ISCTE - IUL

Mathematics

BSc in Management

November 15 2014

Year 2014/2015, 1st semester

Midterm Test | Time length: 1h15 (+15m)

Name:

Student Number:

Class: GAi

Lecturer:

Observations:

1. Do not use calculators.
2. It is not allowed the use of pencil or red ink pen.
3. Do not split off the test booklet.
4. Justify all your answers.
5. Additional draft sheets are not allowed. The last page in the test serves for this purpose. The draft sheet at the end may be used exceptionally to answer any question, since clearly marked
6. Mobile phones must be switched off.
7. Ask no questions. Silence is essential..

Marking:

1.	a)	3.	a)
	b)		b)
	c)		c)
	d)	4.	a)
2.	a)		b)
	b)	5.	
	c)		

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1. Consider the system of linear equations in x, y and z:

$$\begin{cases} -x - y + az = 0 \\ x + 2y + (1-a)z = -1 \\ 2x + y + z = b^2 \end{cases}, \qquad a, b \in \mathbb{R}.$$

(1.0 val.) (a) Write the system in the matrix form $AX = B$.

$$\begin{bmatrix} -1 & -1 & a \\ 1 & 2 & 1-a \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ b^2 \end{bmatrix}$$

(2.0 val.) (b) Discuss the solution set based on the parameters a and b, using Gauss elimination.

$$\left[\begin{array}{ccc|c} -1 & -1 & a & 1 \\ 1 & 2 & 1-a & -1 \\ 2 & 1 & 1 & b^2 \end{array}\right] \xrightarrow[L_3+2L_1]{L_2+L_1} \left[\begin{array}{ccc|c} -1 & -1 & a & 0 \\ 0 & 1 & 1 & -1 \\ 0 & -1 & 2a+1 & b^2 \end{array}\right]$$

$$\xrightarrow[L_3+L_2]{} \left[\begin{array}{ccc|c} -1 & -1 & a & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 2a+2 & b^2-1 \end{array}\right]$$

if $2a+2 \neq 0 \Leftrightarrow a \neq -1$ for all $b \in \mathbb{R}$ system has one single solution

if $a = -1$ then if $b = 1$ or $b = -1$ the system has many infinite solutions

if $a = -1$ and $(b \neq -1$ or $b \neq 1)$ system impossible.

(2.0 val.) (c) Determine for which values of a coefficients matrix A is invertible. Consider $a = 0$ determine A^{-1}.

From a) we know that A is invertible if $a \neq -1$

$a = 0$

$$A = \begin{bmatrix} -1 & -1 & 0 \\ 1 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix} \qquad |A| = -2$$

$$A^{-1} = \frac{1}{-2} \begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ -3 & -1 & -1 \end{bmatrix} \Leftrightarrow A^{-1} = \begin{bmatrix} -1/2 & -1/2 & 1/2 \\ -1/2 & 1/2 & -1/2 \\ 3/2 & 1/2 & 1/2 \end{bmatrix}$$

(1.5 val.) (d) For $a = b = 0$. Determine the system solution $AX = B$ using the result obtained in (c).

For $a = 0$ the system has one single solution

Thus $X = A^{-1}B$

$$X = \begin{bmatrix} -1/2 & -1/2 & 1/2 \\ -1/2 & 1/2 & -1/2 \\ 3/2 & 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/2 \\ -1/2 \\ -1/2 \end{bmatrix}$$

Solution: $(x, y, z) = \left(\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right)$

2. Consider

$$V = \text{span}\{(1,1,1),(2,0,1),(0,2,1)\}.$$

(1.5 val.) (a) Determine a basis for V.

$$\begin{bmatrix} 1 & 2 & 0 \\ 1 & 0 & 2 \\ 1 & 1 & 1 \end{bmatrix} \xrightarrow{\substack{L_2 - L_1 \\ L_3 - L_1}} \begin{bmatrix} 1 & 2 & 0 \\ 0 & -2 & 2 \\ 0 & -1 & 1 \end{bmatrix} \xrightarrow{L_3 - \frac{1}{2}L_2} \begin{bmatrix} 1 & 2 & 0 \\ 0 & -2 & 2 \\ 0 & 0 & 0 \end{bmatrix} \quad R(A) = 2$$

the vectors $(1,1,1)$, $(2,0,1)$ and $(0,2,1)$ are l.D.

Basis for $V = \{(1,1,1), (2,0,1)\}$

(1.5 val.) (b) Verify that vector $(1,1,0)$ does not belong to V.

We can verify if it is possible to obtain $(1,1,0)$ as a linear combination of ~~the~~ two vectors $(1,1,1)$ and $(2,0,1)$)

$$\alpha(1,1,1) + \beta(2,0,1) = (1,1,0)$$

$$\begin{cases} \alpha + 2\beta = 1 \\ \alpha = 1 \\ \alpha + \beta = 0 \end{cases} \quad \begin{cases} 1 + 2(-1) = 1 \\ \alpha = 1 \\ \beta = -1 \end{cases} \quad \begin{cases} 1 = -1 \to \text{impossible condition} \\ a = 1 \\ b = -1 \end{cases}$$

therefore $(1,1,0) \notin V$.

(1.5 val.) (c) Using the results of (a) and (b), indicate a basis of $\mathbb{R}^3$, justify your answer.

in b) it was verified that $(1,1,0) \notin V$, i.e, the vector is not obtained as a linear combination of the vectors $(1,1,1)$ and $(2,0,1)$.

then the vector $(1,1,1)$, $(2,0,1)$ and $(1,1,0)$ are l. i. and span$(\mathbb{R}^3)$

A basis for $\mathbb{R}^3$: $\{(1,1,1), (2,0,1), (1,1,0)\}$

3. Consider the linear transformation $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ given by:

$$T(x, y) = (2x + y, x + y)$$

(1.0 val.) (a) Determine the matrix that represents the transformation in the canonical basis, that is, $M(T, b.c., b.c.)$.

$T(1,0) = (2,1)$

$T(0,1) = (1,1)$

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

(2.0 val.) (b) Determine the Kernel of T, Ker(T). What can be concluded about the transformation T injectivity and about the range(T) ($\dim \operatorname{im}(T)$)? Justify.

$\operatorname{Ker}(T) = \{(x,y) \in \mathbb{R}^2 : T(x,y) = (0,0)\}$

$$[A \mid 0] \rightarrow \left[\begin{array}{cc|c} 2 & 1 & 0 \\ 1 & 1 & 0 \end{array}\right] \xrightarrow{L_2 - 2L_1} \left[\begin{array}{cc|c} 2 & 1 & 0 \\ 0 & -1 & 0 \end{array}\right]$$

$$\begin{cases} -y = 0 \\ 2x + y = 0 \end{cases} \quad \begin{cases} y = 0 \\ 2x = 0 \end{cases} \quad \begin{cases} y = 0 \\ x = 0 \end{cases} \quad \operatorname{Ker}(T) = \{(0,0)\}$$

Since the $\operatorname{Ker}(T) = \{(0,0)\}$ the function is injective.

Dimension theorem

$$\dim(\mathbb{R}^2) = \dim(\operatorname{Ker}(T)) + \underbrace{\dim(\operatorname{Im}(T))}_{\text{Range}(T)}$$

$$2 = 0 + \text{Range}(T)$$

$\text{Range}(T) = 2.$

(2.0 val.) (c) Consider a basis $\mathcal{B} = \{(1,0),(1,1)\}$. Determine the matrix that represents the transformation T in basis $\mathcal{B}$, that is, $M(T,\mathcal{B},\mathcal{B})$.

the change of basis matrix is P^{-1}

$P^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^{-1} \Leftrightarrow P^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$ and $P = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

Call B the matrix that represents the transformation T in basis $\mathcal{B}$.

$B = P^{-1} A P$

$B = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$

4. Considere the matrix:

$$A = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix}.$$

(1.5 val.) (a) Verify that $(2,2,2)$ is an eigenvector of A. What is the eigenvalue associated to that eigenvector. Justify.

Being $\vec{v} = (2,2,2)$

we know that $A\vec{v} = \lambda \vec{v}$

$\begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} = \lambda \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} \Leftrightarrow \begin{bmatrix} 12 \\ 12 \\ 12 \end{bmatrix} = \lambda \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$

$6 \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} = \lambda \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$

then $\lambda = 6$. the eigenvalue associated to the eigenvector $(2,2,2)$ is $\lambda = 6$.

(1.0 val.) (b) Verify that $\lambda = 0$ is another eigenvalue of A.

$$|A - \lambda I_3| = \begin{vmatrix} 2-\lambda & 2 & 2 \\ 2 & 2-\lambda & 2 \\ 2 & 2 & 2-\lambda \end{vmatrix}$$

For $\lambda = 0$ we have

$$\begin{vmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{vmatrix} = 0 \iff |A - 0I_3| = 0 \iff |A| = 0$$

(1.5 val.) 5. Consider a linear transformation $f : \mathbb{R}^4 \longrightarrow \mathbb{R}^4$ and suppose the geometric multiplicity of the eigenvalue $\lambda = -2$ of f is 3 and that $f(0,0,0,1) = (0,0,0,1)$. Being A the matrix that represents the transformation f in the canonical basis, that is, $A = M(f, b.c., b.c.)$. Calculate the determinant of matrix A.

- Since $f(0,0,0,1) = (0,0,0,1)$ then $\lambda = 1$ the eigenvalue associated to the eigenvector $(0,0,0,1)$. thus, $g.m.(1) \geq 1$
- $\lambda = -2$ is an eigenvalue of f with $g.m.(-2) = 3$ we can conclude

Since $g.m.(1) + g.m.(-2) \leq 4$

$g.m.(1) + g.m.(-2) = 4$

then it exists a diagonal matrix D such that $A = PDP^{-1}$ with P a matrix such that the columns of P are the eigenvectors and

$$D = \begin{bmatrix} -2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and $|A| = |PDP^{-1}| = |P|\,|D|\,|P^{-1}| = -8$