ISCTE - IUL

Otimização

Licenciaturas em Finanças e Contabilidade, Gestão e Gestão de Marketing,

1 de Junho de 2015

Ano Lectivo 2014/2015, 2º semestre

Tipo de prova: Exame 1ª Época | Duração: 3h

Nome do aluno:

Número do aluno:

Turma:

Docente:

Observações:

1. A prova deve ser efectuada sem consulta e sem a utilização de máquina de calcular.
2. Não é permitido escrever a lápis ou a caneta de tinta vermelha.
3. Não destaque as folhas do teste.
4. Apresente todas as justificações necessárias.
5. Não são permitidas folhas de rascunho adicionais. A última folha do enunciado serve para esse efeito e pode ser usada excecionalmente para responder a alguma questão, desde que claramente assinalado.
6. Durante a prova o telemóvel deve estar desligado.
7. Não se tiram dúvidas durante a prova.

Cotações:

A resolution

(1.5 pts) 1. Find two unconstrained optima of the following function $f(x, y) = x^4 + y^4 - (x - y)^2$.

Critical points: $\nabla f = \vec{0}$

$$\begin{cases} \frac{\partial f}{\partial x} = 0 \\ \frac{\partial f}{\partial y} = 0 \end{cases} \Leftrightarrow \begin{cases} 4x^3 - 2(x-y) = 0 \\ 4y^3 + 2(x-y) = 0 \end{cases} \Leftrightarrow \begin{cases} (x-y) = 2x^3 \\ 4y^3 + 4x^3 = 0 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} 2x = 2x^3 \\ y = -x \end{cases} \Leftrightarrow \begin{cases} x(x^2-1) = 0 \\ \text{—} \end{cases} \Leftrightarrow \begin{cases} x = 0 \vee x = \pm 1 \\ \end{cases}$$

Critical points: $(0,0)$, $(1,-1)$, $(-1,1)$

Hessian matrix ~~points~~ test:

$$\frac{\partial^2 f}{\partial x^2} = 12x^2 - 2 \qquad \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = 2$$

$$\frac{\partial^2 f}{\partial y^2} = 12y^2 - 2$$

$$H(x,y) = \begin{bmatrix} 12x^2 - 2 & 2 \\ 2 & 12y^2 - 2 \end{bmatrix}$$

$$H(0,0) = \begin{vmatrix} -2 & 2 \\ 2 & -2 \end{vmatrix} \quad |H_1| = -2 \quad |H_2| = 0$$ semidefinite $\Rightarrow$ inconclusive

$$H(1,-1) = \begin{vmatrix} 10 & 2 \\ 2 & 10 \end{vmatrix} \quad |H_1| = 10 \quad |H_2| = 96$$ positive definite $\Rightarrow$ minimum point

$$H(-1,1) = \begin{vmatrix} 10 & 2 \\ 2 & 10 \end{vmatrix} \quad |H_1| = 10 \quad |H_2| = 96$$ positive definite $\Rightarrow$ minimum point

2. The Lagrange company has a production function given by $P(K,L) = 10K^{0.6}L^{0.4}$, where K and L represent, respectively, the capital invested in equipment and work force, in some *m.u.*. In the current year 10000 *m.u.* are available to be invested.

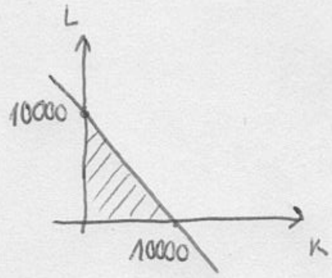
(0.5 pts) (a) Admitting that one wishes to optimize the production, formulate the company problem.

(0.5 pts) (b) Explain why the optimum point belongs to the associated equality constraint.

(1.5 pts) (c) Find the critical points.

a) $\max P(K,L) = 10\,K^{0,6}\,L^{0,4}$ s.t. $K + L \leq 10000$

b)



feasible solution region

As $P(K,L)$ increases when K and L increase, the maximum point will be taken in the frontier of the feasible solution region, specifically in $K+L=10000$

c) $\mathcal{L}(K,L,\lambda) = 10\,K^{0,6}\,L^{0,4} - \lambda\,(K+L-10000)$

Critical points: $\nabla \mathcal{L} = \vec{0}$

$$\begin{cases} \dfrac{\partial \mathcal{L}}{\partial K} = 0 \\ \dfrac{\partial \mathcal{L}}{\partial L} = 0 \\ \dfrac{\partial \mathcal{L}}{\partial \lambda} = 0 \end{cases} \Leftrightarrow \begin{cases} 6\,K^{-0,4}\,L^{0,4} - \lambda = 0 \\ 4\,K^{0,6}\,L^{-0,6} - \lambda = 0 \\ K + L - 10000 = 0 \end{cases} \underset{K,L\neq 0}{\Leftrightarrow} \begin{cases} \lambda = 6\left(\frac{L}{K}\right)^{0,4} \\ \lambda = 4\left(\frac{K}{L}\right)^{0,6} \\ \text{———} \end{cases}$$

$$\Leftrightarrow \begin{cases} 6\left(\frac{L}{K}\right)^{0,4} = 4\left(\frac{K}{L}\right)^{0,6} \\ \text{———} \\ \text{———} \end{cases} \Leftrightarrow \begin{cases} \frac{K}{L} = \frac{3}{2} \\ \text{———} \\ \text{———} \end{cases} \Leftrightarrow \begin{cases} K = \frac{3}{2}L \\ \lambda = 4\left(\frac{\frac{3}{2}4000}{4000}\right)^{0,6} \\ L = 4000 \end{cases}$$

$$\Leftrightarrow \begin{cases} K = 6000 \\ \lambda = 4\left(\frac{3}{2}\right)^{0,6} \\ L = 4000 \end{cases}$$

$\left(6000,\ 4000,\ 4\left(\frac{3}{2}\right)^{0,6}\right)$ critical point

3. Study the nature of the following series

(1.0 pts) (a) $\displaystyle\sum_{n=3}^{+\infty} \frac{n^2+4}{n^3+n+5}$

$$\lim \frac{\frac{n^2+4}{n^3+n+5}}{\frac{1}{n}} = \lim \frac{n^3+4n}{n^3+n+5} = 1 \in \mathbb{R}^+$$

By the limit comparison test, $\displaystyle\sum_{n=3}^{+\infty} \frac{n^2+4}{n^3+n+5}$ and $\sum \frac{1}{n}$ are of the same nature.

As $\sum \frac{1}{n}$ diverges (harmonic series), we conclude that $\displaystyle\sum_{n=3}^{+\infty} \frac{n^2+4}{n^3+n+5}$ also diverges.

(1.0 pts) (b) $\displaystyle\sum_{n=1}^{+\infty} \frac{(-1)^n}{n+3^n}$

$$\left|\frac{(-1)^n}{n+3^n}\right| = \frac{1}{n+3^n} \leq \frac{1}{3^n} = \left(\frac{1}{3}\right)^n$$

$\sum \left(\frac{1}{3}\right)^n$ is convergent because it is a geometric series with ratio $\frac{1}{3} < 1$.

Thus $\displaystyle\sum_{n=1}^{+\infty} \left|\frac{(-1)^n}{n+3^n}\right|$ is convergent, by the comparison test. That is, $\displaystyle\sum_{n=1}^{+\infty} \frac{(-1)^n}{n+3^n}$ is absolutely convergent.

therefore, $\displaystyle\sum_{n=1}^{+\infty} \frac{(-1)^n}{n+3^n}$ is convergent.

(1.0 pts) 4. Compute the convergence interval of the power series $\sum_{n=3}^{+\infty} \underbrace{2(x-1)^n}_{a_n}$

Convergence interval:

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{2|x-1|^{n+1}}{2|x-1|^n} = |x-1|$$

We know that the series converges when $|x-1|<1$, that is, when $x \in]0,2[$, and diverges when $x \in]-\infty, 0[\cup]2,+\infty[$

So the converge interval is $]0,2[$.

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Obs: If they ask the convergence domain ~~interval~~, we should also study what is happening at the frontier points of the convergence interval, $x=0$ and $x=2$.

$\boxed{x=0}$ $\sum_{n=3}^{+\infty} 2(x-1)^n = \sum_{n=3}^{+\infty} 2(-1)^n$ diverges by the divergency test ($2(-1)^n$ has no limit)

$\boxed{x=2}$ $\sum_{n=3}^{+\infty} 2(x-1)^n = \sum_{n=3}^{+\infty} 2(1)^n = \sum_{n=3}^{+\infty} 2$ diverges by the divergency test ($2 \not\to 0$)

So, in this case, the convergence domain coincides with the convergence interval.

5. Consider the following linear programming problem:

$$\begin{aligned} \text{Max } Z &= 2x_1 - x_2 \\ \text{s.t.:}\quad & x_1 + x_2 \leq 30 \\ & 2x_1 - 2x_2 \leq 20 \\ & x_1 \leq 25 \\ & x_1, x_2 \geq 0 \end{aligned}$$

$\longrightarrow Z - 2x_1 + x_2 + 0s_1 + 0s_2 + 0s_3 = 0$

$\longrightarrow x_1 + x_2 + s_1 = 30$

$\longrightarrow 2x_1 - 2x_2 + s_2 = 20$

$\longrightarrow x_1 + s_3 = 25$

(0.5 pts) (a) Write the first Simplex tableaux for this problem.

B.V.	Z	x_1	x_2	s_1	s_2	s_3	I.T.
Z	1	-2	1	0	0	0	0
s_1	0	1	1	1	0	0	30
s_2	0	2	-2	0	1	0	20
s_3	0	1	0	0	0	1	25

(b) Applying the Simplex algorithm, one obtains the following tableaux:

	x_1	x_2	s_1	s_2	s_3	
Z	0	-1	0	1	0	20
s_1	0	2	1	$-1/2$	0	20
x_1	1	-1	0	$1/2$	0	10
s_3	0	1	0	$-1/2$	1	15

(0.5 pts) i. The actual solution is not optimal. Justify this statement.

As in Z row the coefficient of x_2 is negative, by the criterium for optimal solution we conclude that the actual solution is not optimal.

(1.0 pts) ii. Determine the next solution, using the Simplex algorithm. Present the obtained values for the variables and the objective function.

	x_1	$x_2 \downarrow$	s_1	s_2	s_3		
z	0	-1	0	1	0	20	
← s_1	0	$\boxed{2}$	1	-1/2	0	20	(:2)
x_1	1	-1	0	1/2	0	10	
s_3	0	1	0	-1/2	1	15	
z	0	0	1/2	3/4	0	30	
x_2	0	1	1/2	-1/4	0	10	
x_1	1	0	1/2	1/4	0	20	
s_3	0	0	-1/2	-1/4	1	5	

Minimum ratio

$20/2 = 10$

$15/1 = 15$

Optimal solution: $x_1^* = 20$, $x_2^* = 10$

$z^* = 30$

(1.0 pts) (c) A basic variable is always non zero. Justifying, tell if this statement is true or false.

False. A degenerate solution has a basic variable equal to zero.

6. Compute the following primitives

(1.0 pts) (a) $P(\cos(x)\sin^2(x)) = \dfrac{\sin^3(x)}{3} + C$, with $C \in \mathbb{R}$

(1.0 pts) (b) $P(\underbrace{x^2}_{v}\underbrace{e^x}_{u'}) = x^2e^x - P\,2xe^x = x^2e^x - 2\,P\,\underbrace{x}_{v}\underbrace{e^x}_{u'} =$

$$= x^2e^x - 2\left(xe^x - Pe^x\right) =$$

$$= x^2e^x - 2xe^x + 2e^x + C \quad \text{, with } C \in \mathbb{R}$$

(1.0 pts) (c) $P(\sqrt{1-x^2})$, recall that $\cos^2(x) = \frac{\cos(2x)+1}{2}$.

$$P\sqrt{1-x^2} \underset{\substack{\downarrow \\ x = \sin t = \varphi(t) \\ \varphi'(t) = \cos t}}{=} P\left(\sqrt{1-\sin^2 t} \cdot \cos t\right) = P\cos^2 t = P\,\frac{\cos(2t)+1}{2} =$$

$$= \frac{1}{2}P\cos(2t) + P\frac{1}{2} = \frac{1}{2}\cdot\frac{1}{2}P\,2\cos(2t) + \frac{1}{2}t =$$

$$= \frac{1}{4}\sin(2t) + \frac{1}{2}t + C \underset{\substack{\downarrow \\ t = \arcsin x}}{=} \frac{1}{4}\sin\left(2\arcsin x\right) + \frac{1}{2}\arcsin(x) + C$$

with $C \in \mathbb{R}$

(1.0 pts) (d) $P(\frac{x}{(x+1)^2})$.

C.A.

$$\frac{x}{(x+1)^2} = \frac{A}{x+1} + \frac{B}{(x+1)^2}$$

$$x = A(x+1) + B$$

$x = -1 \Rightarrow B = -1$

$x = 0 \Rightarrow 0 = A - 1$

$A = 1$

$$= P\frac{1}{x+1} + P\frac{-1}{(x+1)^2} =$$

$$= \ln|x+1| - \frac{(x+1)^{-1}}{-1} + C,$$

with $C \in \mathbb{R}$

(1.0 pts) 7. Compute $\int_0^{1/2} f(x)\,dx$, where $f(x) = 12x\sqrt{1-4x^2}$.

$$\int_0^{1/2} 12x\sqrt{1-4x^2}\,dx = \frac{12}{-8}\int_0^{1/2} -8x\,(1-4x^2)^{1/2}\,dx =$$

$$= -\frac{3}{2}\left[\frac{(1-4x^2)^{3/2}}{3/2}\right]_0^{1/2} = -\frac{3}{2}\left(\frac{(1-4(\frac{1}{2})^2)^{3/2}}{3/2} - \frac{(1-0)^{3/2}}{3/2}\right)$$

(1.0 pts) 8. Without computing the integral, justify the following inequalities $0 \le \int_1^5 \frac{t}{t+1}\,dt \le 4$.

$\forall t \in [1,5]$, $0 \leq \frac{t}{t+1} \leq 1$

Thus $\underbrace{\int_1^5 0\,dt}_{0} \leq \int_1^5 \frac{t}{t+1}\,dt \leq \underbrace{\int_1^5 1\,dt}_{[t]_1^5 = 4}$,

by monotonecity of Riemann integral.

(2.0 pts) 9. Solve the following differential equation: $x^2y' - 2xy = 3y^4$ and $y(1) = 1$.

$$x^2y' - 2xy = 3y^4 \underset{x \neq 0}{\Leftrightarrow} y' - \frac{2}{x}y = \frac{3}{x^2}y^4 \Leftrightarrow \text{Bernoulli, with } n = 4$$

$$\underset{\text{dividing by } y^4}{\Leftrightarrow} y^{-4}y' - \frac{2}{x}y^{-3} = \frac{3}{x^2} \underset{\substack{v = y^{-3} \\ v' = -3y^{-4}y'}}{\Leftrightarrow} -\frac{v'}{3} - \frac{2}{x}v = \frac{3}{x^2} \Leftrightarrow$$

$$\Leftrightarrow \underbrace{v' + \frac{6}{x}v = -\frac{9}{x^2}}_{\text{Linear}} \Leftrightarrow (x^6 v)' = -\frac{9}{x^2} \cdot x^6 \Leftrightarrow$$

Integrant factor

$$e^{\int \frac{6}{x}dx} = e^{6\ln|x|} = x^6$$

$$\Leftrightarrow x^6 v = \int -9x^4 dx \Leftrightarrow x^6 v = -\frac{9}{5}x^5 + C \Leftrightarrow$$

$$\Leftrightarrow v = -\frac{9}{5x} + \frac{C}{x^6} \underset{v = y^{-3}}{\Leftrightarrow} y^{-3} = -\frac{9}{5x} + \frac{C}{x^6} \Leftrightarrow$$

$$\Leftrightarrow y^3 = \frac{1}{-\frac{9}{5x} + \frac{C}{x^6}} \Leftrightarrow y = \sqrt[3]{\frac{1}{-\frac{9}{5x} + \frac{C}{x^6}}},$$

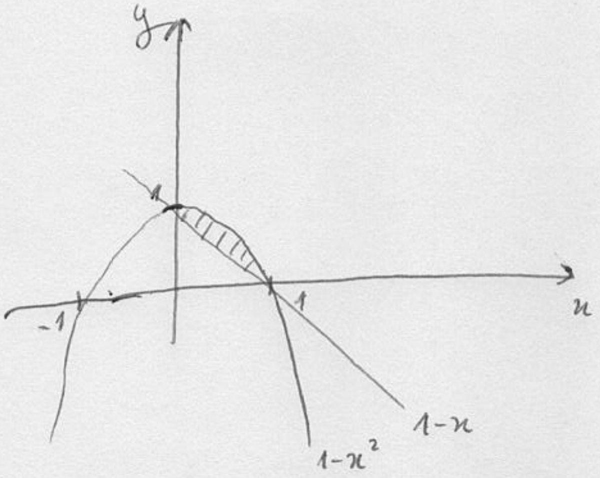
with $C \in \mathbb{R}$

$$y(1) = 1 \Rightarrow 1 = \sqrt[3]{\frac{1}{-\frac{9}{5} + C}} \Leftrightarrow 1 = -\frac{9}{5} + C \Leftrightarrow$$

$$\Leftrightarrow C = \frac{14}{5}$$

Thus $y = \sqrt[3]{\dfrac{1}{-\frac{9}{5x} + \frac{14}{5x^6}}}$ is the wanted solution

(1.0 pts) 10. Compute the area between the curves $y(x) = 1 - x^2$, $y(x) = 1 - x$.



$$A = \int_0^1 1 - x^2 - (1-x)\,dx =$$

$$= \int_0^1 -x^2 + x\,dx =$$

$$= \left[-\frac{x^3}{3} + \frac{x^2}{2} \right]_0^1 =$$

$$= -\frac{1}{3} + \frac{1}{2}$$

(1.0 pts) 11. Admitting that $f(x)$ is a continuous function and $\int_0^\pi f(x)\,dx = 12$, explain why there exists a point $t \in [0, \pi]$ such that $f(t) > 3$.

$$\left.\begin{array}{l} \int_0^\pi f(x)\,dx = 12 \\ f \text{ continuous} \end{array}\right\} \underset{\text{Mean value theorem}}{\Longrightarrow} \exists\, t \in [0,\pi] \text{ such that } \int_0^\pi f(x)dx = f(t)(\pi - 0)$$

So, $12 = f(t)\pi \Leftrightarrow f(t) = \frac{12}{\pi} > 3$