ISCTE Business School - IUL
Lisboa
(1st Semester 2014/2015)

Mathematics

Book 1

Matrices e Determinants

Abdul Suleman
Álvaro Rosa
José Bonito Filipe
Maria João Lopes
Manuel Alberto
Marina Andrade
Nuno Ferreira

Conteúdo

1 Matrices

1.1 Synopsis

1.1.1 Introduction

Definition 1 *Matrix is a rectangular array of numbers, symbols or expressions. The individual items of the matrix are called elements or entries. The generic element* a_{ij} *, has two indexes, in which the first one* $(i = 1, 2, ..., m)$ *indicates the row and the second one* $(j = 1, 2, ..., n)$ *indicates the column:*

$$\begin{bmatrix} a_{11} & a_{12} & ... & a_{1n} \\ a_{21} & a_{22} & ... & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & ... & a_{mn} \end{bmatrix}$$

Definition 2 *A matrix is of type* $m \times n$ *if is has* m *rows e* n *columns.*

1.1.2 Matrix algebra

Definition 3 *Matrix addition is the operation of adding two matrices by adding the corresponding entries together. This operation requires that the adding matrices are of same type.*

Definition 4 *Multiplication of two matrices is defined only if the number of columns of the left matrix is the same as the number of rows of the right matrix. If* A *is an m-by-n matrix and* B *is an n-by-p matrix, then their matrix product* AB *is the m-by-p matrix whose entries are given by dot product of the corresponding row of* A *and the corresponding column of* B*.*

1.1.3 Transpose matrix. Symmetric matrix.

Definition 5 *The transpose matrix of* A *is represented by* A^T *and is formed by turning rows into columns and vice versa.*

Definition 6 *A square matrix* A *that is equal to its transpose, i. e.* $A = A^T$*, is a symmetric matrix. If* $A = -A^T$ *then* A *is said to be skew-symmetric matrix.*

Result 1 *Transposing matrices do verify the following properties:*

1. $(A + B)^T = A^T + B^T$

2. $\left(A^T\right)^T = A$

3. $(AB \cdots C)^T = C^T \cdots B^T A^T$

4. The product of two matrices is a symmetric matrix if and only they are commutable.

1.2 Exercises

1. Consider the matrices

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ -1 & 3 \end{bmatrix} \text{ e } B^T = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 1 \end{bmatrix}.$$

Calculate the matrix X, so that it verifies the following equality: $\left(B^T - 3X\right)^T - 3A = 2B$.

2. Consider

$$A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 0 & 1 \\ 1 & 0 & 2 \end{bmatrix}$$
$$C = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \qquad D = \begin{bmatrix} 2 & 0 \\ 0 & 1 \\ 0 & 2 \end{bmatrix}$$

   (a) Determine, wherever possible AB ; BA ; ABC ; CD.

   (b) Calculate BD and DB and discuss the results.

3. Let $A^T = \begin{bmatrix} 1 & 2 & -1 \end{bmatrix}$ e $B^T = \begin{bmatrix} 3 & -2 & 1 \end{bmatrix}$. Calculate $A^T B$ e AB^T.

4. Let A be a matrix oftype 3×1, with unitary elements. Calculate AA^T e $A^T A$.

5. Consider

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ e } J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$

demonstrate that $[aI + bJ] + [aI - bJ] = 2aI$.

6. Consider the matrices:

$$A = \begin{bmatrix} 2 & -3 & 5 \\ -1 & 4 & 0 \\ -2 & 1 & -5 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 3 \\ 2 & 4 \\ 1 & 2 \end{bmatrix} \qquad C = \begin{bmatrix} -1 & 0 \\ -1 & -3 \\ 1 & -1 \end{bmatrix}$$

(a) Show that $(AB)^T = B^T A^T$.
(b) Calculate $(AB)^T + C^T$.

7. Determine the matrices X and Y that verify the system of equations

$$\begin{cases} 2X + 3Y = B \\ -X - Y = C \end{cases}.$$

Use matrices B and C from previous exercise.

1.3 Solutions

1. $X = \begin{bmatrix} -\frac{4}{3} & 0 & \frac{4}{3} \\ -\frac{8}{3} & -\frac{4}{3} & -\frac{10}{3} \end{bmatrix}$

2. (a) $AB = \begin{bmatrix} 7 & 0 & 8 \\ 6 & 0 & 9 \end{bmatrix}$

   BA can not be done (number of columns of A is not equal to the number of rows of matrix B)

   $ABC = \begin{bmatrix} 15 & 0 & 15 \\ 15 & 0 & 15 \end{bmatrix}$

   $CD = \begin{bmatrix} 2 & 2 \\ 0 & 3 \\ 2 & 2 \end{bmatrix}$

   (b) $BD = \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}$ and $DB = \begin{bmatrix} 4 & 0 & 2 \\ 1 & 0 & 2 \\ 2 & 0 & 4 \end{bmatrix}$, therefore, the product of matrices is not commutative.

3. $A^T B = -2$ and $AB^T = \begin{bmatrix} 3 & -2 & 1 \\ 6 & -4 & 2 \\ -3 & 2 & -1 \end{bmatrix}$

4. $AA^T = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ and $A^T A = 3$

6. (b) $(AB)^T + C^T = \begin{bmatrix} 0 & 6 & -4 \\ 4 & 10 & -13 \end{bmatrix}$

7. $X = \begin{bmatrix} 2 & -3 \\ 1 & 5 \\ -4 & 1 \end{bmatrix}$ and $Y = \begin{bmatrix} -1 & 3 \\ 0 & -2 \\ 3 & 0 \end{bmatrix}$

2 Linear dependence of rows/columns of a matrix; Elementary operations; Rank of a Matrix

2.1 Sypnosis

A linear combination of rows of A is an expression like $\lambda_1 L_1 + \lambda_2 L_2 + ... + \lambda_m L_m$ in which:

- $\lambda_1, \lambda_2, ..., \lambda_m$ are real numbers
- $L_1, L_2, ..., L_m$ are respectively the m rows of the matrix A.

The expression $\lambda_1 L_1 + \lambda_2 L_2 + ... + \lambda_m L_m = 0$ will have one of the following solutions:

1. $\lambda_1 = \lambda_2 = ... = \lambda_m = 0$, in this case, the rows of A are linearly independents;
2. If there is, at least, one $\lambda_i \neq 0$, $(i = 1, 2, ..., m)$, the rows of A are said to be linearly dependents.

The linear dependence of rows/columns of a matrix remains unchanged when the following operations are applied to the matrix (designated as elementary operations):

1. interchange two rows;
2. Multiply a row with a nonzero number;
3. Add a row to another one multiplied by a number (Jacobi operation).

The maximum number of linearly independent rows of A, is caled the rank of A, denoted: rank A. In general, to compute the rank of a matrix, perform elementary row operations until the matrix is left in echelon form; the number of nonzero rows remaining in the reduced matrix is the rank.

2.2 Exercises

1. Consider the matrix $A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & -1 & 0 \\ 3 & 1 & -1 \end{bmatrix}$

   (a) Use the definition of linear dependence of rows of a matrix to verify if A has its rows linearly independents.

   (b) What can you conclude about the columns of A ?

   (c) Write row three as a linear combination of the other rows.

2. Use the definition of linear dependence of columns of a matrix to verify if B has its columns linearly independents:

$$B = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 0 & 2 \\ -1 & -2 & 0 \end{bmatrix}$$

3. Using solely the elementary operations for matrices ($O1, O2$ e $O3$) show how to transform the matrix A into B:

$$\begin{aligned} A &= \begin{bmatrix} 0 & 2 & 3 & 4 \\ 2 & 3 & 5 & 4 \\ 4 & 8 & 13 & 12 \end{bmatrix} \\ B &= \begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix} \end{aligned}$$

4. Determine the rank of the following matrices:

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 & 0 & -1 \\ -1 & 0 & 2 & 1 \\ 0 & 2 & 2 & 0 \\ 1 & 0 & 2 & 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 3 & 1 & 2 & -1 \\ 5 & 5 & 2 & 5 \\ 7 & 4 & 4 & 1 \end{bmatrix}$$

5. Show that the rows of the matrix

$$\begin{bmatrix} 1 & -1 & 0 & 2 & 3 \\ 3 & 2 & 1 & -2 & 0 \\ -1 & 0 & 2 & -3 & 1 \\ 0 & 4 & 3 & -9 & -5 \end{bmatrix}$$

are linearly dependents. How many rows at most are linearly independents?

6. Let

$$A = \begin{bmatrix} 1 & 2 \\ 0 & c \\ 2 & 4 \end{bmatrix}$$

   (a) Consider that $c \neq 0$, which is the highert rank of A?

   (b) Discuss the rank of A, in regard to the parameter c.

7. Show that

$$A_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \quad A_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \quad A_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} \quad A_4 = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

   are linerarly dependents.

8. Consider the matrix

$$A = \begin{bmatrix} 1 & 1 & -3a & 1 \\ 2 & -1 & a & -2 \\ 7 & -2 & 0 & -5 \end{bmatrix}$$

   (a) Write the third row as a linear combination of others.

   (b) Are the columns of A, linearly independents? Explain yourself.

9. Consider the matrix

$$A = \begin{bmatrix} 1 & k & k-1 & 1 \\ -1 & 1 & 0 & k \\ 0 & -k-1 & 1-k & k^2 \end{bmatrix}$$

   (a) Discuss the rank of A relatively to the parameter k.

   (b) Discuss the linear dependence of the rows and the columns of A.

   (c) Let $k = 1$ and prove with the definition, the columns are linearly dependents.

10. A square matrix P is idempotent if $P^2 = P$. Show that if $A.B = A$ e $B.A = B$ then, B e $\left(A^T.B^T\right)$ are idempotents.

11. Prove that the non-null vectors $\vec{v_1}, \vec{v_2}, \ldots, \vec{v_m}$ are linearly dependents if one of them, for instance, $\vec{v_i}$, is a linear combination of the previous $\vec{v_i} = a_i\vec{v_i} + \ldots + a_{i-1}\vec{v_{i-1}}$.

2.3 Solutions

1. (b) They are linearly dependents. The reason is the maximum number of rows that are linarly independents is equal to the maximum number of columns that are linearly independents. As the maximum number of rows that are lineraly independents is less than 3, therefore, the maximum number of columns linearly independents will be less than 3.

   (c) $L_3 = L_1 + L_2$

4. $r_A = 3, \quad r_B = 3 \quad \text{e} \quad r_C = 2$

5. The maximum number of rows linearly independents is 3.

6. (a) $r_A = 2$, for $c \neq 0$

   (b) If $c \neq 0$, $r_A = 2$; Whether $c = 0, \quad r_A = 1$.

8. (a) $L_3 = L_1 + 3L_2$

   (b) The columns are linearly dependents.

9. (a) $r_A = 3, \ \forall k$.

   (b) The rows are linearly independents.The columns are linearly dependents.

3 Matrix inversion

3.1 Sypnosis

An n-by-n (square) matrix A is called invertible if there exists an n-by-n matrix B such that $AB = BA = I$. If B exists, is designated as A^{-1} and previous equality becomes: $AA^{-1} = A^{-1}A = I$. In addition, matrix A must have the rank equal to its dimension in order to be invertible.

3.2 Exercises

1. Let A and B nonsingular matrix of order n prove that:

   (a) $(A^{-1})^{-1} = A$

   (b) $(A.B)^{-1} = B^{-1}.A^{-1}$

   (c) $\left(A^T\right)^{-1} = (A^{-1})^T$

2. Consider A, B and C nonsingular matrices with the same dimension, generalize the result of question 1.b) proving the equality $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$.

3. Prove that, if $A^{-1}.A = I$ e $C.A^{-1} = I$ then $C = A$.

4. Calculate if possible, the inverse of the following matrices:

$$\text{a)} \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix} \qquad \text{b)} \begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 3 & 3 & 2 \\ 2 & 4 & 3 & 3 \\ 1 & 1 & 1 & 1 \end{bmatrix} \qquad \text{c)} \begin{bmatrix} 1 & 3 & 5 \\ 1 & 3 & 5 \\ 1 & 4 & 7 \end{bmatrix}$$

3.3 Solutions

4. (a) $\begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$

   (b) $\begin{bmatrix} 1 & -2 & 1 & 0 \\ 1 & -2 & 2 & -3 \\ 0 & 1 & -1 & 1 \\ -2 & 3 & -2 & 3 \end{bmatrix}$

   (c) The marix has no inverse once the rows are linearly dependents: $(L_1 = L_2)$. The matrix is singular.

4 Determinants

4.1 Sypnosis

Definition 7 *Determinant is a real number associated with a square matrix. It is the sum of the product terms made up from the elements of the matrix. In addtion, the product terms are affected by the positive or negative sign. Those with a even number of inversion of the indexes carry the positive sign; those with a odd number of inversion of the indexes will carry the negative sign.*

4.1.1 Some properties

- $|A^{-1}| = \frac{1}{|A|}$
- $|AB| = |A| \, . \, |B|$

- $|kA| = k^n . |A|$
- $\left|A^T\right| = |A|$
- $|A| + |B| \neq |A + B|$

4.1.2 Computation of determinants

For a 2 x 2 matrix, its determinant is found by subtracting the products of its diagonals.
For a 3 x 3 matrix, the Sarrus rule is widely used.
Another widely used technique is the "Laplacian" minor expansion: The determinat of a square matrix is obtained by summing of the products of the elements of a row (column) with their corresponding co-factors.

4.2 Exercises

1. Determine i and j so that the product term $a_{1i}.a_{32}.a_{4j}.a_{25}.a_{53}$ of a matrix of dimension 5 is even.

2. Calculate and justify the following determinants:

$$A = \begin{vmatrix} 1 & 3 & 0 \\ 2 & 1 & 0 \\ 3 & -1 & 0 \end{vmatrix} \qquad B = \begin{vmatrix} 2 & 2 & 2 \\ 2 & 3 & 4 \\ 1 & 1 & 1 \end{vmatrix}$$

$$C = \begin{vmatrix} 1 & 2 \\ -1 & 2 \end{vmatrix} \qquad D = \begin{vmatrix} 1 & -1 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{vmatrix}$$

   (a) Comment on the linear dependence of the rows and columns of the matrices based on the associated determinant.

   (b) What is the rank of each of the matrices?

3. Compute the determinants:

(a) $\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}$ (b) $\begin{vmatrix} 2 & 3 \\ 1 & 5 \end{vmatrix}$

(c) $\begin{vmatrix} 2 & 3 & 5 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{vmatrix}$ (d) $\begin{vmatrix} 2 & -3 & -4 \\ 1 & 0 & -2 \\ 0 & -5 & -6 \end{vmatrix}$

(e) $\begin{vmatrix} 4 & 2 & -1 & 2 \\ 1 & 3 & 2 & -3 \\ 5 & 1 & -4 & -2 \\ 1 & 0 & 3 & -1 \end{vmatrix}$ (f) $\begin{vmatrix} 2 & -1 & 2 & -1 \\ 2 & 4 & 0 & -2 \\ 1 & -3 & 2 & 0 \\ -1 & 0 & 3 & 4 \end{vmatrix}$

(g) $\begin{vmatrix} a & 0 & 0 & 0 \\ 0 & 0 & 0 & b \\ 0 & c & 0 & 0 \\ 0 & 0 & d & 0 \end{vmatrix}$ (h) $\begin{vmatrix} 1 & -1 & 2 & 0 & 3 \\ 1 & 2 & -2 & 1 & -1 \\ 0 & -3 & -1 & 1 & 2 \\ -1 & 2 & 0 & 3 & -1 \\ 1 & 2 & 1 & -1 & 0 \end{vmatrix}$

(i) $\begin{vmatrix} 1 & 0 & -2 & 0 & 1 \\ 3 & -1 & -2 & 1 & 3 \\ 0 & 0 & 1 & -2 & 1 \\ -1 & 0 & 2 & 3 & -1 \\ 1 & 2 & -1 & -1 & -1 \end{vmatrix}$

4. Solve the equation:

$$\begin{vmatrix} x & -4 & 0 \\ 1 & -3 & -1 \\ 2 & x & 5 \end{vmatrix} = 2$$

5. Solve the following equation:

$$\begin{vmatrix} 0 & x-2 & 0 & 0 \\ x-1 & 0 & x & 0 \\ 0 & x & 0 & x-2 \\ 0 & 0 & x-1 & 0 \end{vmatrix} = 0$$

6. Demonstrate the equality:

$$\begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 1+x_1 & 1 & \cdots & 1 \\ 1 & 1 & 1+x_2 & & 1 \\ \vdots & \vdots & & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 1+x_n \end{vmatrix} = x_1 x_2 \cdots x_n$$

7. Show that

$$\begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 2 & 1 & \cdots & 1 \\ 1 & 1 & 3 & & 1 \\ \vdots & \vdots & & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & n \end{vmatrix} = (n-1)!$$

8. Prove that

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = (c-a)(b-a)(c-b)$$

9. Show that:

$$\begin{vmatrix} a & \frac{b}{2} & c \\ 2a^2 & b^2 & 2c^2 \\ 1 & \frac{1}{2} & 1 \end{vmatrix} = (a-b)(b-c)(c-a)$$

10. Prove the following

$$\begin{vmatrix} ax & a^2+x^2 & 1 \\ ay & a^2+y^2 & 1 \\ az & a^2+z^2 & 1 \end{vmatrix} = \begin{vmatrix} ax & ax^2 & a \\ y & y^2 & 1 \\ z & z^2 & 1 \end{vmatrix}$$

11. Compute the value of x for which the equality is true. justify all your steps:

$$\begin{vmatrix} 3k^2 & 0 & k^2 \\ \frac{3}{2}k-6 & -6 & \frac{k}{2} \\ -6+k^3 & k^3 & -2 \end{vmatrix} = \begin{vmatrix} 2 & -2 & 4 \\ 0 & x & 4 \\ -1 & -x & 2x \end{vmatrix}$$

12. Determine the value of x, z and w that verifies the equation, using the properties of the determinants.

$$\begin{vmatrix} ax^2 & a & 0 \\ 1 & 1+z & w \\ 0 & az & aw \end{vmatrix} = 0$$

13. Consider the

$$|A| = \begin{vmatrix} 6x_3-x_1 & 4y_3-y_1 & 0 & -10w_3-w_1 \\ -w_3^2+x_1 & x_3-1 & 3 & x_2-y_2 \\ 4x_2 & -y_2 & 0 & 2w_2 \\ 3x_3 & 2y_3 & 0 & -5w_3 \end{vmatrix}$$

$$|B| = \begin{vmatrix} y_1 & 3y_2 & 2y_3 \\ x_1 & -12x_2 & 3x_3 \\ w_1 & -6w_2 & -5w_3 \end{vmatrix}$$

Using only the properties of the determinants show that $|A| = |B|$

14. Consider

$$|A| = \begin{vmatrix} 5x_1 + 6x_2 & 5y_1 - 2y_2 & 5z_1 + 4z_2 \\ 3x_2 & -y_2 & 2z_2 \\ 5x_3 & 3y_3 & 7z_3 \end{vmatrix} \text{ e } |B| = \begin{vmatrix} y_1 & 5y_2 & 3y_3 \\ x_1 & -15x_2 & 5x_3 \\ z_1 & -10z_2 & 7z_3 \end{vmatrix}$$

Using only the properties of the determinants show that $|A| = |B|$.

15. Each of the elements of the second column of the matrix

$$B = \begin{bmatrix} -1 & x & 1 \\ 0 & y & 2 \\ 1 & 1 & 1 \end{bmatrix}$$

is equal to the sum of its respective co-factor to the third element of the same column. Determine $|B|$.

16. Consider a square matrix

$$A = \left[\begin{array}{c|c} f & F \end{array}\right]$$

in which

$$f = \begin{bmatrix} 1 & f_2 & f_3 \end{bmatrix}^T,$$

and $f_2 \neq 0$ e $f_3 \neq 0$ e

$$F = \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \\ 0 & 0 \end{bmatrix}$$

(a) Let $|A| = 0$. Show that by having $f_{21}.A_{22} = f_3.A_{31}$ (in which A_{ij} is the co-factor of the element a_{ij} of the matrix A), then $f_{12} = 0 \vee f_{21} = 0$.

(b) Let $f_{12}.f_{21} = \frac{1}{2}$ e $f_{22} \neq 0$. Show that $|A| \neq 0$ sse $f_{11} \neq (2f_{22})^{-1}$.

(c) Let $f_{2j} = f_2.f_{1j}$ $(j = 1, 2)$. Demonstrate with the definition that the columns of A are linearly dependents. Supposing that $f_{11} \neq 0 \vee f_{12} \neq 0$, write down the first column as a linear combination of the others.

17. Let X be a square matrix of dimension n and $X + X^T = \overline{0}$ (null matrix). Show that with n odd, then $|X| = 0$. (Hint: Remember that A, a square matrix of dimension n, then $|kA| = k^n\,|A|$).

4.3 Solutions

1. $i = 1; \;\; j = 4$

2. - For A:
     $|A| = 0$ (there is a null row);
     (a) the rows or the columns are linerarly dependents for the determinant is zero;
     (b) $r = 2$;
   - For B:
     $|B| = 0$ (for row 1 is proportional to row 3);
     (a) The rows and columns are linearly dependents for the determinant is null;
     (b) $r = 2$;
   - Para C:
     $|C| = 4$;
     (a) as the determinant is not null, the rows and the columns are lineraly independents;
     (b) $r = 2$;
   - For D:
     $|D| = 2$ (the determinant of a row echelon form matriz is the term of the main diagonal);
     (a) As the determinant is not null, the rows and the columns are linearly independents;
     (b) $r = 3$;

3. (a) -2 (b) 7 (c) 9 (d) -18 (e) 275 (f) 0 (g) $abcd$ (h) -12 (i) 33

4. $x = 2 \vee x = 13$

5. $x = 1 \vee x = 2$

11. $x = -1 \vee x = -2$

12. $(x, z, 0)\,, \forall x, z$ and $(\pm 1, z, w)\,, \forall z, w$

15. $|B| = 10$

16. (b) i. The first row as a linear combination of the others: $l_1 = \frac{f_{1j}}{f_{2j}} l_2$.

5 Inverse matrix with determinant theory

5.1 Synopsis

An n-by-n (square) matrix A is called invertible if there exists an n-by-n matrix B such that $AB = BA = I$. If B exists, is designated as A^{-1} and previous equality becomes: $AA^{-1} = A^{-1}A = I$.
Accordingly, matrix A have to be square and its rank equal to its dimension; $(|A| \neq 0)$. The inverse of matrix A is $A^{-1} = \frac{1}{|A|}\hat{A}$, being $\hat{A}$ the adjoint matrix, or in the other words, the matrix of co-factors transposed.

5.2 Exercises

1. Consider the matrices

$$A = \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix} \quad B = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 0 & 1 \\ 3 & -1 & 2 \end{bmatrix}$$
$$C = \begin{bmatrix} 1 & 2 \\ -1 & 2 \end{bmatrix} \quad D = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix}$$

   Determine, whenever possible, the inverse of A, B, C and D, by using the adjoint matrices.

2. Consider the following matrix equation: $B^{-1}X^T |A| = B^{-1}\hat{A} - |A| I$. Explicit the matrix X as a function of A e B.

3. A and B are two nonsingular matrices and let $k \in R$. Consider the matrix equation: $(XA^{-1})^T + \left(BA^T\right)^{-1} = KI$, explicit X.

4. Consider the matrix equation $\left[\left(A^T\right)^{-1} X\right]^T + (AB)^{-1} = BB^{-1}$

   (a) Explicit X as a function of A and B.

   (b) Determine X, as $A = \begin{bmatrix} 3 & -4 \\ 0 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 2 \\ -1 & 3 \end{bmatrix}$

5. Explicit X from the following equation $\left(\hat{A}A - I\right)(A - X)^T = I\,|A|$, considering that $|A| \neq 1$

6. Given the following matrix equation $\hat{A}\left(B^{-1}|A|\right)^{-1} = A^{-1}X$ and knowing that A and B are nonsingular matrices

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \text{ e } B = \begin{bmatrix} 2 & -2 \\ 3 & \frac{1}{2} \end{bmatrix}$$

Determine $|X|$.

7. Considering the matrices A and B nonsigular, and

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ e } B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

and do verify the matrix equation: $\left[\left(A^T\right)^{-1} X\right]^T + (AB)^{-1} = A$, determine $|X|$.

8. Consider the matrix $A = \begin{bmatrix} 4K & 0 & 3 \\ -1 & 1 & K+1 \\ K & -K & 0 \end{bmatrix}$

   (a) Calculate the values of K so that A^{-1} will exist. Use the determinant theory for this exercise.

   (b) For $K = 1$ solve the matrix equation $\left(A\hat{A} - I\right)(X - A)^T = I$ $|A|$

9. Consider the following matrix equation $\left[\left(B^T X^{-1}\right)^T\right]^{-1} |A| + A\hat{A} = B^{-1}\hat{A}$ (where $\hat{A}$=adj A). Explicit X as a function of A and B. Let $A = \begin{bmatrix} 1 & -2 \\ 1 & -3 \end{bmatrix}$ e $B = \begin{bmatrix} -4 & 0 \\ 3 & 2 \end{bmatrix}$ and determine X.

10. Prove that:

$$\frac{1}{|B|}\left(\hat{A}\right)^{-1}\left[A\hat{A}A^{-1}\right]\left|B^T A^{-1}\right||A| = I.$$

(Remember that $|AB| = |A|\,|B|$.)

11. Let $|A|\,|B| = |AB|$ prove that $|A^{-1}| = \frac{1}{|A|}$, and $|A| \neq 0$.

12. Let A a nonsingular matrix and $|A| \neq 1$, show that

$$\left(\hat{A} - A^{-1}\right)^{-1} = \frac{A}{|A| - 1}.$$

13. Let A, B and C symmetric matrices of dimension n. The matrices A and B verify the condition $a_{ij} - b_{ij} = \begin{cases} 1 & (i = j) \\ 0 & (i \neq j) \end{cases}$ Discuss the conditions so it is possible to solve for X the following matrix equation: $\frac{A}{|A-B|}\left(C^{-1}X^TC + B\right)^T = A^2$

5.3 Solutions

1.

$$A^{-1} = \begin{bmatrix} 1 & -2 \\ \frac{-1}{2} & \frac{3}{2} \end{bmatrix} \qquad B^{-1} = \begin{bmatrix} \frac{-1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 2 & \frac{-3}{2} \\ 1 & 1 & -1 \end{bmatrix}$$

$$C^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{-1}{2} \\ \frac{1}{4} & \frac{1}{4} \end{bmatrix} \qquad D^{-1} = \begin{bmatrix} 1 & 1 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$$

2. $X = (A^{-1} - B)^T$

3. $X = K.A - (B^{-1})^T$

4. (a) $X = A^T - (B^{-1})^T$

   (b) $X = \begin{bmatrix} 6 & 1 \\ -6 & 1 \end{bmatrix}$

5. $X = A - \frac{|A|}{|A|-1}I$

6. $|X| = 7$, see that $X = B$.

7. $|X| = 0$, see that $|X| = |A^2 - B^{-1}|$

8. (a) $K \neq 0 \;\wedge\; K \neq -1$

   (b) $X = \begin{bmatrix} \frac{36}{7} & 0 & 3 \\ -1 & \frac{15}{7} & 2 \\ 1 & -1 & \frac{8}{7} \end{bmatrix}$, see that $X = A + \frac{|A|}{|A|-1}I$

9. $X = (A^{-1})^T - B^T$, $X = \begin{bmatrix} 7 & -2 \\ -2 & -3 \end{bmatrix}$

13. $X = 1$, if there is A^{-1}.

6 System of linear equations; Cramer's Rule; Rouché Theorem

6.1 Synopsis

6.1.1 Taxonomy

A general system of m linear equations with n unknowns can be written as

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \quad & \ldots + a_{1n}x_1 = b_1 \\ a_{21}x_1 + a_{22}x_2 + \quad & \ldots + a_{2n}x_2 = b_2 \\ \vdots \quad & \\ a_{m1}x_1 + a_{m2}x_+ \quad & \ldots + a_{mn}x_n = b_n \end{aligned}$$

Here, the x_1,x_2,..., x_n are the unknowns, a_{11}, a_{12}, ... a_{mn} are the coefficents of the system and the b_1, b_2, ..., b_n are the constant terms.
This system of linear equations can be written in the matrix form $AX = b$, so that A is the matrix of the coefficents, X, the column matrix of the unknowns and b is the column matrix of the constant terms.
A solution of a linear system is an assignment of values to the variables x_1, x_2,..., x_n such that each of the equation is satisfied. The set of all possible solutions is call solution set.
This linear system may behave in any one of three possible ways:

1. The system has a single unique solution
2. The system has infinite many solutions
3. The system has no solution

In order to determine the nature of the solution set of the linear system we start with calculating the rank of the matrix A, which is the highest dimension sub-matrix within A with non-null determinant. Then, compare the rank with the number of equations and the number of unknowns. The outcome is described as follows:

1. $r = m = n$ The system has a single unique solution
2. $r = m < n$ The system has a infinite many solutions
3. $r < m$ Check with Rouché Theorem.

6.1.2 Rouché Theorem

Consider a system of linear equations with n unknows with matrix equation $AX = b$. The system

1. has a solution: $rank(A|b) = rank(A)$
2. has a unique single solution: $rank(A|b) = rank(A) = n$

A system of linear equations with the matrix equation $AX = b$ is homogeneous if b is a matrix of zeros. We write $AX = 0$. A homogenous system of linear equations has always zero solution (trivial solution) or it has a non-trivial solution, and therefore infinite many solutions, if and only if $rank(A|b) = rank(A) < n$.

6.1.3 Cramer's Rule

Consider a system of linear equations represented in matrix multiplication form:

$$AX = b$$

Where the square matrix A is invertible and the vector $X = [x_1, ...x_n]^T$ is a column vector of the variables. Then, the Cramer's Rule states that

$$x_i = \frac{detA_i}{detA} \qquad i = 1 \dots n$$

Where A_i is the matrix formed by replacing th ith column of A by the column vector b.

6.2 Exercises

1. Solve the following sytems of equations:

(a) $\begin{cases} x + 2y + z + 2t = 16 \\ x + y + z + t = 10 \\ 4x + 2y + 3z + 4t = 33 \\ 4x + 3y + 2z + t = 20 \end{cases}$

(b) $\begin{cases} x + y + 2z = 9 \\ 3x - 4y + 5z = 10 \\ 2x + y - z = 1 \\ 6x - 2y + 6z = 20 \\ x - 5y + 6z = 9 \end{cases}$

(c) $\begin{cases} x - 5y + z = 4 \\ 2x + y - z = 1 \\ x + 6y - 2z = -3 \\ 4x - 4y + z = 9 \\ x - 16y + 4z = 9 \end{cases}$

(d) $\begin{cases} x + 2y + z + t = 0 \\ x - y + z + t = 0 \\ 2x + y + z + t = 0 \\ 3x + y + z + t = 0 \end{cases}$

(e) $\begin{cases} x - y - z = 2 \\ 2x + y - z = 1 \\ 3x - y - z = -2 \end{cases}$

(f) $\begin{cases} 2x + y + z = 4 \\ x + y + 2z = 1 \end{cases}$

2. Study the nature of the solutions of the systems of equations as a function o f parameter k:

$$\text{(a)} \begin{cases} x_1 + x_2 + x_3 = 1 \\ 2x_1 + x_2 - x_3 = -1 \\ x_1 + 3x_2 + 2x_3 = 2 \\ 2x_1 + 2x_2 + 3x_3 = 3 \\ x_1 + x_2 - 4x_3 = k \end{cases} \quad \text{(b)} \begin{cases} kx + y + z = 1 \\ x + ky + z = k \\ x + y + kz = k^2 \end{cases}$$

3. Study the linear system as a function of the parameter m:

$$\begin{cases} x + y - mz + mt = 0 \\ y + z + 2t = 1 \\ x + 2y + (2 + m)t = 1 \end{cases}$$

Let $m = 1$, use the Cramer's Rule to solve the linear system.

4. Consider the following system of linear equations

$$\begin{cases} x + y + (1 - m)z = m + 2 \\ (1 + m)x - y + 2z = 0 \\ 2x - my + 3z = m + 2 \end{cases}$$

(a) Write the system in the form of $AX = B$ e compute the determinant of A.

(b) Discuss the nature of the system as a function of the parameter m.

(c) Solve the linear system

   i. for the case in which there is a single and unique solution
   ii. for the case of homegeneous system
   iii. for which it has a infinite many solutions.

5. Consider the linear system with the variables x, y, z and w:

$$\begin{cases} x - y = a \\ y - z = b \\ z + w = -b \\ x - aw = a \end{cases}$$

   Discuss the system as a function of the parameters a and b. Solve it by using Cramer's Rule with the values of a e b such that the system has an infinite many solutions.

6. Consider the sistem of equations

$$\begin{cases} -2x_1 + 3x_2 - x_3 = 0 \\ x_1 + 3x_3 = 0 \\ 3x_1 - x_2 + ax_3 = b \end{cases}$$

   Determine a and b such that the system has an infinite many solutions and solve it with Cramer's rule.

7. Consider the system of equations with the unknowns x, y and z:

$$\begin{cases} ax - z = 0 \\ \frac{3}{4}x + (a-1)y = a \\ y - b = -z \end{cases}$$

   (a) Write its equivalente in the matrix form.
   (b) Analyse the nature of the system as a function of a and b.

8. Consider the following simplified Keynesian model:

$$\begin{cases} Y = C + I_0 + G_0 \\ C = a + bY \end{cases}$$

   Use Cramer's Rule to compute the endogenous variables Y and C.

item Consider the system S, with four equations and four unknowns (p, q, r e s), and the main determinant

$$\triangle = \begin{array}{ccc} (p) & (q) & (s) \\ \end{array}$$
$$\begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix}$$

Consider yet non principal equation (the third) $\quad q + r + s = 2b \quad$ and let $\begin{bmatrix} 1 & 0 & 2b & a \end{bmatrix}^T$ the matrix of constant terms.

Discuss the nature of S as function of the parameters a and b.

9. Consider the linear system with unknowns x, y, z and t and the parameters c, $d \in R$

$$\begin{cases} x - y + z = c + d + t \\ x + y - z - t = c - d \\ cx - cz - dt = 4 - dy \end{cases}$$

   (a) Write the equivalent in matrix form.
   (b) Study the system as a function of the parameters c and d.
   (c) Let $\quad c = 1 \quad$ e $\quad d = 0$
      i. Solve the system with Cramer's Rule.
      ii. Verify, if the equation $x + y - z - t = -1$ is compatible with the system.

10. Consider the linear system with x, y, z and t:

$$\begin{cases} x + y = m \\ y + z = -n \\ z + t = p \\ x + mt = m \end{cases}$$

Study the system as function of m, n and p. Solve it with Cramer's Rule and with the values of m, n and p in which the system has an infite many solutions.

11. Consider the system S with the unknowns x, y, w and t:

$$\begin{cases} x + w + kt = 0 \\ y + t = 1 \\ y + w + kt = 2a \\ y + w + t = b \end{cases}$$

Discuzz the nature of the system according to the parameters k, a and b . Solve it with Cramer's Rule for the case of infinite many solutions.

12. Consider the following system of linear equations with the unknowns x, y and z:
$$\begin{cases} 2x + 3y - z = \beta \\ x + 2y - 2z = 1 \\ x + \alpha y + z = -1 \end{cases}$$
    (a) Write it in the matrix form.
    (b) Discuss the nature of the system as a function of α and β.
    (c) Let $\alpha = 1$ and $\beta = 0$. Solve it with Cramer's Rule.

13. Determine the values of k for which the system
$$\begin{cases} x + (k-1)z = 0 \\ ky + z = 0 \\ kx + z = 0 \end{cases}$$
has non-trivial solution.Choose a vale for K and solve the system.

14. Consider the homogenous system for x, y e z (real numbers):
$$\begin{cases} x + ay + a^2 z = 0 \\ x + by + b^2 z = 0 \\ x + cy + c^2 z = 0 \end{cases}$$
Discuss the nature of the system for a, b, and c.

    Solve with Cramer's Rule for the case of infinite many solutions.

15. Consider the system of linear equations $Ax = b$ in the version of homogenous system $Ax = 0$. Show that if y_1 and y_2 are two solutions of the complete system $Ax = b$ then $z = y_1 - y_2$ is solution of $Ax = 0$.

6.3 Solutions

1. (a) Single and unique solution (SPD): $(1, 2, 3, 4)$
    (b) Single and unique solution (SPD): $(1, 2, 3)$
    (c) No solution (SI)
    (d) Infinite many solutions (SPI) with d=1: $(0, 0, -t, t), \forall t$
    (e) Single and unique solution (SPD): $\left(-2, \frac{1}{2}, -\frac{9}{2}\right)$

(f) Infinite many solutions (SPI) com d=1: $(3+z,\ -2-3z,\ z), \forall z$

2. (a) $k \neq -4$: No solution (SI); $k=4$: Single and unique solution (SPD)

   (b) $k \neq 1 \wedge k \neq -2$: SPD; $k=1$: SPI (d=1); $k=-2$: SI

3. $m \neq 1$: SPI (d=1); $m=1$: SPI (d=2). For $m=1$ we get: $(-1+2z+t, 1-z-2t, z, t), \forall z,$

4. (a) $\begin{bmatrix} 1 & 1 & 1-m \\ m+1 & -1 & 2 \\ 2 & -m & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} m+2 \\ 0 \\ m+2 \end{bmatrix}$

   $|A| = m(m+2)(m-2)$

   (b) $m \neq 0 \wedge m \neq \pm 2$: SPD; $m=0$: SPI (d=1); $m=2$: SI; $m=-2$: SPI (d=1)

      i. $\left(\frac{1}{m-2}, -\frac{m+3}{m-2}, -\frac{m+2}{m-2}\right)$, for $m \neq 0 \wedge m \neq \pm 2$;

      ii. $(x, -x, 0), \forall x$, for $m=-2$;

      iii. $\left(1-\frac{3z}{2}, 1+\frac{z}{2}, z\right), \forall z$, for $m=0$.

5. $a \neq -1 \wedge \forall b$: SPD; $a=-1 \wedge \forall b$: SPI (d=1). For SPI we get $(-1-w, -w, -b-w, w), \forall w$.

6. $a = \frac{22}{3} \wedge b = 0$: The solution is $\left(-3x_3, -\frac{5x_3}{3}, x_3\right), \forall x_3$.

7. (a) $\begin{bmatrix} a & 0 & -1 \\ \frac{3}{4} & a-1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ a \\ b \end{bmatrix}$

   (b) $a \neq -\frac{1}{2} \wedge a \neq -\frac{3}{2} \wedge \forall b$: SPD; $a=-\frac{1}{2} \wedge b=\frac{1}{3}$: SPI (d=1); $a=\frac{3}{2} \wedge b=3$: SPI (d=1); $a=-\frac{1}{2} \wedge b \neq \frac{1}{3}$: SI; $a=\frac{3}{2} \wedge b \neq 3$: SI.

8. $b \neq 1$ (SPD): $Y = \frac{I_0+G_0+a}{1-b} \wedge C = \frac{a+b(I_0+G_0)}{1-b}$;
   $b=1 \wedge I_0+G_0+a=0$ (SPI): $Y = C-a \wedge \forall C$.

9. $a=2b$: SPI (d=1); $a \neq 2b$: SI.

10. (a) $\begin{bmatrix} 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ c & d & -c & -d \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} c+d \\ c-d \\ 4 \end{bmatrix}$

    (b) $d \neq c$: SPI (d=1); $d=c$: SI

    (c) i. $(1+t, -3+t, -3+t, t), \forall t$.

ii. The equation is incompatible.

11. $m \neq 1 \wedge \forall p \wedge \forall n$: SPD; $m = 1 \wedge p = -n \wedge \forall n$: SPI (d=1); $m = 1 \wedge p \neq -n \wedge \forall n$: SI; When it is infinite and many solutions, the solution is $(1-t, t, -n-t, t), \forall t$.

12. $k \neq 1 \wedge \forall a \wedge \forall b$: SPD; $k = 1 \wedge \forall a \wedge b = 2a$: SPI (d=1); $k = 1 \wedge \forall a \wedge b \neq 2a$: SI; For SPI the solution is $(1-2a-t, 1-t, 2a-1, t), \forall t$.

13. (a) $\begin{bmatrix} 2 & 3 & -1 \\ 1 & 2 & -2 \\ 1 & \alpha & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \beta \\ 1 \\ -1 \end{bmatrix}$

    (b) $\alpha \neq 1$: SPD; $\alpha = 1 \wedge \beta = 0$: SPI (d=1); $\alpha = 1 \wedge \beta \neq 0$: SI.

    (c) SPI (d=1) and the solution is $(-4z-3, 3z+2, z)\, \forall z$.

14. $k = 0 \vee k = \frac{1 \pm \sqrt{5}}{2}$

    For $k = 0$, the solution is $(0, y, 0), \forall y$;

    For $k = \frac{1+\sqrt{5}}{2}$, the solution is $\left(\frac{1-\sqrt{5}}{2}z, \frac{1-\sqrt{5}}{2}z, z\right), \forall z$;

    For $k = \frac{1-\sqrt{5}}{2}$, the solution is $\left(\frac{1+\sqrt{5}}{2}z, \frac{1+\sqrt{5}}{2}z, z\right), \forall z$.

15. $a \neq b \wedge b \neq c \wedge a \neq c$: SPD; $a \neq b \wedge (a = c \vee b = c)$: SPI (d=1);

    $a \neq c \wedge (a = b \vee b = c)$: SPI (d=1); $b \neq c \wedge (a = b \vee c = a)$: SPI (d=1); $a = b = c$: SPI (d=2)

    For $a \neq b \wedge (a = c \vee b = c)$, the solution is $(abz, -(a+b)z, z), \forall z$

    For $a \neq c \wedge (a = b \vee b = c)$, the solution is $(acz, -(a+c)z, z), \forall z$

    For $b \neq c \wedge (a = b \vee c = a)$, the solution is $(bcz, -(b+c)z, z), \forall z$

    For $a = b = c$, the solution is $(-ay - a^2 z, y, z), \forall y, z$.

7 Revision Exercises

1.

    (a) Prove that

$$\frac{1}{|B|} \cdot \left(\hat{A}\right)^{-1} \cdot \left[A.\hat{A}.A^{-1}\right] \cdot \left|B^T.A^{-1}\right| \cdot |A| = I,$$

    The matrices A and B are square of dimension n and $\hat{A}$ is the adjoint matrix of A.

(b) Solve the matrix equation for X

$$\begin{vmatrix} x & 0 & -1 & 1 & 0 \\ 1 & x & -1 & 1 & 0 \\ 1 & 0 & x-1 & 0 & 1 \\ 0 & 1 & -1 & x & 1 \\ 0 & 1 & -1 & 0 & x \end{vmatrix} = 0$$

2. Consider the matrix equation $\quad B^{-1}X^T\,|A| = B^{-1}\hat{A} - |A|\,I.$

   (a) Explicit the matrix X as a function of A and B. Justify.

   (b) Consider the matrix A and

$$A = \begin{bmatrix} \alpha & 1 & \alpha \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

   Determine α such that $A\hat{A} = I$, without computing the adjoint matrix o A $(\hat{A})$.

   (c) Use the value of α from previous exercise to determine $|B^{-1}|$. Consider that $B = 2A$ and make use of the properties of the determinants. Justify adequaltely.

3. Let A, X and I square matrices of dimension n, and I the identity matric. Consider that $|A| = 3$ and:

$$AX\,(I + A) - \left(X^T A^T\right)^T A = A^2\hat{A},$$

   and, $\hat{A}$ is the adjoint matrix of A.

   (a) Show that it is false the following proposition: $AX = X \Longrightarrow A = I$. What conditions have to be imposed on X so that the proposition becomes true?

   (b) Calcule $|X|$.

4. Consider the following determinants

$$|A| = \begin{vmatrix} 1 & 2 & 1 & b \\ 5 & 3 & a & 3 \\ -1 & 2 & 1 & b \\ 2 & -1 & a & 1 \end{vmatrix} \text{ e } |B| = \begin{vmatrix} 1 & 5 & -1 & 2 \\ -2 & -3 & -2 & 1 \\ 1 & a & 1 & a \\ b & 3 & b & 1 \end{vmatrix}$$

   Determine, without computing directly the determinants $|A|+|B|$, and $|B|$ is obtained from $|A|$.

5. Considerthe following system of equations for the unknowns x, y, z e w

$$\begin{cases} y = z - w \\ ax = b - w \\ x - ay + z = a \end{cases}$$

   (a) Write the system in the matrix form.

   (b) Study the system according to the paramenters a and b.

   (c) Let $a = 2$ and $b = 1$ solve the system with Cramer's Rule.

6. Consider the linear system

$$\begin{cases} 4x + 4y - 3z = 0 \\ x + y - z = 0 \\ kx + 2y + 2z = 0 \end{cases}$$

   (a) Determine k such that the system has non-trivial solutions. In this case, what can be said about the linear dependence of the rows of the coeficent matrix?

   (b) Let $k = 2$ solve the system with Cramer's Rule.

8 Solutions

1. (b) $x = -1$

2. (a) $X = (A^{-1})^T - B^T$

   (b) $\alpha = 2$

   (c) $|B^{-1}| = \frac{1}{8}$

3. (a) It is necessary that X be nonsingular to be invertible.

   (b) $|X| = |3I| = 3^n.|A| + |B| = 0$

4. $|A| + |B| = 0$

5. (b) Se $a \neq 0 \wedge a \neq 1$: SPI, d=1.
   Se $a = 0$: SPI, d=1.
   Se $a = 1 \wedge b = 1$, SPI, d=2.
   Se $a = 1 \wedge b \neq 1$, SI.

   (c) $z = -\frac{3}{2} + \frac{3}{2}w$, $x = \frac{1}{2} - \frac{1}{2}w$, $y = -\frac{3}{2} + \frac{1}{2}w$

6. (a) $k = 2$: $\quad C_1 = C_2$ e $|A| = 0$. Rows are linearly dependents.
   (b) $(x, y, z) = (x, -x, 0)\,, \forall x$.