Yin-Yang Problem

In the figure, arc (AXO) and arc (OYB) form semicircles with diameters equal to the radius of circle O. Two interesting mathematical properties can be shown:

(1) the area of region *R* equals the area of

region *S*; (2) the length of the curved path from *A* to *B* through points *X*, *O*, and *Y* equals one-half the circumference of the circle *O*. The basic definitions follow with a geometric proof for you to complete on page 50.

## Definitions

- 1. Let A(R), A(S) be the areas of regions R and S, respectively.
- 2. Let A(AXO), A(OYB), A(ALB), and A(AMB) be the areas of respective semi-circles.
- Let ⊥(AXO) and ⊥(OYB) be defined as arc lengths of respective semicircles. ⊥(AXOYB) is the length of the curved path from A to B through points X, O, and Y.

| <b>-</b><br>Υin-Yang  | g Proof         |
|---|-----------------|
| Given (Statements)  | Prove (Reasons) |
| 1. Circle <i>O</i> with diameter <i>AB</i> ;<br>semicircles <i>AXO</i> and <i>OYB</i> with<br>diameters equal to $\frac{AB}{2}$ . | 1.              |
| 2. $A(S) = A(AXO) + A(ALB) - A(OYB);$<br>A(R) = A(OYB) + A(AMB) - A(AXO)  | 2.              |
| 3. $A(AXO) = A(OYB);$<br>A(ALB) = A(AMB)  | 3.              |
| 4. $A(R) = A(AXO) + A(ALB) - A(OYB)$  | 4.              |
| 5. $A(S) = A(R)$ $Q. \in D.$  | 5.              |
| 6. $\bot(AXOYB) = \bot(AXO) + \bot(OYB)$  | 6.              |
| 7. $\bot(AXO) = \pi \frac{AB}{4};$  | 7.              |
| $\bot(OYB) = \pi \frac{AB}{4}$  |                 |
| 8. $\perp$ ( <i>AXOYB</i> ) = $\pi \frac{AB}{4} + \pi \frac{AB}{4}$   | 8.              |
| 9. $\bot(AXOYB) = \frac{\pi}{2}$ (AB)   | 9.              |
| 10. $\perp(AXOYB) = \frac{1}{2}$ (circumference)  | 10.             |
| Q.E.D.  |                 |
|   |                 |
|   |                 |



Another Case of Vin-Yang! Given: Circle O with diameter AD containing points A, B, C, D such that AB = OB = OC = CD: semicircles: AXB; AYO; AZC; CJD; OID: BHD. Prove: A(R) = A(S) = A(T) = A(U); $\bot$ (*AXBHD*) =  $\bot$ (*AYOID*)  $= \bot (AZCJD).$ Four A Instructions 1. Study the Four Regions figure, what R М is given, and what you need to prove S 0 in the problem. Use the suggestions that follow to help you construct H T a proof. U 2. State the formulas for the area and circumference of a circle. Apply the D formulas to the four paths and the Regions four regions of area. 3. Develop a set of algebraic steps from the given to the prove. 4. Translate the algebraic steps into a formal proof by asking yourself for the reasons in each step. 5. Decide whether you need any additional theorems or definitions between steps. Supply the needed steps in your proof.









- 1. Study the Zhou diagrams in the figures on page 55. In each diagram, assume that each of the square grids, underlying the bold-lined patterns, consists of unit squares. Explore possible numerical relations for the lengths and areas of the figures.
- 2. In Figure 1, find the length and area of the square-board grid and various bold-lined geometrical shapes on the board. Write down any numerical relationships that satisfy the Pythagorean theorem.
- 3. In Figure 2, can you find any geometrical relationships between the two bold-lined squares? Are there any numerical relationships that satisfy the Pythagorean theorem?
- 4. In Figure 3, the smaller bold-lined square is inscribed in the larger one. Find the numerical relationships between the sides and areas of the geometrical shapes formed on the board. Use the Pythagorean theorem and the quadratic formula to find the lengths of the legs of any of the four congruent right triangles formed between the two bold-lined squares.

Hint

Let one leg be defined as having length xand the other leg 5 - x. Below.

Shang dynasty stone column with symbols for heaven and earth (Shang dynasty, circa 1500–1045 в.с.).

From ancient times, the numbers 3, 4, and 5 have had important religious significance in China. The number 3 is a symbol for heaven, and the number 4 represents the earth. The number 5 is significant because it is the number of the Five Phases: Wood, Fire, Earth, Metal, and Water.



Gou-Gu Theorem

On page 55, we presented the three mathematical figures of Shang Gao. Figure 1 demonstrates the *Gou-gu* theorem, the Chinese version of the Pythagorean theorem. In ancient China, *Gou-gu* referred to the properties of right-angled triangles. *Gou* referred to the base of a right triangle (horizontal leg), while *Gu* signified vertical measure—the altitude of the triangle (vertical leg). The hypotenuse of the right triangle is called *xian*, or "bow string." The *Gou-Gu* theorem states that in a right triangle, the sum of the squares of the *Gou* and the *Gu* is equal to the square of the *xian*. Here is the diagram of the third-century Chinese commentator of the *Gou-Gu* theorem, Liu Hui:



If we examine the figure without moving the pieces, it is clear that the square of side c has area equal to the sum of the areas of one yellow square and four red triangles. Using modern algebra, we write

$$c^{2} = (b-a)^{2} + 4R$$
  $R = \frac{ab}{2}$  (1)

which simplifies to the familiar  $a^2 + b^2 = c^2$ .

Another way to look at the above figure is to think of the area of the large square with side a + b as the sum of areas of the yellow square and four rectangles of base a and altitude b. Again, using algebra, we write

$$(a + b)^{2} = (b - a)^{2} + 4ab$$
(2)

Equation (2) was used in ancient China to solve certain quadratic equations.

# Liu Hui's Proof

In Liu Hui's third-century *Commentary* on the Jiuzhang suanshu, we find the *Gou-Gu* theorem in a form similar to that of the Pythagorean theorem:

In a right triangle, the area of the square on the hypotenuse (*xian*) equals the sum of the areas of the squares on the base (*gou*) and the altitude (*gu*). (See Figure 1.)

Lui Hui provides a simple proof based on the Chinese dissection process using Figure 2. We have shaded the right triangle having legs a and b and hypotenuse c. Start with square *ABCD* whose side is c and area is  $c^2$ . Remove pieces 1, 2, and 3 by translation and rotation to form pieces 4, 5, and 6, respectively. The original square has now been transformed into two new squares: (1) square *EFBG* of side b and area  $b^2$ ; (2) square *GHIC* of side a and area  $a^2$ . We write

$$A(GHIC) + A(EFBG) = A(ABCD)$$

$$a^2 + b^2 = c^2$$

Can you prove that pieces 1, 2, and 3 are congruent to pieces 4, 5, and 6, respectively?





# Ambitious Horse Solution

This problem can be readily solved using the formula for the sum of a finite arithmetic series:

$$S_n = a + (a + d) + (a + 2d) + \dots + (a + (n - 1)d)$$

where a is the first term, n is the number of terms, and d is the constant difference. It can be shown that

$$S_n = \frac{n}{2}(a + a_n) \text{ and } a_n = a + (n - 1)d$$
 (1)

where  $a_n$  is the *n*th or last term of the series. In the problem, a = 193 li, n = 15, and d = 15. Hence,  $a_n = 403$  and  $S_n = 4,470$  li.

The Chinese solution makes use of Yang Hui's diagram (Figure 1). Each bar represents the distance the horse traveled on a particular day. The diagram is abbreviated in that five bars represent all fifteen bars or terms of an arithmetic series. The area of the entire figure represents the total distance the horse traveled on the journey. The area in Figure 1 is equivalent to that of the triangle and rectangle shown in Figure 2 on page 61. Using algebraic notation, we find the distance the horse traveled by combining the areas of the triangle and the rectangle in Figure 2:

$$S_n = A_1 + A_2$$
, where  $A_1 = \frac{1}{2}(a_n - a)(n)$  and  $A_2 = an$ 







Chinese mathematicians of the early Han dynasty (circa 200 B.C.) were proficient in extracting square roots. So ingenious were their methods that they led to the solution of quadratic equations and polynomial equations of higher degree.

The ancient Chinese method of extracting square roots relies on division and estimation, similar to algorithms of modern Western mathematics. However, in the Chinese technique, a strong component of geometry clarifies the mechanical steps. Since these steps were done on a calculating board with calculating rods, the theory can be better understood if algebra is used.

The fundamental principle of extracting square roots, in ancient Chinese and modern Western mathematics, can be explained by the two algebraic identities:

$$(a+b)^2 = a^2 + 2ab + b^2$$
(1)

$$(a + b + c)^{2} = (a + b)^{2} + 2(a + b)c + c^{2}$$
(2)

Equations (1) and (2), along with an iteration process, govern the extraction of square root of any rational number. The following explanation presents the principles of square root extraction in modern Western format, supplemented by the Chinese geometric dissection process. Fundamentally, the two processes are the same: one done on a calculating board, the other by written arithmetic.

#### Below.

Han dynasty books consisted of vertical writing on bamboo slips about 23.1 cm in length. As the drawing shows, the bundle could be rolled up and tied for convenience.



Assume *n* represents a whole number whose positive square root is precisely a two-digit number a + b, where *a* is the value of the tens digit and *b* is the value of the ones digit. Our task is to derive a + b from *n*. Since a + b is the square root of *n*,  $n = a^2 + 2ab + b^2$ . In Chinese mathematics, this fact was represented geometrically (see Figure 1). The area of the large square represents *n*, and the side of this square is a + b or  $\sqrt{n}$ . Each of the cells of Figure 1 relates to the expression  $a^2 + 2ab + b^2$ .

There are two square cells of area  $a^2$  and  $b^2$ , respectively; and there are rectangular cells of length a and width b, having a total area of 2ab. As we explore the steps in extracting a + b using the written algebraic process, we can see the steps unfold geometrically through the Chinese dissection process.


Step 1

Set the number  $n = a^2 + 2ab + b^2$  as the dividend. Find the principal square root of the first term and subtract its square from the dividend.

In Figure 1 on page 63, this operation corresponds to finding the shaded square area  $a^2$  and subtracting it from the area  $(a + b)^2$  of the large square. The result,  $2ab + b^2$ , is the area of the L-shaped region called a *gnomon*.

Estimate the second digit. Take twice *a*, the partial root, as a trial divisor. Divide the first term of the remainder. Write the quotient above the second term of the dividend.

$$a + b$$

$$a^{2} + 2ab + b^{2}$$

$$a^{2}$$
Trial divisor  $\rightarrow 2a \quad 2ab + b^{2}$ 

When working with numerical values, the structure of the precise factors is unknown. So, when dividing by a trial divisor, the quotient may be too large so that the subtraction yields a negative number. In that case, a smaller quotient would be tested. In this case, *b* is the precise number needed.

#### Step 3

Add the quotient b to the trial divisor to obtain the actual divisor. Multiply the second term of the root b by the actual divisor. Subtract the product from the last remainder.

$$a + b$$

$$a^{2} + 2ab + b^{2}$$

$$a^{2}$$
Trial divisor  $\rightarrow 2a$ 

$$2ab + b^{2}$$

$$2ab + b^{2}$$

$$2ab + b^{2}$$

$$a^{2}$$

$$b^{2}$$

$$b^{2}$$

$$a^{2}$$

As previously mentioned, in working with numerical values, it is not always obvious what value to use for *b*. Re-examining the L-shaped gnomon of Figure 1 on page 63 reveals an important relationship Chinese mathematicians used to estimate *b*. By inspecting the relationship between the areas of the cells, it follows that

$$2ab \le 2ab + b^{2}$$
  
But  $2ab + b^{2} = n - a^{2}$   
$$2ab \le n - a^{2}$$
  
$$b \le \frac{n - a^{2}}{2a}$$
(3)

This formula provides the largest possible value of *b* as a trial number. After *a* is found, Formula (3) is easily evaluated since  $n - a^2$  is the remainder in step 1. If the value of *b* is too large, the subtraction is negative and a smaller number must be tried.

## Square Root with Three Digits

The process of extracting the square root of a number can be extended to a root with three or more digits. Assume *n* represents a whole number whose positive square root is precisely a three-digit number a + b + c, where *a* is the value of the hundreds digit, *b* is the value of the tens digit, and *c* is the value of the ones digit. We need to derive a + b + c, from *n*.

If  $\sqrt{n} = a + b + c$ , then  $n = (a + b + c)^2$  by definition. Using Identity (2) on page 57 and expanding the terms, it follows that

$$(a + b + c)^{2} = (a + b)^{2} + 2(a + b)c + c^{2}$$
$$(a + b + c)^{2} = a^{2} + b^{2} + c^{2} + 2(ab + ac + bc)$$
(4)

Geometrically, Equation (4) can be represented by Figure 2 on page 66. The area of the large square represents  $n = (a + b + c)^2$  with side equal to  $\sqrt{n} = a + b + c$ . Equation (4) conforms to the Chinese pattern of geometric dissection. The area of the large square is the sum of the areas of three squares and six rectangles as can be seen in the figure.



Example

Find:  $\sqrt{121,104}$ 

Step 1

Set 121,104 as the dividend, marking off the digits in pairs from right to left. The number of pairs is the same as the number of digits in the square root, in this case three.

12 ' 11 ' 04

#### Step 2

Examine the pair 12, representing 120,000. The largest whole-number square root contained in 12 is 3. The 3 is placed above the 12 as a partial square root and represents 300. The square of 3 is 9, but 9 actually represents  $(300)^2$ , or 90,000. We write 90,000 under the dividend and subtract, obtaining a remainder of 31,104.

| 3  |        |    |
|----|--------|----|
| 12 | ' 11 ' | 04 |
| 9  | 00     | 00 |
| 3  | 11     | 04 |

In most square root algorithms, 9 would be written under the 12 in abbreviated form, rather than its full place value of 90,000, but we wish to demonstrate the Chinese dissection process, showing the geometric regions corresponding to the numbers in the algorithm, beginning with Figure 3 on page 68.

In Figure 3, the number 121,104 is the area of square *ABCD*, while 90,000 is the area of shaded square *EFGD*. When the shaded area is removed from the figure, the L-shaped gnomon *ABCGFE* remains and has area equal to 31,104.







Step 4

We are looking for the ones digit c to complete the square root. In Figure 5 on page 70, c is the unknown side of the gnomon. If we can find a digit c that makes the gnomon have an area of exactly 5,504 square units, we will have found a square root of exactly three digits. If there were no ones digit c giving us the precise area of 5,504 square units, we could continue the iterations and find subsequent place values of tenths, hundredths, and so on, or resort to approximating the root with a fractional remainder.

Continuing our search for the ones digit c, we apply the Chinese estimation formula, Equation (3) on page 65, to Figure 6.

$$c \le \frac{n - (a + b)^2}{2(a + b)}$$
$$c \le \frac{5,504}{2(340)} \approx 8.09$$

The value of *c* cannot be greater than 8. If we use c = 8, the gnomon *ABCLJH* has area 5,504, and we have found the last digit of the square root. See Figure 6.



Returning to the corresponding steps of the algorithm, take twice 340 as the trial divisor to find c. Divide the trial divisor into the remainder to get the estimate for the ones digit. The result

 $5,504 \div 680 \approx 8.09$ 

corresponds to the Chinese formula, Equation (3), we used when we looked at the geometric process. The final steps of the algorithm are written as

|                | _                             | 3    | 4      | 8  | ← Square root     |
|----------------|-------------------------------|------|--------|----|-------------------|
|                |                               | 12   | ' 11 ' | 04 | ← Dividend        |
|                |                               | 9    | 00     | 00 | ← Square          |
|                | 600                           | 3    | 11     | 04 | Remainder         |
|                | 640                           | 2    | 56     | 00 | ← Product         |
| Trial divisor  | $\longrightarrow$ 2(340) = 68 | sō - | 55     | 04 | Remainder         |
| Actual divisor | $\rightarrow$ 680 + 8 = 68    | 8    | 55     | 04 | ← Product         |
|                |                               |      |        | 0  | - Final remainder |

By comparing step 4 and Figure 6 with step 3 and Figure 3, we see that the algorithmic steps or geometric procedure form an iteration process. Chinese mathematicians of the early Han dynasty were capable of extracting the square root of any rational number. They had a choice of continuing the iteration process to whatever place value they thought sufficient or using approximation formula to terminate the process. At this early period of history, a well-developed decimal system existed in China. Sometimes, the square root of a number is an irrational number. Although Chinese mathematicians did not develop the concept of irrational numbers completely, they recognized them as being different from other numbers. They called irrational numbers *side numbers* in the context of extracting square roots.



Han dynasty character

From the *Nine Chapters of Mathematical Art*, we learn that ancient Chinese mathematicians of the early Han dynasty solved quadratic equations of the form  $x^2 + mx - n = 0$ , where *m* and *n* are positive rational numbers. Their technique was based on geometrical dissection and the square root extraction process performed on the calculating board. In all Chinese examples, problems were numerically specific and did not have the benefit of a written algebra.

We can best understand the ancient Chinese method by outlining the process for a solution having a three-digit answer. However, this process generalizes to any rational number solution or irrational number, when used with the approximation formula for the remainder.

Suppose  $x_1 = a + b + c$  is a positive root of Equation (1), where *a* is the hundreds digit, *b* is the tens digit, and *c* is the ones digit:

$$x^2 + mx - n = 0$$
 (1)

Substituting the root  $x_1 = a + b + c$  into Equation (1), we have the following new Equation (2) involving the three digits of the root:

$$(a + b + c)2 + m(a + b + c) - n = 0$$
(2)

Equation (2) can be expanded and terms rearranged to be compatible with the Chinese square root extraction process:

$$(a + b + c)^{2} = a^{2} + b^{2} + c^{2} + 2(ab + ac + bc) + m(a + b + c) - n = 0$$
(3)

Impression made from an early bronze seal belonging to a Han dynasty general.



Equation (1) has a special geometric relation to Figure 1 below. The area of square *AIJD* equals  $x^2$ . The area of the shaded rectangle *IBCJ* equals *mx*, and the area of rectangle *ABCD* equals *n*. Furthermore, Equation (1), as a whole, is a statement about the dissection process: When all the component rectangles are removed from rectangle *ABCD*, the area of rectangle *ABCD* is reduced to zero.



FIGURE 1

We begin solving Equation (1) by focusing our attention on Figure 1. The essential idea is to begin the process of extracting the square root on square *AIJD* with the value n = area of *ABCD*—not the area of square *AIJD* as in the case of a regular square root extraction. Keep in mind that calculating the square root this way has an element of trial and error. We are forcing the solution of the quadratic equation into the form of square root extraction. As we find *a*, *b*, and *c* through the square root process, we successively remove the parts of the shaded rectangles, that is, *ma*, *mb*, *mc*. These steps are outlined on the next page, followed by a numerical example.







#### Step 3

Find *c*. By using Formula (5) on page 75, we estimate the value of *c*. Accordingly, if a = 300 and b = 40,  $c \le 8.09$ . We use c = 8 and compute  $c^2 + 2ac + 2bc + 15c$  and subtract from the last remainder. If a = 300, b = 40, and c = 8,  $c^2 + 2ac + 2bc + 15c = 5,624$ . Since the final remainder is zero, x = 348 is the solution of the quadratic equation  $x^2 + 15x - 126,324 = 0$ .



If the last remainder of this process had not been zero, two additional zeros could have been added to the right of 24 in the dividend. The process can be extended along the established pattern to find the tenths place. There is no limit to this process, except perhaps fatigue. As many decimal place values as are needed can be obtained by adding more pairs of zeros to the right side of the dividend. If the root is irrational, the process can be terminated after reaching a desired accuracy. In such cases, the final place value can be approximated by remainder formula. Ancient Chinese mathematicians did not develop the concept of irrational number, but recognized that such nonterminating decimals were different and called them *side numbers*.

# Advanced Mathematics of Medieval China

During the Song and Yuan dynasties (960–1368 A.D.), China developed an impressive amount of advanced mathematics. One of those achievements was the solution of polynomial equations of higher degree. As early as the mid-eleventh century, the illustrious Chinese mathematician Jia Xian introduced an iterated multiplication process on the calculating board for extracting roots or solving polynomial equations of arbitrary degree. His method was equivalent to the Ruffini-Horner method developed about 800 years later in Europe.

Since Jia Xian's work on solving polynomial equations was done on a calculating board with movable calculating rods, we can best understand his mathematics using modern algebraic techniques. The logical basis for justifying Jia Xian's method rests on the *diminished root theorem* from the part of algebra known as the *theory of equations*.

### Diminished Root Theorem

Let  $f(x) = a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \dots + a_n = 0$  be a rational integral polynomial defined over  $x \in R$ . For any  $c \in R$ , f(x) is successively divided by x - c as follows:

 $Q_1(x)$  is the quotient and  $r_1$  is the remainder of  $f(x) \div (x - c)$ ;  $Q_2(x)$  is the quotient and  $r_2$  is the remainder of  $Q_1(x) \div (x - c)$ ;  $Q_3(x)$  is the quotient and  $r_3$  is the remainder of  $Q_2(x) \div (x - c)$ ; and so on, until the process ends with  $Q_n(x) = a_0$  and  $r_n$ . If a new polynomial equation f(y) is formed such that

 $f(y) = a_0 y^n + r_n y^{n-1} + r_{n-1} y^{n-2} + \dots + r_2 y + r_1 = 0$ , then the roots of the f(y) = 0 equation are *c* less than those of the f(x) = 0 equation, that is, y = x - c.

A simple example of a type of problem encountered by Jia Xian was the extraction of a fourth root of a number or the solution of a fourth-degree polynomial equation. The two processes can be interchangeable as the following example illustrates.

Example 1

Find 
$$x = \sqrt[4]{279,841}$$
 (3)

or solve for x, 
$$x^4 - 279,841 = 0$$
 (4)

At right.

Jia Xian was familiar with the binomial expansion diagram shown here. The European version known as Pascal's triangle was discovered 600 years later by the French mathematician Blaise Pascal (1623–1662).

The illustration shown was first published in Zhu Shijie's *Siyuan yujian xicao* in 1303.



Solution

In Chinese mathematics, Equations (3) and (4) are equivalent as only a single positive root was sought for Equation (4).

In our solution to the problem, we apply the diminished root theorem to Equation (4) and use synthetic division to simplify the process.

Step 1

Our process begins by making an estimate. We know that 20 < x < 30 because  $(20)^4 < x^4 < (30)^4$ . Assume that x = 20 + y, where 0 < y < 10. This transformation implies that the solution x is at least 20 plus some positive number less than 10. If we apply the diminished root theorem to Equation (4) with c = 20, we obtain a new equation in y, whose solution is a number less than 10. Using synthetic division, we get

| 1 | 0  | 0     | 0      | -279,841 | 20 |
|---|----|-------|--------|----------|----|
|   | 20 | 400   | 8,000  | 160,000  | L  |
| 1 | 20 | 400   | 8,000  | -119,841 |    |
|   | 20 | 800   | 24,000 |          |    |
| 1 | 40 | 1,200 | 32,000 |          |    |
|   | 20 | 1,200 |        |          |    |
| 1 | 60 | 2,400 |        |          |    |
|   | 20 |       |        |          |    |
| 1 | 80 |       |        |          |    |

Therefore, if y = x - 20, the new equation in *y* is

$$g(y) = y^4 + 80y^3 + 2,400y^2 + 32,000y - 119,841 = 0$$
 (5)

#### Step 2

Since Chinese mathematics used only positive roots, we can assume that all terms involving y in Equation (5) are positive. Therefore,

 $32,000y - 119,841 \le 0$  (6)  $y \le 3.7$  Accordingly,  $y \le 3.7$  can be used to approximate the next digit of the root. We use y = 3 as an approximation for the ones digit of the root. Applying the diminished root theorem to Equation (5), we set up the transformation y = z + c, where c = 3, to find a new equation in *z* whose solution we anticipate to have a value in the tenths place, that is, z < 1.

| 1 | 80<br>3 | 2,400<br>249 | 32,000<br>7,947 | –119,841<br>119,841 | 3 |
|---|---------|--------------|-----------------|---------------------|---|
| 1 | 83      | 2,649        | 39,947          | 0                   |   |

Applying the diminished root theorem, we find that after the first synthetic division, the first remainder is zero. We need not continue to divide synthetically because the first division tells us that g(3) = 0. This fact follows by use of the remainder theorem of modern algebra: If  $g(y) \div (y - c)$  has remainder *r*, then g(c) = r.

Since y = 3 is the solution of Equation (5) and y = x - 20, it follows by substitution that x = 23 is the solution to Equations (3) or (4). Thus we found that 23 is the fourth root of 279,841 or x = 23 is the solution of the polynomial equation  $x^4 - 279,841 = 0$ .

This first example was intended to be very simple to illustrate the basic process of solving higher-degree polynomials. As a second example that is more typical of using the diminished root theorem and Jia Xian's method, we return to the Quadratic Equation (6) on page 75, where we solved the equation using the geometric dissection method of the Han dynasty. It is interesting to compare the numerical values generated between the two processes.

 $\mathcal{E}_{xample 2}$ Solve  $x^2 + 15x - 126,324 = 0$  for x.

Step 1

We write the equation in functional notation to better communicate the reasoning:

Solve 
$$f(x) = x^2 + 15x - 126,324 = 0$$
 for x.

(1)

We estimate the solution of Equation (1) is such that 300 < x < 400. In either Chinese or Western mathematics, this is a trial process. In modern algebra, the *location principle* would be used, that is, substitute arbitrary values of x in  $f(x) = x^2 + 15x - 126,324$  and look for sign changes in f(x). In this case, f(300) = -31,824 and f(400) = 39,676. Hence, a root for Equation (1) falls between x = 300 and x = 400. The steps using synthetic division follow:

|   | Location | Principle |     |
|---|----------|-----------|-----|
| 1 | 15       | -126,324  | 300 |
|   | 300      | 94,500    |     |
| 1 | 315      | -31,824   |     |
|   |          |           |     |
| 1 | 15       | -126,324  | 400 |
|   | 400      | 166,000   |     |
| 1 | 415      | +39,676   |     |
|   |          |           |     |

 $f(x) = x^{2} + 15x - 126,324$ f(300) = -31,824f(400) = +39,676

Step 2

We apply the diminished root theorem to Equation (1) with x = y + 300, where 0 < y < 100. In the transformation to the new equation in *y*, the solution, *y*, is reduced one place value.

| 1 | 15<br>300  | -126,324<br>94,500 | _300 |
|---|------------|--------------------|------|
| 1 | 315<br>300 | -31,824            |      |
| 1 | 615        |                    |      |

According to the diminished root theorem, we have the new equation in y with root reduced by 300:

$$g(y) = y^2 + 615y - 31,824 = 0$$

Step 3

Since  $y^2 \ge 0$ , Equation (2) can be used to make an estimate for the solution of *y*. We write the following inequality based on this assumption:

$$615y - 31,824 \le 0$$
 (3)  
 $y \le 51.7$ 

Thus, by trial we determine that y is at least 40. Using the location principle, we have g(50) = +1,426 and g(40) = -5,624. The sign change between y = 40 and y = 50 implies that the solution for y is such that 40 < y < 50. The steps using synthetic division follow:

| 1 | 615<br>50 | -31,824<br>33,250 | 50 |
|---|-----------|-------------------|----|
| 1 | 665       | +1,426            |    |
| 1 | 615<br>40 | -31,824<br>26,200 | 40 |
| 1 | 655       | -5,624            |    |

$$g(y) = y^{2} + 615y - 31,824$$
$$g(50) = +1,426$$
$$g(40) = -5.624$$

Step 4

We apply the diminished root theorem to Equation (2) with y = z + 40, where 0 < z < 10. We obtain a new equation in *z* whose solution is 40 less than the solution of the equation in *y*. Applying the diminished root theorem using synthetic division, we have

| 1 | 615<br>40 | -31,824<br>26,200 | 40 |    |
|---|-----------|-------------------|----|----|
| 1 | 655<br>40 | -5,624            |    |    |
| 1 | 695       |                   |    | ŧ, |

Taking the values from the previous synthetic division, we can write the following new function in *z*:

$$h(z) = z^2 + 695z - 5624 = 0 \tag{4}$$

Since  $z^2 \ge 0$ , Equation (4) can be used to estimate the solution of *z*. We write the following inequality:

$$695z - 5624 \le 0$$
 (5)  
 $z \le 8.09$ 

Thus, by trial we determine that z is at least 8. We start by applying the location principle. We have h(8) = 0. There is no need to continue as we have found the exact solution for Equation (4). The steps are as follows:

Location Principle

| 1 | 695<br>8 | -5,624<br>5,624 | 8 |
|---|----------|-----------------|---|
| 1 | 703      | 0               |   |

$$h(z) = z^2 + 695z - 5,624$$
$$h(8) = 0$$

Since z = 8 and y = z + 40, then y = 48.

But x = 300 + y.

Therefore, x = 348 and is the solution of

$$f(x) = x^2 + 15x - 126,324 = 0.$$

We have used modern algebra and synthetic division to perform the steps of the solution of Quadratic Equation (1). Jia Xian and other medieval Chinese mathematicians used the calculating board. Although there were no permanent recorded steps during the process, the actual numbers on the calculating board correspond to the written numbers in our synthetic division steps.



As early as the fifth century A.D., Chinese mathematicians solved problems with indeterminate solutions. We find the intriguing problem known as the "Problem of the One Hundred Fowls" in the Chinese classic *Zhang Qiujian suanjing (The Computational Classic of Zhang Qiujian)*. The problem follows, expressed in contemporary language. Can you find a solution?

#### The Problem

A rooster costs 5 copper coins called *qian*. A hen costs 3 *qian*, while three chicks can be purchased for 1 *qian*. You have 100 *qian* to buy exactly 100 fowls. How many roosters, hens, and chicks are there?

#### Hint

There are several possible solutions. Assume that the solution has at least one of each type of fowl.



Solution to the 100 Fowls Problem x be the number of roosters: Let: *y* be the number of hens: z be the number of chicks. x + u + z = 100Then: (1) $5x + 3y + \frac{1}{3}z = 100$ (2)Solving Equation (1) for z and substituting into Equation (2), we have  $5x + 3y + \frac{1}{3}(100 - x - y) = 100$  $y=25-\frac{7}{4}x$ (3)Equation (3) does not have a unique solution. Since any solution in x, y, and z must be a positive integer, we introduce the parameter  $a, a \in N$ defined by (4) x = 4aSubstituting Equation (4) into Equation (3), we have u = 25 - 7a(5)Substituting both Equations (4) and (5) into Equation (1), we have z = 75 + 3a(6)Thus, we can redefine the equation set of (1) and (2) by the following parametric set in a. x = 4a(4)y = 25 - 7a(5)z = 75 + 3a(6)

The solution set in (x, y, z) is infinite. If a = 0, there can be no roosters. If a = 1, we have Zhang Qiujian's classic solution of 4 roosters, 18 hens, and 78 chicks. What other solutions are possible?

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# Chinese Remainder Theorem

Solutions of problems such as "The Emperor of Qin Secretly Counts His Soldiers" were so prominent in the early history of Chinese mathematics that even Western historians refer to the basis of the method of solution as the *Chinese remainder theorem*. We utilize modern congruence notation of number theory to first state the Chinese remainder theorem and then apply it to the Emperor of Qin problem.

#### Definition:

 $a \equiv b \pmod{m}$  if and only if a = b + mk, where a, b, k, and  $m \in N$ 

Suppose x is an integer such that, when it is divided by  $m_1$ , the remainder is  $r_1$ . If x is divided by  $m_2$ , the remainder is  $r_2$ . If the process is repeated an arbitrary number of times, say k, the divisor of x is  $m_k$ , the remainder is  $r_k$ ,  $m_k$ ,  $r_k \in N$ . This process is equivalent to the following set of congruences:

$$x \equiv r_1 \pmod{m_1}$$
$$x \equiv r_2 \pmod{m_2}$$
$$x \equiv r_3 \pmod{m_3}$$

$$x \equiv r_k \pmod{m_k} \tag{1}$$

If  $m_1, m_2, m_3, \ldots, m_k$  are pairwise relatively prime and  $M = m_1 m_2 m_3 \ldots m_k$ , then there exists a set of integers  $a_1, a_2, a_3, \ldots, a_k$  such that for each  $a_k$ ,

$$a_{k} \frac{M}{m_{k}} \equiv 1 \pmod{m_{k}}$$
(2)

and

$$x \equiv a_1 r_1 \frac{M}{m_1} + a_2 r_2 \frac{M}{m_2} + \dots + a_k r_k \frac{M}{m_k} \pmod{M}$$

(3)

Solution to the Emperor of Qin Problem Let: x be the number of soldiers. Then:  $x \equiv 2 \pmod{3}$ (4)  $x \equiv 3 \pmod{5}$  $x \equiv 2 \pmod{7}$ Applying the Chinese remainder theorem from page 88, we have  $m_1 = 3$  $M = 3 \cdot 5 \cdot 7 = 105$  $r_{1} = 2$  $m_{2} = 5$   $r_{2} = 3$  $m_{3} = 7$   $r_{3} = 2$ The three values of  $a_{\mu}$  are defined by Equation (2) on page 88. It follows by substitution that  $a_1 \frac{M}{m_1} = a_1 \frac{105}{3} = 35a_1 \equiv 1 \pmod{3}$ (5) $a_2 \frac{M}{m_2} = a_2 \frac{105}{5} = 21a_2 \equiv 1 \pmod{5}$ (6) $a_3 \frac{M}{m_2} = a_3 \frac{105}{7} = 15a_3 \equiv 1 \pmod{7}$ (7)Applying the definition of congruence to Equations (5), (6), and (7), it follows that  $a_1 = 2$ ,  $a_2 = 1$ ,  $a_3 = 1$  satisfy the congruences. Substituting the values of  $a_{\mu}$  in Equation (3) on page 88, we have  $x \equiv 2 \cdot 2 \cdot 35 + 1 \cdot 3 \cdot 21 + 1 \cdot 2 \cdot 15 \pmod{105}$  $x \equiv 23 \pmod{105}$ Thus, 23 is the smallest number of soldiers. However, by adding integral multiples of 105 to 23, you can have additional solutions.



The following problem from ancient China is based on the Han classic, the *Nine Chapters of Mathematical Art.* 

#### The Problem

Resting on the floor of a pavillion is a cylindrical pillar 20 ft high and 3 ft in circumference. A floral pattern with a vine wraps uniformly about the pillar. One end of the vine-pattern starts at the base of the pillar and winds so that the other end just reaches to the very top of the pillar. Can you find the length of the vine wound about the pillar?

#### Hint

The ancient text of the *Nine Chapters* of *Mathematical Art* provides a rule that can help you solve the problem:

"Take seven times the circumference of the pillar for the second side of a right triangle and the pillar's height for the first side. The hypotenuse is the length of the vine."

What assumptions did the Chinese mathematicians make in using this method?

# 明

#### Above.

The character for *Ming* of the Ming dynasty (1368–1644 A.D.).

#### Below.

Drawing based on a Ming dynasty painting.





In Figure 3, the seven turns of the vine are shown as a composite of seven right triangles, each with hypotenuse x. Therefore, the length of the entire vine is 7x or 29 ft.







The Ambitious Horse: Ancient Chinese Mathematics Problems















The stretched position of the rope is shown in the figure on the right. The leg *AB* of the right triangle *ABC* is *x*, determined by the first contact position. Leg *BC* is 8 ft, the distance from the first contact position to the end of the rope in the stretched position. The hypotenuse *AC* is the length of the entire rope, or x + 3. If we apply the Pythagorean theorem to the right triangle *ABC*, it follows that

$$(x + 3)^{2} = 8^{2} + x^{2}$$

$$x^{2} + 6x + 9 = 64 + x^{2}$$

$$6x = 55$$

$$x = 9 \frac{1}{6} \text{ ft}$$

$$x + 3 = 12 \frac{1}{6} \text{ ft}, \text{ the length of the rope}$$

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The Party's Over

The following problem from ancient China is found in the Han dynasty classic, the *Nine Chapters of Mathematical Art*.

The Problem

A bureaucrat going about his business of overseeing water use spotted a woman washing dishes in the river. Immediately, he demanded of her: "Why are there so many dishes here?" She replied, "There was a dinner party in the house." His next question was "How many guests attended the party?" She did not know but gave this reply: "Every two guests shared one dish for rice; every three guests used one dish for broth; every four guests used one dish for meat; and altogether, sixty-five dishes were used at the party."

If you had been at the river, could you have answered the questions of the overseer? How many guests attended the party?



Solution to The Party's Over Problem The solution of this problem is based on fractional parts or ratio. From the information in the problem, we construct the following ratios and convert them to a form in which they share a lowest common denominator. number of plates/guest =  $\frac{1}{2} = \frac{6}{12}$ Rice: number of plates/guest =  $\frac{1}{3} = \frac{4}{12}$ Broth: number of plates/guest =  $\frac{1}{4} = \frac{3}{12}$ Meat: (Lowest common denominator, or LCD = 12) By adding the three fractions (expressed with the LCD), we find the number of plates for rice, broth, and meat used by 12 guests.  $\frac{6}{12} + \frac{4}{12} + \frac{3}{12} = \frac{13}{12}$  plates/guest Thus, it takes 13 plates to serve a complete meal for 12 guests. Interpret the fraction  $\frac{13}{12}$  as a rate, r, or  $r = \frac{13}{12}$  plates/guest. As a rate problem, the following simple equation can be written: A = rn, where A = number of plates for all guests r = number of plates/guest n = number of guests  $n = \frac{A}{r} = \frac{65 \text{ plates}}{13/12 \text{ plates/guests}} = 60 \text{ guests}$ Therefore,




#### The Ambitious Horse: Ancient Chinese Mathematics Problems





#### The Ambitious Horse: Ancient Chinese Mathematics Problems





The Ambitious Horse: Ancient Chinese Mathematics Problems



Page 32, Exercise with the Dog

You could define a single closed region as a closed region in which you can connect any two points with a path that only contains points belonging to the closed region.

- 2. 8 square units
- 3.–7. Answers vary with the figures chosen in construction.
- 8. The square;  $4 \cdot \sqrt{8} = 8\sqrt{2}$  units
- 9. An arrangement where the pieces pairwise only shared one point;  $10(2 + \sqrt{2})$  units

## Page 33, Chinese Dissections

As shown in the diagram at the left of the problem, dissect the parallelogram into a trapezoid and a triangle. In the diagram below, these dissections are trapezoid *ABFD* 

and triangle BCF.

 $FB \perp CD$  and  $EA \perp CD$ , because FB is the altitude and EA is constructed that way.



FB = EA, because both are the distances between the same parallel lines.

AD = BC, because ABCD is a parallelogram.

 $\Delta BCF \cong \Delta ADE$ , because of HL congruence. You can translate  $\Delta BCF$  into  $\Delta ADE$  and form a rectangle with base *b* and height *h*. Since both the parallelogram and the rectangle consist of the same two congruent figures, their areas are the same. Thus, the area of the parallelogram is  $A_p = bh$ .

Dissect the parallelogram into two triangles as shown in the diagram below.  $\triangle ABC \cong \triangle ADC$ , because of SSS congruence. Thus, their areas are equal—and therefore  $A_t = \frac{1}{2}bh$ .



Page 34, Classic Theorem

Explanations can vary, but the following is one possible solution:

Draw altitude *CL* of  $\triangle ABC$  from vertex *C*. Label the intersection of *CL* and *MN*, point *K*, as shown in the diagram.

*L* is also *C*'s mirror image over *MN*, because  $\triangle ACL \sim \triangle MCK$  (*MN* || *AB* and *M* is a midpoint of *AC*).

Thus, ML = MC = AM, due to the property of reflection and because *M* is the midpoint of *AC*. So,  $\triangle AML$  is isosceles. For similar reasons, LN = CN = NB, so  $\triangle LNB$  is isosceles. *MX* and *NY* are perpendicular to *AB*,

so they are the axes of symmetry for the isosceles triangles  $\Delta AML$  and  $\Delta LNB$ . Therefore, *L* is the mirror image of *A* over *MX* and of *B* over *NY*.

Because of the property of reflection as a transformation that preserves angle measures,  $\angle 1 = \angle A$ ,  $\angle 2 = \angle B$ , and  $\angle 3 = \angle C$ . Applying the angle addition postulate and substitution, we find that  $\angle 1 + \angle 2 + \angle 3 = 180^{\circ}$ , so  $\angle A + \angle B + \angle C = 180^{\circ}$ .



Page 35, Problem 1

A. Dissecting along the diagonal:

Refer to the diagram below.  $A(ABCD) = A(\Delta ABC) + A(\Delta ACD)$ .  $\Delta ABC$ 's base is  $b_1$  and its height is h, so  $A(\Delta ABC) = \frac{1}{2}b_1h$ . For  $\Delta ACD$ , we can choose  $b_2$  (the trapezoid's second base) as its base. The height of the triangle will also be h, because the distance of vertex A from DC is the same as the distance between C and AB.  $A(\Delta ACD) = \frac{1}{2}b_2h$  and therefore,  $A(ABCD) = \frac{1}{2}b_1h + \frac{1}{2}b_2h = \frac{1}{2}(b_1 + b_2)h$ .



B. Dissecting through its bases:

Refer to the diagram below. Dissect the trapezoid into a parallelogram and a triangle. So, A(ABCD) = A(ADCE) + A(ECB).  $A(ADCE) = b_2h$  and  $A(ECB) = \frac{1}{2}(b_1 - b_2)h$ . Thus,  $A(ABCD) = b_2h + \frac{1}{2}(b_1 - b_2)h$  $= \frac{1}{2}(b_1 + b_2)h$ .



C. Dissecting along the median:

Refer to the diagram below. Draw in the perpendicular line segments through the endpoints of the median labeled *E* and *F*.  $\triangle AGE \cong \triangle DJE$ , because AE = DE (*EF* is a median) and  $\angle AEG = \angle DEJ$  (vertical angles). Both triangles have a right angle at *G* and *J* respectively, because *JG* was constructed that way. Thus, A(AGE) = A(DJE). The same can be said of  $\triangle HBF$  and  $\triangle ICF$ , therefore A(HBF) = A(ICF). So, A(ABCD)= A(GHIJ).

And 
$$A(GHIJ) = EF \cdot h$$
, and because  $EF = JI = GH$ ,  
 $2 \cdot EF = b_1 - x - y + b_2 + x + y = b_1 + b_2$ ,  
so  $EF = \frac{1}{2}(b_1 + b_2)$  and  $A(GHIJ) = EF \cdot h = \frac{1}{2}(b_1 + b_2)h$ 





Interior angles of an obtuse triangle:

Either draw in the midsegment just as you did on page 34, connecting the two shorter sides (turn the triangle on its longest side as base), or draw in the midsegment as shown on the diagram below. *CD* is the altitude from vertex *C*. *EF* || *AB*, so *CD*  $\perp$  *EF*. Because  $\triangle CDA \sim \triangle CGE$ , and *E* is a midpoint, *CG* = *GD*. Therefore, *C*'s mirror image over midline *EF* is *D*. Thus,  $\triangle CEF \cong \triangle DEF$  and  $\angle ECF = \angle EDF$ .

CF = DF due to the property of reflection and CF = FB because *F* is a midpoint, so DF = FB and  $\Delta DFB$  is isosceles. Because the base angles of an isosceles triangle are equal,  $\angle FBD = \angle FDB$ .

By the same reasoning, CE = DE because of reflection and CE = EA because *E* is a midpoint, so DE = EA and  $\Delta DEA$  is isosceles. Since the base angles of an isosceles triangle are equal,  $\angle EDA = \angle EAD$ .

 $\angle CAD = 180^{\circ} - \angle CAB$ , because they are supplementary.

From the last two steps, knowing that  $\angle EDA = \angle EAD$ , we find that  $\angle EDF + \angle FDA = 180^{\circ} - \angle CAB$ , therefore  $\angle EDF + \angle FDA + \angle CAB = 180^{\circ}$ . And because  $\angle EDF = \angle ACB$  and  $\angle FDA = \angle B$ , we get  $\angle ACB + \angle B + \angle CAB = 180^{\circ}$ .



Page 35, Problem 3

Refer to the figure on page 34 for the Classic Theorem problem.

Label the common vertex of angles 1, 2, and 3, point *L*.

 $\triangle AMX \cong \triangle LMX, \ \triangle CMN \cong \triangle LMN, \ \text{and} \ \triangle BYN \cong \triangle LYN, \ \text{therefore:}$  $\Rightarrow (XYNM) = \frac{1}{2} \Rightarrow (ABC). \ \Rightarrow (XYNM) = XY \cdot YN. \ YN = \frac{1}{2}h \ \text{and} \ XY = \frac{1}{2}AB, \ \text{so} \ \Rightarrow (ABC) = 2 \cdot \frac{1}{2}h \cdot \frac{1}{2}AB = \frac{1}{2}AB \cdot h.$ 













Page 51, More Yin-Yang! Part 2 Algebraic Proof for  $\bot$ (*AXBHD*) =  $\bot$ (*AYCID*) 1.  $\bot$ (*AXBHD*) =  $\bot$ (*AXB*) +  $\bot$ (*BHD*)  $\bot(AYCID) = \bot(AYC) + \bot(CID)$ 2.  $\perp$ (*AXB*) =  $\pi \frac{AB}{2}$  $\bot(CID) = \pi \frac{AB}{2}$  $\bot(BHD) = \pi \frac{BD}{2} = \pi(AB)$  $\bot(AYC) = \pi \frac{AC}{2} = \pi(AB)$ 3.  $\perp$ (AXBHD) =  $\frac{\pi}{2}$ (AB) +  $\pi$ (AB) =  $\frac{3\pi}{2}$ (AB)  $\bot(AYCID) = \pi(AB) + \frac{\pi}{2}(AB) = \frac{3\pi}{2}(AB)$ 4.  $\perp$ (AXBHD) =  $\perp$ (AYCID) =  $\frac{\pi}{2}$ (3AB) =  $\frac{\pi}{2}$ (AD)  $=\frac{1}{2}[\pi(AD)]$  $=\frac{1}{2}$ [circumference of circle *O*] Q.E.D.







And from the Pythagorean theorem:  $3^2 + 4^2 = 5^2$ 



## Page 58, Gou-Gu Theorem

The congruence of triangles 2 and 5 is true for the following reasons:

AB = CD, because ABCD is a square.  $\angle DKC = \angle AFB = 90^\circ$ , because EGBF is a square and DK is constructed to be perpendicular to EC. And  $\angle KCD = \angle FBA$ , because they are corresponding angles.

The congruence of triangles 1 and 4, and 3 and 6:

Because of the congruence of  $\Delta DKC$  and  $\Delta AFB$ , DK = AF and EL = DK, so AF = EL and therefore EF = AL. Also GB = EF, so GB = AL. AD = BC, because ABCD is a square, so  $\Delta ALD \cong \Delta BGC$ . IC = GC, because HGCI is a square. From the congruence of  $\Delta AFB$  and  $\Delta CGB$ , GC = AF, and AF = EL = DK, so GC = DK and CI = DK. Thus  $\Delta BGC$  and square HGCI can be translated by vector CD and the images will be  $\Delta ALD$  and square ELDK.



Page 99, The Party's Over

Alternate solution for the problem:

Let n be the number of guests.

So, 
$$\frac{n}{2} + \frac{n}{3} + \frac{n}{4} = 65$$
  
 $6n + 4n + 3n = 780$   
 $13n = 780$   
 $n = 60$ 



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# **Pronunciation Guide**

| Vowels and Diphth    | nongs                   | Consonants            |                             |
|----------------------|-------------------------|-----------------------|-----------------------------|
| а                    | f <b>a</b> ther         | b                     | <b>b</b> ike                |
| e                    | h <b>er</b>             | р                     | <b>p</b> ike                |
| i                    | see                     | m                     | mother                      |
| u                    | h <b>oo</b> f           | f                     | five                        |
| ü                    | b <b>ui</b> ck          | d                     | dog                         |
| ai                   | b <b>i</b> ke           | t                     | tea                         |
| ei                   | p <b>a</b> ge           | n                     | <b>n</b> ight, i <b>n</b> k |
| ao                   | h <b>ou</b> se          | 1                     | luck                        |
| ou                   | <b>oa</b> th            | g                     | give                        |
| an                   | l <b>an</b> d           | k                     | <b>k</b> ind                |
| en                   | m <b>en</b> tion        | h                     | house                       |
| in                   | t <b>in</b>             | j                     | <b>g</b> in                 |
| ang                  | H <b>ong</b> Kong       | q [ch]                |                             |
| eng                  | l <b>eng</b> th         | x [hs]                |                             |
| ing                  | th <b>ing</b>           | z [tz]                |                             |
| ong                  | bass <b>oon[g]</b>      | c [ts]                |                             |
| er                   | f <b>ar</b>             | S                     | see                         |
| ia                   | Ma <b>ya</b>            | zh [dj]               | george                      |
| ie                   | hear                    | ch                    | <b>ch</b> icken             |
| iu                   | you                     | sh                    | she                         |
| ian                  | Ian                     |                       |                             |
| iang                 | young                   |                       |                             |
| ua                   | q <b>ua</b> druple      |                       |                             |
| ui                   | q <b>ui</b> ck          |                       |                             |
| uan                  | t <b>wen</b> ty         |                       |                             |
| un                   | soon                    |                       |                             |
| Note: The i after "2 | z," "c," "s," "zh," "ch | ," and "sh" is pronot | unced like ə (schwa).       |

| 聖 | ታ<br>ቻ              |                    |                        |                              |
|---|---------------------|--------------------|------------------------|------------------------------|
| ſ | ba                  | ba                 | Rong Fang              | rong-fang                    |
|   | Ba Gua              | ba-gwa             | san                    | san                          |
|   | bai<br>Beijing      | bai                | Siyuan yujian<br>xicao | si-yuan-yü-jian-<br>hsi-tsao |
|   | Chop Zi             | obon tai           | Shang                  | shang                        |
|   |                     |                    | Shang Gao              | shang-gao                    |
|   | cneng               | cneng              | shang wei              | shang-wei                    |
|   | Daxie snumu zi      | da-nsie-snu-mu-tzi | shi                    | shi                          |
|   | er                  | ar                 | shou                   | shou                         |
|   | gou                 | gou                | Shushu jiuzhang        | shu shu jin diang            |
| 1 | Gou-Gu              | gou-gu             | Silusilu jiuzilalig    | shu-shu-jiu-ujang            |
|   | gu                  | gu                 | SI                     | SI                           |
|   | Haidao suanjing     | hai-dao-swan-jing  | Song                   | song                         |
| ł | Han                 | han                | Sun                    | sun                          |
|   | Han Gaozu           | han-gao-tzu        | Sunzi suanjing         | sun-tzi-swan-jing            |
|   | Jia Xian            | jia-hsian          | Tang                   | tang                         |
| 1 | Jing xi zi zhi      | jing-hsi-tzi-dji   | wan                    | wan                          |
|   | jiu                 | jiu                | Wei                    | wei                          |
| 1 | Jiuzhang suanshu    | jiu-djang-swan-shu | wu                     | wu                           |
|   | Kangxi              | kang-hsi           | xia wei                | hsia-wei                     |
|   | Kong Qiu            | kong-chiu          | xian                   | hsian                        |
|   | kung-fu             | kung-fu            | Xiangjie               | hsiang-jie-                  |
|   | le                  | le                 | jiuzhang suania        | jiu-djang-swan-ia            |
|   | li                  | li                 | Yang Hui               | yang-hui                     |
|   | ling                | ling               | yi                     | yi                           |
|   | liu                 | liu                | Yin-Yang               | yin-yang                     |
|   | Liu Hui             | liu-hui            | Yongzheng              | yong-djeng                   |
|   | Manchu              | man-chu            | yu                     | yü                           |
|   | Meng                | meng               | Yuan                   | yüan                         |
|   | Meng Tian           | meng-tian          | Zhang Qiujian          | djang chiu-jian              |
|   | Ming                | ming               | Zhang Qiujian          | djang chiu-jian-             |
| 1 | qi                  | chi                | suanjing               | swan-jing                    |
| 1 | qian                | chian              | znong wei              | ajong-wei                    |
|   | Qin                 | chin               | Znou                   | ajou .                       |
|   | Qin Jiushao         | chin-jiu-shao      |                        | ajou-bi                      |
| 1 | -<br>Qin Shihuangdi | chin-shi-hwang-di  | Zhoubi suanjing        | ajou-bi-swan-jing            |
| L | Qing                | ching              | Zhu Shijie             | aju-shi-jie                  |
| 畁 |                     | 0                  |                        | <u>ل</u>                     |
| 點 | <b>t</b> h          |                    |                        |                              |