University Institute of Lisbon

Mathematics Departament

Differential calculus in R^n exercises

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1 Topology in $\mathbb{R}^n$. Domains and graphic representation

1. Consider the following functions:

   (a) $f:\mathbb{R}^2\to\mathbb{R},\quad f(x,y)=\sqrt{1-x^2-y^2}$
   (b) $f:\mathbb{R}^2\to\mathbb{R},\quad f(x,y)=\ln(1-x+y),\quad x,y\geq 0$
   (c) $f:\mathbb{R}^2\to\mathbb{R},\quad f(x,y)=\ln(x+y)$
   (d) $f:\mathbb{R}^3\to\mathbb{R},\quad f(x,y,z)=\frac{1}{\sqrt{4-x^2-y^2-z^2}}$

   For each function determine:

   (i) the domain of definition and represent the graph of each function.
   (ii) interior, exterior and the border.
   (iii) if the domain of definition if an open set, a closed and bounded set.

2. Determine the domain of definition D and represent the graph of each of the following functions:

   (a) $f:D\subseteq\mathbb{R}^3\to\mathbb{R},\quad f(x,y,z)=\dfrac{1}{x^2+y^2+z^2}$
   (b) $f:D\subseteq\mathbb{R}^2\to\mathbb{R},\quad f(x,y)=1+\sqrt{-(x-y)^2}$
   (c) $f:D\subseteq\mathbb{R}^2\to\mathbb{R},\quad f(x,y)=\dfrac{1}{\sqrt{y-\sqrt{x}}}$
   (d) $f:D\subseteq\mathbb{R}^2\to\mathbb{R}^2,\quad f(x,y)=(e^{x-y},\ \ln(x+y))$
   (e) $f:D\subseteq\mathbb{R}^2\to\mathbb{R}^3,\quad f(x,y)=\left(x,\ y,\ x^2+y^2\right)$
   (f) $f:D\subseteq\mathbb{R}^3\to\mathbb{R},\quad f(x,y,z)=\dfrac{1}{\sqrt{1-x^2-y^2-z^2}}$
   (g) $f:D\subseteq\mathbb{R}^2\to\mathbb{R},\quad f(x,y)=\sqrt{x^2+y^2-1}-\sqrt{9-x^2-y^2}$

3. Determine the domain of definition of each of the following functions:

   (a) $$f(x,y)=\begin{cases}\dfrac{1}{\ln(x+y)}, & (x,y): x+y>0\\ \sqrt{1-x-y}, & (x,y): x+y\leq 0\end{cases}$$

   (b) $$f(x,y)=\begin{cases}\dfrac{2x^3+3y^4}{2x^3-y^3}, & (x,y)\neq(0,0)\\ 1, & (x,y)=(0,0)\end{cases}$$

   (c) $$f(x,y)=\begin{cases}\ln(3x+y), & (x,y): 3x+y>0\\ \dfrac{1}{x+y}, & (x,y): 3x+y\leq 0\end{cases}$$

(d) $f(x,y) = \begin{cases} \dfrac{\sqrt{x^2+y^2}}{3y^2-x} & , \quad (x,y) : x \neq 3y \\ 0 & , \quad (x,y) : x = 3y \end{cases}$

4. Consider the function $f(x,y) = \ln(xy-1) + \sqrt{9-(x-1)^2-y^2}$.
   Determine its domain of definition and represent it graphically.

5. Given the ser $A = \left\{(x,y) \in \mathbb{R}^2 : x+y \leq 1 \ \wedge \ y-x \leq 1 \ \wedge \ y \geq 0\right\}$, consider the function:

$$f(x,y) = \begin{cases} \dfrac{x}{y^2+1} & , \quad (x,y) \in A \\ 1 & , \quad (x,y) \notin A \end{cases}$$

   Determine the domain of definition.

2 Limits and continuity

1. Consider the function $f(x,y) = \dfrac{x-y}{x+y}$.
   (a) Determine the domain of definition f.
   (b) Calculate the limit in the point $(1,2)$.
   (c) Calculate, if possible, the limit of the function at the point $(0,0)$.

2. Proof that the funtion $f(x,y) = \dfrac{xy}{x^2+y^2}$ does not have limit at $(0,0)$.

3. Proof that the funtion $f(x,y) = \dfrac{xy^2}{x^2+y^4}$ does not have limit at $(0,0)$.

4. Consider

$$f(x,y) = \begin{cases} \frac{xy}{\sqrt{x^2+y^2}} & , \quad (x,y) \neq (0,0) \\ 1 & , \quad (x,y) = (0,0) \end{cases}$$

   Verify if there is a limit at the point $(0,0)$..

5. Proof by definition that

$$\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{(x^2+y^2)^3}} \neq 0.$$

6. Calculate, if possible, the limit of the function $f(x,y) = 2\dfrac{x^2-y^2}{\sqrt{x^2+y^2}}$ at the point $(0,0)$.

7. Consider

$$f(x,y) = \begin{cases} \frac{xy}{x^2-y^2} & , \quad (x,y):\ x \neq \pm y \\ 1 & , \quad (x,y): x = \pm y \end{cases}$$

   Verify if the function has limit at $(0,0)$.

8. Consider the functions in exercises 4 and 5. What can be concluded about the continuity of each function at point $(0,0)$.

9. Consider

$$f(x,y) = \begin{cases} \frac{x^3+4y^2}{x^2-5y^2} & , \quad (x,y) \neq (0,0) \\ 0 & , \quad (x,y) = (0,0) \end{cases}$$

   Verify if the function is continuous at point $(0,0)$.

10. Study the continuity of the function:

$$f(x,y)=\begin{cases} \dfrac{y-2}{x+3} & , \quad (x,y)\neq(0,0) \\ 0 & , \quad (x,y)=(0,0) \end{cases}$$

11. Given the function $f:\mathbb{R}^2\to\mathbb{R}^2$:

$$f(x,y)=\big(f_1(x,y),f_2(x,y)\big)=\begin{cases} f_1(x,y)=\dfrac{x-4}{2y+2} \\ f_2(x,y)=\dfrac{y-3}{x^2+1} \end{cases}$$

Study its continuity at point $(0,0)$.

12. Consider the functions of exercises 5 and 6. Verify, for each function, if it is possible to make it continuous at ponit $(0,0)$.

13. Consider

$$f(x,y)=\begin{cases} \dfrac{3x^3+2y^3}{x^2+y^2} & , \quad (x,y)\neq(0,0) \\ 0 & , \quad (x,y)=(0,0) \end{cases}.$$

Study its continuity.

14. Consider the function:

$$f(x,y)=\begin{cases} \dfrac{x^2y}{y+x\sin x} & , \quad (x,y):y\neq -x\sin x \\ 1 & , \quad (x,y):y=-x\sin x \end{cases}$$

Proof that the function is not continuous at $(0,0)$. Justify.

3 Partial derivatives of 1st order. Differentiation. Directional derivatives. Gradient and Jacobian matrix

1. Calculate the partial derivatives of 1st order of the following functions:

   (a) $f(x,y) = x^2 + y^2$

   (b) $f(x,y) = x^2 + \sin(xy)$

   (c) $f(x,y) = \dfrac{x^4 - y^4}{xy}$

   (d) $f(x,y) = \sqrt{e^{x-5y^2} - y^2}$

   (e) $f(x,y) = \ln \sin\left(\dfrac{x}{\sqrt{y}}\right)$

   (f) $f(x,y,z) = \sqrt{4 - x^2 - y^2 - z^2}$

2. Consider the function $f(x,y) = \frac{2x}{x^2+y^2}$. Calculate, by definition, $\left(\frac{\partial f}{\partial y}\right)_{(1,1)}$ and $\left(\frac{\partial f}{\partial x}\right)_{(1,2)}$.

3. Consider

$$f(x,y) = \begin{cases} \dfrac{x+y}{x^2+y^2} & , \quad (x,y) \neq (0,0) \\ 0 & , \quad (x,y) = (0,0) \end{cases}$$

   Calculate $\left(\frac{\partial f}{\partial x}\right)_{(0,0)}$ and $\left(\frac{\partial f}{\partial y}\right)_{(0,0)}$.

4. Consider

$$f(x,y) = \begin{cases} \dfrac{xy}{x^2 - y^2} & , \quad x \neq \pm y \\ 4 & , \quad x = \pm y \end{cases}$$

   Calculate $\left(\frac{\partial f}{\partial x}\right)_{(-2,-2)}$ and $\left(\frac{\partial f}{\partial y}\right)_{(-2,-2)}$.

5. Consider

$$f(x,y) = \begin{cases} \dfrac{2x^3+3y^4}{2x^3 - y^3} & , \quad (x,y) \neq (0,0) \\ 1 & , \quad (x,y) = (0,0) \end{cases}$$

   Determine $\left(\frac{\partial f}{\partial x}\right)_{(0,0)}$.

6. Given the function $z = xy \tan\left(\frac{y}{x}\right)$. Verify that $x\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 2z$.

7. Consider

$$f(x,y) = \begin{cases} \dfrac{x^2 y}{x^4 + y^2} & , \quad (x,y) \neq (0,0) \\ \\ 0 & , \quad (x,y) \neq (0,0) \end{cases}$$

Determine $\left(\dfrac{\partial f}{\partial y}\right)_{(a,b)}$ para todo o $(a,b) \in \mathbb{R}^2$.

8. Consider the function:

$$f(x,y) = \begin{cases} \dfrac{2x^2 - y^3}{x^2 + y^2} & , \quad (x,y) \neq (0,0) \\ \\ 0 & , \quad (x,y) = (0,0) \end{cases}$$

Calculate $\left(\dfrac{\partial f}{\partial x}\right)_{(a,b)}$ for all $(a,b) \in \mathbb{R}^2$.

9. Consider the function $f(x,y) = xy$.
   (a) Show that f it is a differentiable function at point $(0,0)$.
   (b) Calculate the gradient of f.
   (c) Conclude that f is a differentiable function in $\mathbb{R}^2$.
   (d) Using c), calculate the directional derivative at point $(1,-1)$ according the vector $(0,2)$.
   (e) Calculate the normal derivative of f at point $(1,-1)$ according the vector $(0,2)$.

10. Consider the following function:

$$f(x,y) = \begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}} & , \quad (x,y) \neq (0,0) \\ \\ 0 & , \quad (x,y) = (0,0) \end{cases}$$

Verify if the function is differentiable at point $(0,0)$.

11. Consider

$$f(x,y) = \begin{cases} \dfrac{2y^5 + x^2 y^3}{\sqrt{x^2+y^2}} & , \quad (x,y) \neq (0,0) \\ \\ 1 & , \quad (x,y) = (0,0) \end{cases}$$

(a) Study the function continuity at point $(0,0)$

(b) With the result of a) what can be concluded about the differentiability of the function at $(0,0)$? Justify.

12. Consider the function:

$$f(x,y) = \begin{cases} \dfrac{x}{y-1} & , \quad y \neq 1 \\ 0 & , \quad y = 1 \end{cases}$$

Verify that $f(x,y)$ is not differentiable at point $(2,1)$.

13. Consider the differentiable function $g(x,y) = \sin(xy)$.

    (a) Determine the gradient vector of g at point $(0,0)$.

    (b) Calculate the directional derivative of g at point $(0,0)$, according the direction of vector $(1,2)$.

    (c) Calculate the normal derivative of g at point $(0,0)$, according the direction of vector $(1,2)$..

14. Determine the gradient of the following functions:

    (a) $f(x,y,z) = (x-1)^2 + (y+2)^2 + (z-3)^2$, at point $(1,-2,0)$.

    (b) $f(x,y) = 2x^2 - 3xy + y^2 + 4x - 3y$ at the point where $f'_x = 0$ and $f'_y = 0$.

15. Consider the function $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ given by $f(x,y) = x\sin^2 y + xy^2$. Justify in which direction the normal derivative has the expression:

$$f'_{\vec{u}}(x,y) = \sin^2 y + y^2.$$

16. Consider $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ given by:

$$f(x,y) = \left(x^2 + y^2, -3x^2, 2xy\right).$$

Determine the jacobian matrix of f.

17. Consider the functions f and g of exercises 9 and 13. Consider $h : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ a differentiable function given by:

$$h(x,y) = (f(x,y), g(x,y)).$$

    (a) Determine the jacobian matrix of h.

    (b) Determine h'_x, using a).

    (c) Calculate the directional derivative of h at point $(1,2)$ according the vector $(1,1)$.

4 Chain rule

1. Consider the function $g(t) = f(x(t), y(t))$ where $f(x, y) = 2x + 2y$, $x(t) = \sin(t)$ and $y(t) = \cos(t)$.

   (a) Determine the analytic expression of $g(t)$.

   (b) Calculate $\frac{\mathrm{d}g}{\mathrm{d}t}$, using the analytic expression of $g(t)$.

   (c) Calculate $\frac{\mathrm{d}g}{\mathrm{d}t}$, using the chain rule.

2. Being $z = \tan\left(x^2 + y^2\right)$ with $x = t^2 - 3t, y = \log t$. Calculate $\frac{\partial z}{\partial t}$.

3. Being $z = f(u(x, y), v(x, y))$ with $u(x, y) = x^2 - y^2$ and $v(x, y) = e^{xy}$, determine the expression of each partial derivatives $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

4. Show that the function $F(x, y, z) = f(x - y, y - z, z - x)$ verifies the equation:

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y} + \frac{\partial F}{\partial z} = 0$$

   whatever the function f is.

5. Show that for function $z = yf\left(x^2 - y^2\right)$ it is obtained:

$$\frac{1}{x}\frac{\partial z}{\partial x} + \frac{1}{y}\frac{\partial z}{\partial y} = \frac{z}{y^2}$$

6. For function $z = x^\alpha g\left(\frac{y}{x}\right)$, with α a constant, calculate $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

7. Being $V(x, y, z) = xy^2 h\left(\frac{y}{x}, \frac{x}{z}\right)$, show that:

$$x\frac{\partial V}{\partial x} + y\frac{\partial V}{\partial y} + z\frac{\partial V}{\partial z} = 3V.$$

5 Higher order derivatives

Notation: $\dfrac{\partial^2 f}{\partial y \partial x} = \dfrac{\partial}{\partial y}\left(\dfrac{\partial f}{\partial x}\right)$ and $\dfrac{\partial^2 f}{\partial x \partial y} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial f}{\partial y}\right)$.

1. Consider the following functions:

   (a) $f(x, y) = \ln(x^2 + y^2)$.

   (b) $f(x, y) = x^2 + \sin(xy)$.

   (c) $f(x, y) = \dfrac{x}{\sqrt{y}}$.

   For each function calculate:

   (i) $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$.

   (ii) $\dfrac{\partial^2 f}{\partial x^2}$, $\dfrac{\partial^2 f}{\partial x \partial y}$, $\dfrac{\partial^2 f}{\partial y \partial x}$ and $\dfrac{\partial^2 f}{\partial y^2}$.

   (iii) Calculate the hessian matrix.

2. Consider exercise 7 of Section 3. Calculate $\dfrac{\partial^2 f}{\partial y^2}(x, y)$ and $\dfrac{\partial^2 f}{\partial x \partial y}(x, y)$.

3. Being $f(x, y, z) = ye^{x-y} + xz$. Calculate the hessian matrix.

4. Consider $g(x, y) = 2xy^2 + 4\ln(4x)$. Calculate the partial derivatives of 2nd order of g.

5. Being $h(x, y) = y^2e^x + x^2y^3 - 1$. Determine the expression of the 3rd order partial derivatives $\dfrac{\partial^3 h}{\partial x^2 \partial y}(x, y)$ and $\dfrac{\partial^3 h}{\partial x^3}$.

6. Consider the function $g(x, y) = e^x \ln(y) + \sin(x)\ln(y)$. Determine partial derivatives $\dfrac{\partial^2 g}{\partial y^2}(x, y)$ and $\dfrac{\partial^3 g}{\partial y \partial x \partial y}(x, y)$.

7. Consider the function:

$$f(x, y) = \begin{cases} \dfrac{xy}{x^2 - y^2} & , \quad x \neq \pm y \\ 0 & , \quad x = \pm y \end{cases}$$

   Calculate the value of the partial derivatives $\dfrac{\partial^2 f}{\partial x^2}(0, 0)$ and $\dfrac{\partial^2 f}{\partial y^2}(0, 0)$.

6 Homogeneous function

1. Show that the following functions are homogeneous. Determine the homogeneous degree and verify the Euler theorem:
   (a) $f(x, y) = x^2 + 4xy + 4y^2$
   (b) $f(x, y) = \ln\left(\frac{(x+y)^2}{yx}\right)$
   (c) $f(x, y, z) = \sin\left(\frac{x+y}{z}\right)$
   (d) $f(x, y) = \sqrt[3]{yx^2}$

2. Consider $f(x, y)$ an homogeneous function with homogeneous degree 2. Consider the function:
$$g(x, y) = xf(x, y).$$
   (a) What is the homogeneous degree of $g(x, y)$?
   (b) Show that g'_x and g'_y are homogeneous functions with homogeneous degree 2.
   (c) Show that $g(x, y)$ verifies the Euler theorem.

3. Being
$$f(x, y) = x^k y^{2+k} + yx.$$
   (a) For which values of k the function is homogeneous? What is its homogeneous degree?
   (b) For the value of k found, proof the Euler theorem.

4. Being $V(x, y, z) = xy^2 h\left(\frac{y}{x}, \frac{x}{z}\right)$, show that:
$$x\frac{\partial V}{\partial x} + y\frac{\partial V}{\partial y} + z\frac{\partial V}{\partial z} = 3V.$$

5. Consider the following production function $Y = AK^{\alpha}L^{1-\alpha}, k > 0, L > 0$, the Cobb-Douglas function with two production factors: Capital (K) and Labor (L).
   (a) Determine its degree of homogeneity.
   (b) Suppose $\alpha = 0.75$, verify the Euler theorem.
   (c) Proof that the marginal productivity of capital $\frac{\partial Y}{\partial K}$ is an homogeneous function with homogeneous degree zero.

6. Consider $z = f(u, v)$ a compound function in which $u = x^3$ and $v = x^2y$. It is known that $f(u, v)$ is an homogeneous function with homogeneous degree 2 and of class C^2. Still consider that
$$\left(\frac{\partial f}{\partial u}\right)_{(8,4)} = 1 \quad \text{and} \quad \left(\frac{\partial f}{\partial v}\right)_{(8,4)} = 2.$$

(a) Calculate $\left(\frac{\partial z}{\partial x}\right)_{(2,1)}$ and $\left(\frac{\partial z}{\partial y}\right)_{(2,1)}$.

(b) Determine $f(8,4)$.

(c) What is the value of the derivative of z, at point $(x,y) = (2,1)$, according the direction of the vector $(-1,0)$? What is the name of this derivative?