Theoretical Manual of SEAWAY

(Release 4.19, 12-02-2001)

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Report1216a February 2001

Abstract

This report describes in detail the theoretical backgrounds of the six degrees of freedom ship motions program SEAWAY.

SEAWAY is a frequency-domain ship motions computer code, based on the linear strip theory, to calculate the wave-induced loads, motions, added resistance and internal loads for six degrees of freedom of displacement ships, yachts and barges, sailing in regular and irregular waves. When not taking into account interaction effects between the two individual hulls, these calculations can be carried out for twin-hull ships, such as semi-submersibles and catamarans, too. This potential theory program is suitable for deep water as well as for shallow water. Viscous roll damping, bilge keels, anti-roll tanks, free surface effects and linear springs can be added. A dedicated editor takes care for a simple input of data.

This report and other information on the strip theory program SEAWAY can be found on the Internet at web site http://dutw189.wbmt.tudelft.nl/~johan, which can also be reached by a link to this site from http://www.shipmotions.nl.

Aditional information can be obtained by e-mail (J.M.J.Journee@wbmt.tudelft.nl) from the author.

The last revision of this report is dated: 3 July 2001. Remarks and errata are very welcome!

Contents

Chapter 1

Introduction

This report aims being a guide and an aid for those who want to study the theoretical backgrounds and the algorithms of a ship motions computer program based on the strip theory.
The present report describes in detail the theoretical backgrounds and the algorithms of a six degrees of freedom ship motions personal computer program, named SEAWAY. A User Manual is given by [Journée, 2001a]. Extensive verifications and validations of this program have been presented by [Journée, 2001b].
This program, based on the ordinary and the modified strip theory, calculates the wave-induced loads and motions with six degrees of freedom of mono-hull ships and barges, sailing in a seaway. When not taking into account interaction effects between the two individual hulls, these calculations can be carried out for twin-hull ships, such as semi-submersibles and catamarans, too.
In the past a preliminary description of all algorithms, used in strip theory based ship motions calculations, has been given by the author, see [Journée, 1992]. Since then, this program has been extended and adapted considerably, so a revised report is presented here.

Chapter 1, this introduction, gives a short survey of the contents of all chapters in this report.

Chapter 2 gives a general description of the various strip theory approaches. A general description of the potential flow theory is given. The derivations of the hydromechanical forces and moments, the wave potential and the wave and diffraction forces and moments have been described.
The equations of motion are given with solid mass and inertia terms and hydromechanical forces and moments in the left hand side and the wave exciting forces and moments in the right hand side.
The principal assumptions are a linear relation between forces and motions and the validity of obtaining the total forces by a simple integration over the ship length of the two-dimensional cross sectional forces.
This includes for all motions a forward speed effect caused by the potential mass, as it has been defined by [Korvin-Kroukovsky and Jacobs, 1957] for the heave and pitch motions. This approach is called the "Ordinary Strip Theory Method". Also an inclusion of the

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forward speed effect caused by the potential damping, as for instance given by [Tasai, 1969], is given. This approach is called the "Modified Strip Theory Method".
The inclusion of so-called "End-Terms" has been described.

Chapter 3 describes several conformal mapping methods. For the determination of the two-dimensional hydrodynamic potential coefficients for sway, heave and roll motions of ship-like cross sections, these cross sections are conformal mapped to the unit circle. The advantage of conformal mapping is that the velocity potential of the fluid around an arbitrary shape of a cross section in a complex plane can be derived from the more convenient circular section in another complex plane. In this manner hydrodynamic problems can be solved directly with the coefficients of the mapping function.
The close-fit multi-parameter conformal mapping method is given. A very simple and straight on iterative least squares method, used to determine the conformal mapping coefficients, has been described. Two special cases of multi-parameter conformal mapping have been described too: the well known classic Lewis transformation ([Lewis, 1929]) with two parameters and an Extended-Lewis transformation with three parameters, as given by [Athanassoulis and Loukakis, 1985].

Chapter 4 describes the determination of the two-dimensional potential mass and damping coefficients for the six modes of motions at infinite and finite water depths.
At infinite water depths, the principle of the calculation of these potential coefficients is based on work of [Ursell, 1949] for circular cylinders and [Frank, 1967] for any arbitrary symmetric cross section.
Starting from the velocity potentials and the conjugate stream functions of the fluid with an infinite depth as have been given by [Tasai, 1959], [Tasai, 1960], [Tasai, 1961] and [Jong, 1973] and using the multi-parameter conformal mapping technique, the calculation routines of the two-dimensional hydrodynamic potential coefficients of ship-like cross sections are given for the sway, heave and roll motions.
For shallow water, the method of [Keil, 1974] - based on a variation of the theory of [Ursell, 1949] - has been given.
The pulsating sources method of [Frank, 1967] for deep water has been described too.
Because of using the strip theory approach here, the pitch and yaw coefficients follow from the moment about the ship's centre of gravity of the heave and sway coefficients, respectively.
Approximations are given for the surge coefficients.

Chapter 5 gives some corrections on the hydrodynamic damping due to viscous effects. The surge damping coefficient is corrected for viscous effects by an empirical method, based on a simple still water resistance curve as published by [Troost, 1955].
The analysis of free-rolling model experiments and two (semi-)empirical methods published by [Miller, 1974] and [Ikeda et al., 1978], to determine a viscous correction of the roll damping coefficients are described in detail.

Chapter 6 describes the determination of the hydromechanical forces and moments in the left hand side of the six equations of motion of a sailing ship in deep water for both the ordinary and the modified strip theory method.

Chapter 7 describes the wave exciting forces and moments in the right hand side of the six equations of motion of a sailing ship in water with an arbitrarily depth, using the relative motion concept for both the ordinary and the modified strip theory method.

First, the classical approach has been described. Then, an alternative approach - based on diffraction of waves - with equivalent accelerations an velocities of the water particles has been described.

Chapter 8 describes the solution of the equations of motion. The determination of the frequency characteristics of the absolute displacements, rotations, velocities and accelerations and the vertical relative displacements. The use of a wave potential valid for any arbitrary water depth makes the calculation method, with deep water coefficients, suitable for ships sailing with keel clearances down to about 50 percent of the ship's draft.

Chapter 9 describes some anti-rolling devices. A description is given of an inclusion of passive free-surface tanks as defined by the experiments of [Bosch and Vugts, 1966] and by the theory of [Verhagen and van Wijngaarden, 1965]. Active fin stabilizers and active rudder stabilizers have been described too.

Chapter 10 describes the inclusion of linear spring terms to simulate the behavior of anchored or moored ships.

Chapter 11 describes two methods to determine the transfer functions of the added resistances due to waves. The first method is a radiated wave energy method, as published by [Gerritsma and Beukelman, 1972]. The second method is an integrated pressure method, as published by [Boese, 1970].

Chapter 12 describes the determination of the frequency characteristics of the lateral and vertical shear forces and bending moments and the torsional moments in a way as presented by [Fukuda, 1962] for the vertical mode. Still water phenomena are described too.

Chapter 13 describes the statistics in irregular waves, by using the superposition principle. Three examples of normalized wave spectra are given: the somewhat wide wave spectrum of Neumann, an average wave spectrum of Bretschneider and the more narrow Mean JONSWAP wave spectrum.
A description is given of the calculation procedure of the energy spectra and the statistics of the ship motions for six degrees of freedom, the added resistances, the vertical relative motions and the mechanic loads on the ship in waves coming from any direction.
For the calculation of the probability of exceeding a threshold value by the motions, the Rayleigh probability density function has been used.
The static and dynamic swell up of the waves, of importance when calculating the probability of shipping green water, are defined according to [Tasaki, 1963].
Bow slamming phenomena are defined by both the relative bow velocity criterium of [Ochi, 1964] and the peak bottom impact pressure criterium of [Conolly, 1974].

Chapter 14 describes the additions to the algorithms in case of twin- hull ships, such as semi-submersibles and catamarans. For interaction effects between the two individual hulls will not be accounted here.

Chapter 15 shows some typical numerical recipes, as used in program SEAWAY.

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Chapter 2

Strip Theory Method

The ship is considered to be a rigid body floating in an ideal fluid: homogeneous, incompressible, free of surface tension, irrotational and without viscosity. It is assumed that the problem of the motions of this floating body in waves is linear or can be linearized. As a result of this, only the external loads on the underwater part of the ship are considered and the effect of the above water part is fully neglected.

The incorporation of seakeeping theories in ship design has been discussed clearly by [Faltinsen and Svensen, 1990]. An overview of seakeeping theories for ships were presented and it was concluded that - nevertheless some limitations - strip theories are the most successful and practical tools for the calculation of the wave induced motions of the ship, at least in an early design stage of a ship.
The strip theory solves the three-dimensional problem of the hydromechanical and exciting wave forces and moments on the ship by integrating the two-dimensional potential solutions over the ship's length. Interactions between the cross sections are ignored for the zero-speed case. So each cross section of the ship is considered to be part of an infinitely long cylinder.

The strip theory is a slender body theory, so one should expect less accurate predictions for ships with low length to breadth ratios. However, experiments showed that the strip theory appears to be remarkably effective for predicting the motions of ships with length to breadth ratios down to about 3.0, or even sometimes lower.
The strip theory is based on the potential flow theory. This holds that viscous effects are neglected, which can deliver serious problems when predicting roll motions at resonance frequencies. In practice, for viscous roll damping effects can be accounted fairly by empirical formulas.
Because of the way that the forced motion problems are solved generally in the strip theory, substantial disagreements can be found between the calculated results and the experimental data of the wave loads at low frequencies of encounter in following waves. In practice, these "near zero frequency of encounter problems" can be solved by forcing the wave loads to go to zero artificially.

For high-speed vessels and for large ship motions, as appear in extreme sea states, the strip theory can deliver less accurate results. Then the so-called "end-terms" can be important too.

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The strip theory accounts for the interaction with the forward speed in a very simple way. The effect of the steady wave system around the ship is neglected and the free surface conditions are simplified, so that the unsteady waves generated by the ship are propagating in directions perpendicular to the centre plane of the ship. In reality the wave systems around the ship are far more complex. For high-speed vessels, unsteady divergent wave systems become important. This effect is neglected in the strip theory.
The strip theory is based on linearity. This means that the ship motions are supposed to be small, relative to the cross sectional dimensions of the ship. Only hydrodynamic effects of the hull below the still water level are accounted for. So when parts of the ship go out of or into the water or when green water is shipped, inaccuracies can be expected. Also, the strip theory does not distinguish between alternative above water hull forms.
Because of the added resistance of a ship due to the waves is proportional to the relative motions squared, its inaccuracy will be gained strongly by inaccuracies in the predicted motions.

Nevertheless these limitations, seakeeping prediction methods based upon the strip theory provide a sufficiently good basis for optimization studies at an early design stage of the ship. At a more detailed design stage, it can be considered to carry out additional model experiments to investigate for instance added resistance or extreme event phenomena, such as shipping green water and slamming.

2.1 Definitions

Figure 2.1 shows a harmonic wave as seen from two different perspectives. Figure 2.1-a shows what one would observe in a snapshot photo made looking at the side of a (transparent) wave flume; the wave profile is shown as a function of distance x along the flume at a fixed instant in time. Figure 2.1-b is a time record of the water level observed at one location along the flume; it looks similar in many ways to the other figure, but time t has replaced x on the horizontal axis.

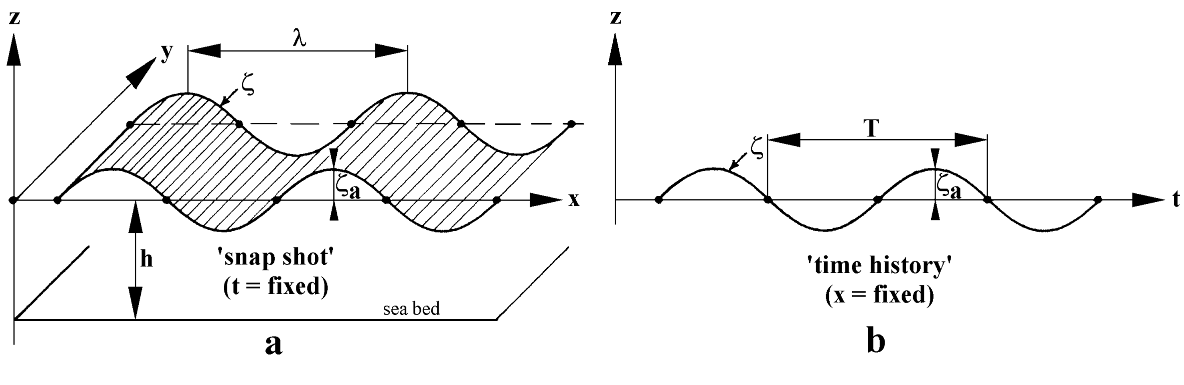

Figure 2.1: Harmonic Wave Definitions

Notice that the origin of the coordinate system is at the still water level with the positive z-axis directed upward; most relevant values of z will be negative. The still water level is the average water level or the level of the water if no waves were present. The x-axis is positive in the direction of wave propagation. The water depth, h, (a positive value) is measured between the sea bed ($z = -h$) and the still water level ($z = 0$).

The highest point of the wave is called its crest and the lowest point on its surface is the trough. If the wave is described by a harmonic wave, then its amplitude ζ_a is the distance from the still water level to the crest, or to the trough for that matter. The subscript "a" denotes amplitude, here.
The horizontal distance (measured in the direction of wave propagation) between any two successive wave crests is the wave length, λ. The distance along the time axis is the wave period, T. The ratio of wave height to wave length is often referred to as the dimensionless wave steepness, $2\zeta_a/\lambda$.
Since the distance between any two corresponding points on successive sine waves is the same, wave lengths and periods are usually actually measured between two consecutive upward (or downward) crossings of the still water level. Such points are also called zero-crossings, and are easier to detect in a wave record.
Since sine or cosine waves are expressed in terms of angular arguments, the wave length and period are converted to angles using:

$$
\begin{aligned}
k\lambda &= 2\pi \qquad \text{or:} \qquad k = \frac{2\pi}{\lambda} \\
\omega T &= 2\pi \qquad \text{or:} \qquad \omega = \frac{2\pi}{T}
\end{aligned}
\tag{2.1}
$$

in which k is the wave number (rad/m) and ω is the circular wave frequency (rad/s).
Obviously, the wave form moves one wave length during one period so that its speed or phase velocity, c, is given by:

$$
c = \frac{\lambda}{T} = \frac{\omega}{k} \tag{2.2}
$$

Suppose now a sailing ship in waves, with coordinate systems as given in figure 2.2.

A right-handed coordinate system $S(x_0, y_0, z_0)$ is fixed in space. The (x_0, y_0)-plane lies in the still water surface, x_0 is directed as the wave propagation and z_0 is directed upwards.

Another right-handed coordinate system $O(x, y, z)$ is moving forward with a constant ship speed V. The directions of the axes are: x in the direction of the forward speed V, y in the lateral port side direction and z upwards. The ship is supposed to carry out oscillations around this moving $O(x, y, z)$ coordinate system. The origin O lies above or under the time-averaged position of the centre of gravity G. The (x, y)-plane lies in the still water surface.

A third right-handed coordinate system $G(x_b, y_b, z_b)$ is connected to the ship with G at the ship's centre of gravity. The directions of the axes are: x_b in the longitudinal forward direction, y_b in the lateral port side direction and z_b upwards. In still water, the (x_b, y_b)-plane is parallel to the still water surface.

If the wave moves in the positive x_0-direction (defined in a direction with an angle μ relative to the ship's speed vector, V), the wave profile - the form of the water surface - can now be expressed as a function of both x_0 and t as follows:

$$
\zeta = \zeta_a \cos(kx_0 - \omega t) \qquad \text{or:} \qquad \zeta = \zeta_a \cos(\omega t - kx_0) \tag{2.3}
$$

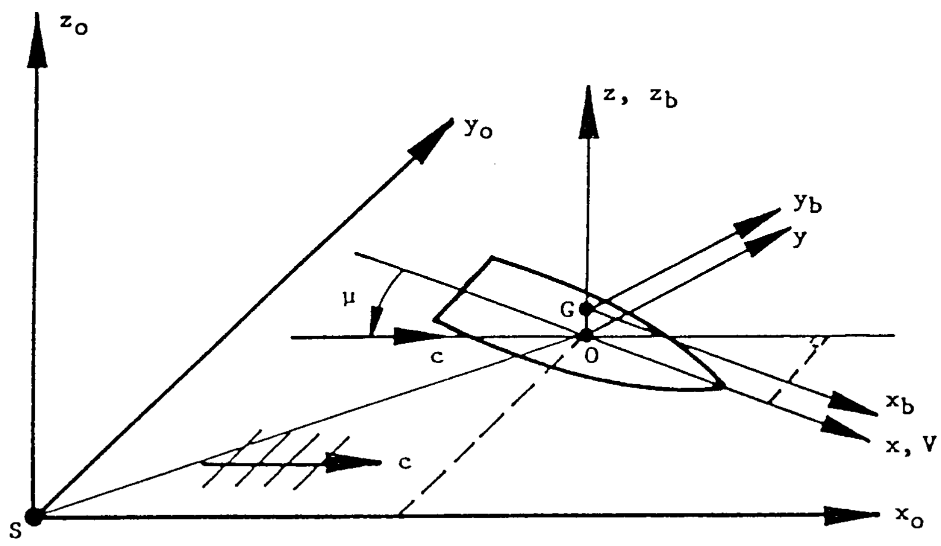

Figure 2.2: Coordinate System

The right-handed coordinate system $O(x, y, z)$ is moving with the ship's speed V, which yields:

$$x_0 = Vt \cos \mu + x \cos \mu + y \sin \mu \tag{2.4}$$

From the relation between the frequency of encounter ω_e and the wave frequency ω:

$$\omega_e = \omega - kV \cos \mu \tag{2.5}$$

follows:

$$\zeta = \zeta_a \cos(\omega_e t - kx \cos \mu - ky \sin \mu) \tag{2.6}$$

The resulting six ship motions in the $O(x, y, z)$ system are defined by three translations of the ship's centre of gravity in the direction of the x-, y- and z-axes and three rotations about them:

$$\begin{array}{rcl}
\text{surge} & : & x = x_a \cos(\omega_e t + \varepsilon_{x\zeta}) \\
\text{sway} & : & y = y_a \cos(\omega_e t + \varepsilon_{y\zeta}) \\
\text{heave} & : & z = z_a \cos(\omega_e t + \varepsilon_{z\zeta}) \\
\text{roll} & : & \phi = \phi_a \cos(\omega_e t + \varepsilon_{\phi\zeta}) \\
\text{pitch} & : & \theta = \theta_a \cos(\omega_e t + \varepsilon_{\theta\zeta}) \\
\text{yaw} & : & \psi = \psi_a \cos(\omega_e t + \varepsilon_{\psi\zeta})
\end{array} \tag{2.7}$$

The phase lags of these motions are related to the harmonic wave elevation at the origin of the $O(x, y, x)$ system, the average position of the ship's centre of gravity:

$$\text{wave:} \qquad \zeta = \zeta_a \cos(\omega_e t) \tag{2.8}$$

The harmonic velocities and accelerations in the $O(x, y, z)$ system are found by taking the derivatives of the displacements, for instance for surge:

$$\text{surge displacement} \quad : \qquad x = x_a \cos(\omega_e t + \varepsilon_{x\zeta})$$

$$\begin{array}{rcl} \text{surge velocity} & : & \dot{x} = -\omega_e x_a \sin(\omega_e t + \varepsilon_{x\zeta}) \\ \text{surge acceleration} & : & \ddot{x} = -\omega_e^2 x_a \cos(\omega_e t + \varepsilon_{x\zeta}) \end{array} \tag{2.9}$$

2.2 Incident Wave Potential

In order to use the linear potential theory with waves, it will be necessary to assume that the water surface slope is very small. This means that the wave steepness is so small that terms in the equations of motion of the waves with a magnitude in the order of the steepness-squared can be ignored.
Suppose a wave moving in the x-z plane. The profile of a simple wave with a small steepness looks like a sine or a cosine and the motion of a water particle in a wave depends on the distance below the still water level. This is reason why the wave potential is written as:

$$\Phi_w(x, z, t) = P(z) \cdot \sin(kx - \omega t) \tag{2.10}$$

in which $P(z)$ is an (as yet) unknown function of z.
The velocity potential $\Phi_w(x, z, t)$ of the harmonic waves has to fulfill four requirements:

1. Continuity condition or Laplace equation
2. Sea bed boundary condition
3. Free surface dynamic boundary condition
4. Free surface kinematic boundary condition.

These requirements lead to a more complete expression for the velocity potential as will be explained in the following sections. The relationships presented in these sections are valid for all water depths, but the fact that they contain so many hyperbolic functions makes them cumbersome to use. Engineers - as opposed to (some) scientists - often look for ways to simplify the theory. The simplifications stem from the following approximations for large and very small arguments, s, as shown in figure 2.3:

$$\begin{array}{lrcl} \text{for large arguments, } s & \sinh(s) & \approx & \cosh(s) \gg s \\ & \tanh(s) & \approx & 1 \end{array} \tag{2.11}$$

$$\begin{array}{lrcl} \text{for small arguments, } s & \sinh(s) & \approx & \tanh(s) \approx s \\ & \cosh(s) & \approx & 1 \end{array} \tag{2.12}$$

2.2.1 Continuity Condition

The velocity of the water particles (u, v, w) in the three translational directions, or alternatively (v_x, v_y, v_z), follow from the definition of the velocity potential, Φ_w:

$$u = v_x = \frac{\partial \Phi_w}{\partial x} \qquad v = v_y = \frac{\partial \Phi_w}{\partial y} \qquad w = v_z = \frac{\partial \Phi_w}{\partial z} \tag{2.13}$$

Since the fluid is homogeneous and incompressible, the continuity condition becomes:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{2.14}$$

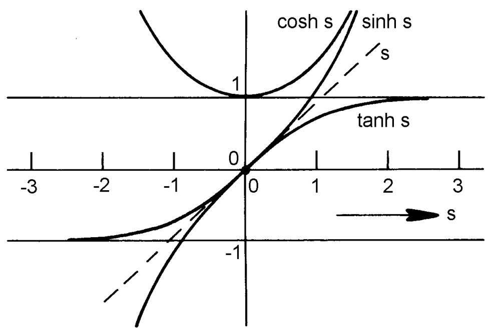

Figure 2.3: Hyperbolic Functions Limits

2.2.2 Laplace Equation

This continuity condition results in the Laplace equation for potential flows:

$$\nabla^2\Phi_w = \frac{\partial^2\Phi_w}{\partial x^2} + \frac{\partial^2\Phi_w}{\partial y^2} + \frac{\partial^2\Phi_w}{\partial z^2} = 0 \tag{2.15}$$

Water particles move here in the x-z plane only, so in the equations above:

$$v = \frac{\partial\Phi_w}{\partial y} = 0 \quad \text{and} \quad \frac{\partial v}{\partial y} = \frac{\partial^2\Phi_w}{\partial y^2} = 0 \tag{2.16}$$

Taking this into account, a substitution of equation 2.10 in equation 2.15 yields a homogeneous solution of this equation:

$$\frac{d^2P(z)}{dz^2} - k^2P(z) = 0 \tag{2.17}$$

with as solution for $P(z)$:

$$P(z) = C_1e^{+kz} + C_2e^{-kz} \tag{2.18}$$

Using this result from the first boundary condition, the wave potential can be written now with two unknown coefficients as:

$$\Phi_w(x,z,t) = \left(C_1e^{+kz} + C_2e^{-kz}\right) \cdot \sin\left(kx - \omega t\right) \tag{2.19}$$

in which:

$\Phi_w(x,z,t)$	=	wave potential (m^2/s)
e	=	base of natural logarithms (-)
C_1, C_2	=	as yet undetermined constants (m^2/s)
k	=	wave number (1/m)
t	=	time (s)
x	=	horizontal distance (m)
z	=	vertical distance, positive upwards (m)
ω	=	wave frequency (1/s)

2.2.3 Sea Bed Boundary Condition

The vertical velocity of water particles at the sea bed is zero (no-leak condition):

$$\frac{\partial \Phi_w}{\partial z} = 0 \qquad \text{for: } z = -h \tag{2.20}$$

Substituting this boundary condition in equation 2.19 provides:

$$kC_1 e^{-kh} - kC_2 e^{+kh} = 0 \tag{2.21}$$

By defining:

$$C = 2C_1 e^{-kh} = 2C_2 e^{+kh} \tag{2.22}$$

or:

$$C_1 = \frac{C}{2} e^{+kh} \qquad \text{and} \qquad C_2 = \frac{C}{2} e^{-kh} \tag{2.23}$$

it follows that $P(z)$ in equation 2.18 can be worked out to:

$$\begin{aligned} P(z) &= \frac{C}{2} \left(e^{+k(h+z)} + e^{-k(h+z)} \right) \\ &= C \cosh k \left(h + z \right) \end{aligned} \tag{2.24}$$

and the wave potential with only one unknown becomes:

$$\Phi_w(x, z, t) = C \cdot \cosh k \left(h + z \right) \cdot \sin \left(kx - \omega t \right) \tag{2.25}$$

in which C is an (as yet) unknown constant.

2.2.4 Free Surface Dynamic Boundary Condition

The pressure, p, at the free surface of the fluid, $z = \zeta$, is equal to the atmospheric pressure, p_0. This requirement for the pressure is called the dynamic boundary condition at the free surface.

The Bernoulli equation for an instationary irrotational flow (with the velocity given in terms of its three components) is in its general form:

$$\frac{\partial \Phi_w}{\partial t} + \frac{1}{2} \left(u^2 + v^2 + w^2 \right) + \frac{p}{\rho} + gz = 0 \tag{2.26}$$

In two dimensions, $v = 0$ and since the waves have a small steepness (u and w are small), this equation becomes:

$$\frac{\partial \Phi_w}{\partial t} + \frac{p}{\rho} + gz = 0 \tag{2.27}$$

At the free surface this condition becomes:

$$\frac{\partial \Phi_w}{\partial t} + \frac{p_0}{\rho} + g\zeta = 0 \qquad \text{for: } z = \zeta \tag{2.28}$$

The constant value p_0/ρ can be included in $\partial \Phi_w / \partial t$; this will not influence the velocities being obtained from the potential Φ_w. With this the equation becomes:

$$\frac{\partial \Phi_w}{\partial t} + g\zeta = 0 \qquad \text{for: } z = \zeta \tag{2.29}$$

The potential at the free surface can be expanded in a Taylor series, keeping in mind that the vertical displacement ζ is relatively small:

$$\begin{aligned} \left\{\Phi_w(x,z,t)\right\}_{z=\zeta} &= \left\{\Phi_w(x,z,t)\right\}_{z=0} + \zeta \cdot \left\{\frac{\partial \Phi_w(x,z,t)}{\partial z}\right\}_{z=0} + \ldots\ldots \\ \left\{\frac{\partial \Phi_w(x,z,t)}{\partial t}\right\}_{z=\zeta} &= \left\{\frac{\partial \Phi_w(x,z,t)}{\partial t}\right\}_{z=0} + O(\varepsilon^2) \end{aligned} \tag{2.30}$$

which yields for the linearized form of the free surface dynamic boundary condition:

$$\frac{\partial \Phi_w}{\partial t} + g\zeta = 0 \qquad \text{for: } z = 0 \tag{2.31}$$

With this, the wave profile becomes:

$$\zeta = -\frac{1}{g} \cdot \frac{\partial \Phi_w}{\partial t} \qquad \text{for: } z = 0 \tag{2.32}$$

A substitution of equation 2.10 in equation 2.32 yields the wave profile:

$$\zeta = \frac{\omega C}{g} \cdot \cosh kh \cdot \cos\left(kx - \omega t\right) \tag{2.33}$$

or:

$$\zeta = \zeta_a \cdot \cos\left(kx - \omega t\right) \quad \text{with:} \quad \zeta_a = \frac{\omega C}{g} \cdot \cosh kh \tag{2.34}$$

With this the corresponding wave potential, depending on the water depth h, is given by the relation:

$$\Phi_w = \frac{\zeta_a g}{\omega} \cdot \frac{\cosh k(h+z)}{\cosh kh} \cdot \sin(kx - \omega t) \tag{2.35}$$

or when ωt is the first of the sine function arguments, as generally will be used in ship motion equations:

$$\Phi_w = \frac{-\zeta_a g}{\omega} \cdot \frac{\cosh k(h+z)}{\cosh kh} \cdot \sin(\omega t - kx) \tag{2.36}$$

In deep water, the expression for the wave potential reduces to:

$$\Phi_w = \frac{-\zeta_a g}{\omega} \cdot e^{kz} \cdot \sin(\omega t - kx) \qquad \text{(deep water)} \tag{2.37}$$

2.2.5 Free Surface Kinematic Boundary Condition

So far the relation between the wave period T and the wave length, λ, is still unknown. The relation between T and λ (or equivalently ω and k) follows from the boundary condition that the vertical velocity of a water particle in the free surface of the fluid is identical to the vertical velocity of that free surface itself (no-leak condition); this is a kinematic boundary condition.
Using the equation of the free surface 2.34 yields:

$$\begin{aligned} \frac{dz}{dt} &= \frac{\partial \zeta}{\partial t} + \frac{\partial \zeta}{\partial x} \cdot \frac{dx}{dt} \qquad \text{for the wave surface: } z = \zeta \\ &= \frac{\partial \zeta}{\partial t} + u \cdot \frac{d\zeta}{dx} \end{aligned} \tag{2.38}$$

The second term in this expression is a product of two values, which are both small because of the assumed small wave steepness. This product becomes even smaller (second order) and can be ignored, see figure 2.4.

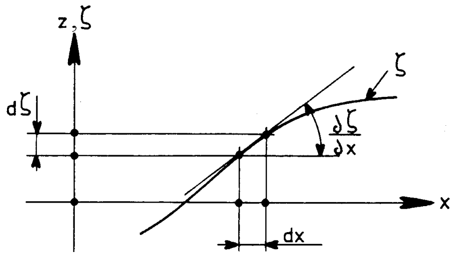

Figure 2.4: Kinematic Boundary Condition

This linearization provides the vertical velocity of the wave surface:

$$\frac{dz}{dt} = \frac{\partial \zeta}{\partial t} \qquad \text{for the wave surface: } z = \zeta \tag{2.39}$$

The vertical velocity of a water particle in the free surface is then:

$$\frac{\partial \Phi_w}{\partial z} = \frac{\partial \zeta}{\partial t} \qquad \text{for } z = 0 \tag{2.40}$$

Analogous to equation 2.31 this condition is valid for $z = 0$ too, instead of for $z = \zeta$ only. A differentiation of the free surface dynamic boundary condition (equation 2.31) with respect to t provides:

$$\frac{\partial^2 \Phi_w}{\partial t^2} + g \frac{\partial \zeta}{\partial t} = 0 \qquad \text{for } z = 0 \tag{2.41}$$

or after re-arranging terms:

$$\frac{\partial \zeta}{\partial t} + \frac{1}{g} \cdot \frac{\partial^2 \Phi_w}{\partial t^2} = 0 \qquad \text{for } z = 0 \tag{2.42}$$

Together with equation 2.39 this delivers the free surface kinematic boundary condition or the Cauchy-Poisson condition:

$$\frac{\partial z}{\partial t} + \frac{1}{g} \cdot \frac{\partial^2 \Phi_w}{\partial t^2} = 0 \qquad \text{for: } z = 0 \tag{2.43}$$

2.2.6 Dispersion Relationship

The information is now available to establish the relationship between ω and k (or equivalently T and λ) referred to above. A substitution of the expression for the wave potential (equation 2.35) in equation 2.43 gives the dispersion relation for any arbitrary water depth h:

$$\omega^2 = k\ g \cdot \tanh kh \tag{2.44}$$

In many situations, ω or T will be know; one must determine k or λ. Since k appears in a nonlinear way in 2.44, that equation will generally have to be solved iteratively.
In deep water ($\tanh kh = 1$), equation 2.44 degenerates to a quite simple form which can be used without difficulty:

$$\omega^2 = k\, g \qquad \text{(deep water)} \tag{2.45}$$

When calculating the hydromechanical forces and the wave exciting forces on a ship, it is assumed that $x \approx x_b$, $y \approx y_b$ and $z \approx z_b$. In case of forward ship speed, the wave frequency ω has to be replaced by the frequency of encounter of the waves ω_e. This leads to the following expressions for the wave surface and the first order wave potential in the $G(x_b, y_b, z_b)$ system:

$$\zeta = \zeta_a \cos(\omega_e t - k x_b \cos\mu - k y_b \sin\mu) \tag{2.46}$$

and the expression for the velocity potential of the regular waves, Φ_w, becomes:

$$\Phi_w = \frac{-\zeta_a g}{\omega} \cdot \frac{\cosh k(h + z_b)}{\cosh(kh)} \sin(\omega_e t - k x_b \cos\mu - k y_b \sin\mu) \tag{2.47}$$

2.2.7 Relationships in Regular Waves

Figure 2.5 shows the relation between λ, T, c and h for a wide variety of conditions. Notice the boundaries $\lambda/h \approx 2$ and $\lambda/h \approx 20$ in this figure between short (deep water) and long (shallow water) waves.

2.3 Floating Rigid Body in Waves

Consider a rigid body, floating in an ideal fluid with harmonic waves. The water depth is assumed to be finite. The time-averaged speed of the body is zero in all directions. For the sake of simple notation, it is assumed here that the $O(x, y, z)$ system is identical to the $S(x_0, y_0, z_0)$ system. The x-axis is coincident with the undisturbed still water free surface and the z-axis and z_0-axis are positive upwards.
The linear fluid velocity potential can be split into three parts:

$$\Phi(x, y, z, t) = \Phi_r + \Phi_w + \Phi_d \tag{2.48}$$

in which:

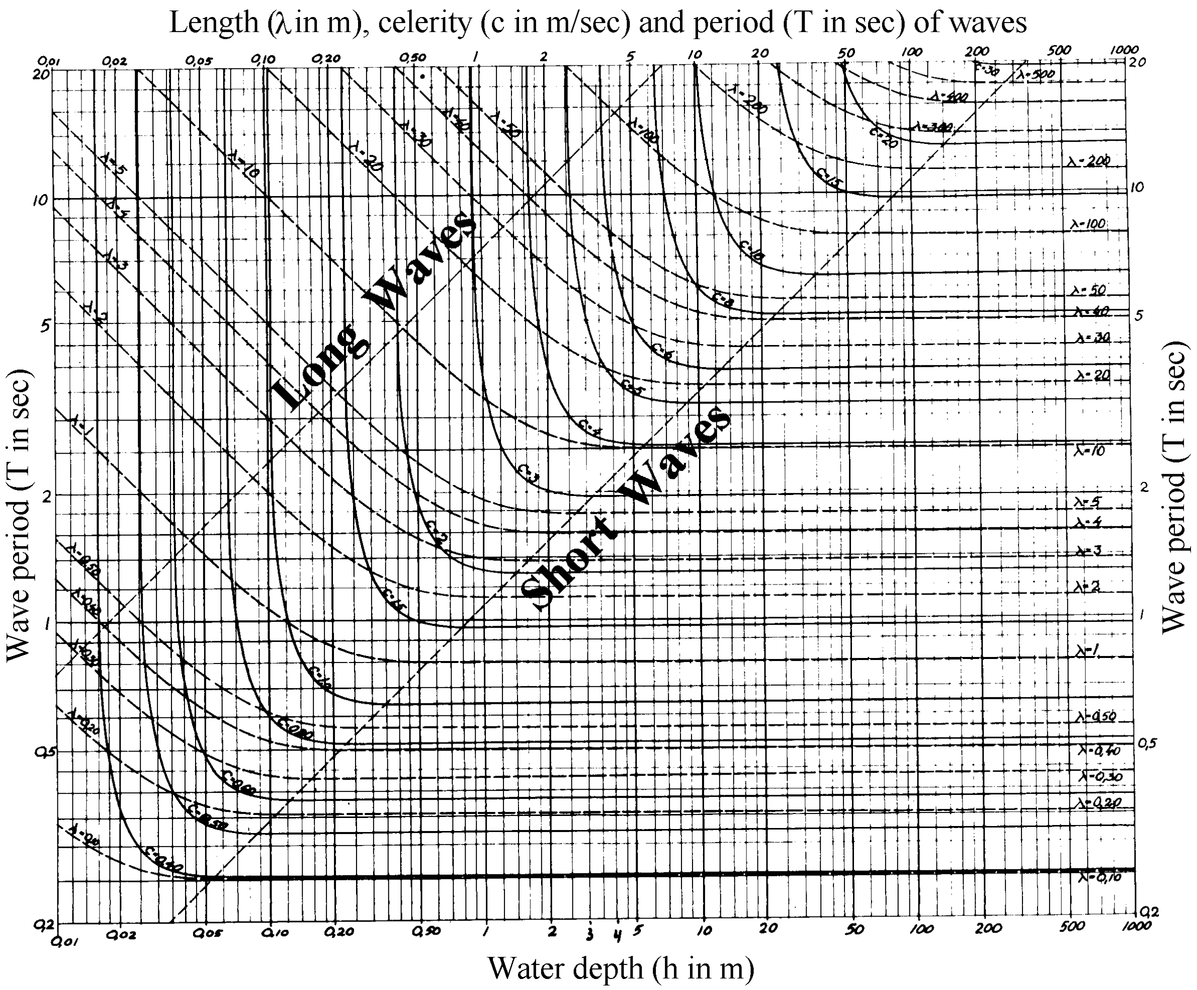

Figure 2.5: Relationships between λ, T, c and h

Φ_r = the radiation potential for the oscillatory motion of the body in still water
Φ_w = the incident undisturbed wave potential
Φ_d = the diffraction potential of the waves about the restrained body

2.3.1 Fluid Requirements

From the definition of a velocity potential Φ follows the velocity of the water particles in the three translational directions:

$$v_x = \frac{\partial \Phi}{\partial x} \qquad v_y = \frac{\partial \Phi}{\partial y} \qquad v_z = \frac{\partial \Phi}{\partial z} \tag{2.49}$$

The velocity potentials, $\Phi = \Phi_r + \Phi_w + \Phi_d$, have to fulfill a number of requirements and boundary conditions in the fluid. Of these, the first three are identical to those in the incident undisturbed waves. Additional boundary conditions are associated with the oscillating floating body.

1. **Continuity Condition or Laplace Equation**

As the fluid is homogeneous and incompressible, the continuity condition:

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = 0 \tag{2.50}$$

results into the equation of Laplace:

$$\nabla^2 \Phi = \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2} = 0 \tag{2.51}$$

2. **Sea Bed Boundary Condition**
The boundary condition on the sea bed, following from the definition of the velocity potential, is given by:

$$\frac{\partial \Phi}{\partial z} = 0 \qquad \text{for: } z = -h \tag{2.52}$$

3. **Boundary Condition at the Free Surface**
The pressure in a point $P(x, y, z)$ is given by the linearized Bernoulli equation:

$$p = -\rho \frac{\partial \Phi}{\partial t} - \rho g z \qquad \text{or:} \qquad \frac{\partial \Phi}{\partial t} + g\zeta = \frac{-p}{\rho} \tag{2.53}$$

At the free surface of the fluid, so for $z = \zeta(x, y, z, t)$, the pressure p is constant. Because of the linearization, the vertical velocity of a water particle in the free surface becomes:

$$\frac{dz}{dt} = \frac{\partial \Phi}{\partial z} \approx \frac{\partial \zeta}{\partial t} \tag{2.54}$$

Combining these two conditions provides the boundary condition at the free surface:

$$\frac{\partial^2 \Phi}{\partial t^2} + g \frac{\partial \Phi}{\partial z} = 0 \qquad \text{for: } z = 0 \tag{2.55}$$

4. **Kinematic Boundary Condition on the Oscillating Body Surface**
The boundary condition at the surface of the rigid body plays a very important role. The velocity of a water particle at a point at the surface of the body is equal to the velocity of this (watertight) body point itself. The outward normal velocity, v_n, at a point $P(x, y, z)$ at the surface of the body (positive in the direction of the fluid) is given by:

$$\frac{\partial \Phi}{\partial n} = v_n(x, y, z, t) \tag{2.56}$$

Because the solution is linearized, this can be written as:

$$\frac{\partial \Phi}{\partial n} = v_n(x, y, z, t) = \sum_{j=1}^{6} v_j \cdot f_j \tag{2.57}$$

in terms of oscillatory velocities, v_j, and generalized direction-cosines, f_j, on the surface of the body, S, given by:

$$\begin{aligned}
f_1 &= \cos(n, x) \\
f_2 &= \cos(n, y) \\
f_3 &= \cos(n, z) \\
f_4 &= y \cos(n, z) - z \cos(n, y) = y \cdot f_3 - z \cdot f_2 \\
f_5 &= z \cos(n, x) - x \cos(n, z) = z \cdot f1 - x \cdot f_3 \\
f_6 &= x \cos(n, y) - y \cos(n, x) = x \cdot f_2 - y \cdot f_1
\end{aligned} \tag{2.58}$$

The direction cosines are called generalized, because f_1, f_2 and f_3 have been normalized (the sum of their squares is equal to 1) and used to obtain f_4, f_5 and f_6.
Note: The subscripts $1, 2, ...6$ are used here to indicate the mode of the motion. Also displacements are often indicated in literature in the same way: $x_1, x_2, ...x_6$.

5. **Radiation Condition**
The radiation condition states that when the distance R of a water particle to the oscillating body tends to infinity, the potential value tends to zero:

$$\lim_{R\to\infty} \Phi = 0 \tag{2.59}$$

6. **Symmetric or Anti-symmetric Condition**
Since ships and many other floating bodies are symmetric with respect to its middle line plane, one can make use of this to simplify the potential equations:

$$\begin{aligned}
\Phi^{(2)}(-x,y) &= -\Phi^{(2)}(+x,y) && \text{for sway} \\
\Phi^{(3)}(-x,y) &= +\Phi^{(3)}(+x,y) && \text{for heave} \\
\Phi^{(4)}(-x,y) &= -\Phi^{(4)}(+x,y) && \text{for roll}
\end{aligned} \tag{2.60}$$

in which $\Phi^{(i)}$ is the velocity potential for the given direction i.
This indicates that for sway and roll oscillations, the horizontal velocities of the water particles, thus the derivative $\partial\Phi/\partial x$, at any time on both sides of the body must have the same direction; these motions are anti-symmetric. For heave oscillations these velocities must be of opposite sign; this is a symmetric motion. However, for all three modes of oscillations the vertical velocities, thus the derivative $\partial\Phi/\partial y$, on both sides must have the same directions at any time.

2.3.2 Forces and Moments

The forces $\vec{F}$ and moments $\vec{M}$ follow from an integration of the pressure, p, over the submerged surface, S, of the body:

$$\begin{aligned}
\vec{F} &= -\iint_S (p \cdot \vec{n}) \cdot dS \\
\vec{M} &= -\iint_S p \cdot (\vec{r} \times \vec{n}) \cdot dS
\end{aligned} \tag{2.61}$$

in which $\vec{n}$ is the outward normal vector on surface dS and $\vec{r}$ is the position vector of surface dS in the $O(x, y, z)$ coordinate system.
The pressure p - via the linearized Bernoulli equation - is determined from the velocity potentials by:

$$\begin{aligned}
p &= -\rho\frac{\partial\Phi}{\partial t} - \rho g z \\
&= -\rho\left(\frac{\partial\Phi_r}{\partial t} + \frac{\partial\Phi_w}{\partial t} + \frac{\partial\Phi_d}{\partial t}\right) - \rho g z
\end{aligned} \tag{2.62}$$

which can obviously be split into four separate parts, so that the hydromechanical forces $\vec{F}$ and moments $\vec{M}$ can be split into four parts too:

$$\begin{aligned}\vec{F} &= \rho \iint_S \left(\frac{\partial \Phi_r}{\partial t} + \frac{\partial \Phi_w}{\partial t} + \frac{\partial \Phi_d}{\partial t} + gz \right) \vec{n} \cdot dS \\ \vec{M} &= \rho \iint_S \left(\frac{\partial \Phi_r}{\partial t} + \frac{\partial \Phi_w}{\partial t} + \frac{\partial \Phi_d}{\partial t} + gz \right) (\vec{r} \times \vec{n}) \cdot dS \end{aligned} \tag{2.63}$$

or:

$$\begin{aligned}\vec{F} &= \vec{F}_r + \vec{F}_w + \vec{F}_d + \vec{F}_s \\ \vec{M} &= \vec{M}_r + \vec{M}_w + \vec{M}_d + \vec{M}_s \end{aligned} \tag{2.64}$$

2.3.3 Hydrodynamic Loads

The hydrodynamic loads are the dynamic forces and moments caused by the fluid on an oscillating body in still water; waves are radiated from the body. The **radiation potential**, Φ_r, which is associated with this oscillation in still water, can be written in terms, Φ_j, for 6 degrees of freedom as:

$$\begin{aligned}\Phi_r(x,y,z,t) &= \sum_{j=1}^{6} \Phi_j(x,y,z,t) \\ &= \sum_{j=1}^{6} \phi_j(x,y,z) \cdot v_j(t) \end{aligned} \tag{2.65}$$

in which the space and time dependent potential term, $\Phi_j(x,y,z,t)$ in direction j, is now written in terms of a separate space dependent potential, $\phi_j(x,y,z)$ in direction j, multiplied by an oscillatory velocity, $v_j(t)$ in direction j.
This allows the normal velocity on the surface of the body to be written as:

$$\begin{aligned}\frac{\partial \Phi_r}{\partial n} &= \frac{\partial}{\partial n} \sum_{j=1}^{6} \Phi_j \\ &= \sum_{j=1}^{6} \left\{ \frac{\partial \phi_j}{\partial n} \cdot v_j \right\} \end{aligned} \tag{2.66}$$

and the generalized direction cosines are given by:

$$f_j = \frac{\partial \phi_j}{\partial n} \tag{2.67}$$

With this the radiation terms in the hydrodynamic force becomes:

$$\begin{aligned}\vec{F} &= \rho \int\int_S \left(\frac{\partial \Phi_r}{\partial t} \right) \vec{n} \cdot dS \\ &= \rho \int\int_S \left(\frac{\partial}{\partial t} \sum_{j=1}^{6} \phi_j v_j \right) \vec{n} \cdot dS \end{aligned} \tag{2.68}$$

and the moment term:

$$\begin{aligned}\vec{M} &= \rho \iint_S \left(\frac{\partial \Phi_r}{\partial t} \right) (\vec{r} \times \vec{n}) dS \\ &= \rho \iint_S \left(\frac{\partial}{\partial t} \sum_{j=1}^{6} \phi_j v_j \right) (\vec{r} \times \vec{n}) \cdot dS \end{aligned} \tag{2.69}$$

The components of these radiation forces and moments are defined by:

$$\vec{F}_r = (X_{r_1}, X_{r_2}, X_{r_3}) \qquad \text{and} \qquad \vec{M}_r = (X_{r_4}, X_{r_5}, X_{r_6}) \tag{2.70}$$

with:

$$\begin{aligned}\vec{X}_{r_k} &= \rho \iint_S \left(\frac{\partial}{\partial t} \sum_{j=1}^{6} \phi_j v_j \right) f_k \cdot dS \\ &= \rho \iint_S \left(\frac{\partial}{\partial t} \sum_{j=1}^{6} \phi_j v_j \right) \frac{\partial \phi_k}{\partial n} \cdot dS \qquad \text{for: } k = 1, ...6 \end{aligned} \tag{2.71}$$

Since ϕ_j and ϕ_k are not time-dependent in this expression, it reduces to:

$$X_{r_k} = \sum_{j=1}^{6} X_{r_{kj}} \qquad \text{for: } k = 1, ...6 \tag{2.72}$$

with:

$$X_{r_{kj}} = \frac{dv_j}{dt} \rho \iint_S \phi_j \frac{\partial \phi_k}{\partial n} \cdot dS \tag{2.73}$$

This radiation force or moment $X_{r_{kj}}$ in the direction k is caused by a forced harmonic oscillation of the body in the direction j. This is generally true for all j and k in the range from 1 to 6. When $j = k$, the force or moment is caused by a motion in that same direction. When $j \neq k$, the force in one direction results from the motion in another direction. This introduces what is called **coupling** between the forces and moments (or motions).
The above equation expresses the force and moment components, $X_{r_{kj}}$ in terms of still unknown potentials, ϕ_j; not everything is solved yet! A solution for this will be found later in this chapter.

Oscillatory Motion

Now an oscillatory motion is defined; suppose a motion (in a complex notation) given by:

$$s_j = s_{a_j} e^{-i\omega t} \tag{2.74}$$

Then the velocity and acceleration of this oscillation are:

$$\begin{aligned}\dot{s}_j &= v_j &&= -i\omega s_{a_j} e^{-i\omega t} \\ \ddot{s}_j &= \frac{dv_j}{dt} &&= -i\omega^2 s_{a_j} e^{-i\omega t}\end{aligned} \tag{2.75}$$

The hydrodynamic forces and moments can be split into a load in-phase with the acceleration and a load in-phase with the velocity:

$$\begin{aligned} X_{r_{kj}} &= -M_{kj}\ddot{s}_j - N_{kj}\dot{s}_j \\ &= \left(s_{a_j}\omega^2 M_{kj} + i s_{a_j}\omega N_{kj}\right) e^{-i\omega t} \\ &= \left((-s_{a_j}\omega^2 \rho \iint\limits_S \phi_j \frac{\partial \phi_k}{\partial n} dS) \right) e^{-i\omega t} \end{aligned} \tag{2.76}$$

So in case of an oscillation of the body in the direction j with a velocity potential ϕ_j, the hydrodynamic mass and damping (coupling) coefficients are defined by:

$$M_{kj} = -\Re\left\{\rho \iint\limits_S \phi_j \frac{\partial \phi_k}{\partial n} \cdot dS\right\} \qquad \text{and} \qquad N_{kj} = -\Im\left\{\rho\omega \iint\limits_S \phi_j \frac{\partial \phi_k}{\partial n} \cdot dS\right\} \tag{2.77}$$

In case of an oscillation of the body in the direction k with a velocity potential ϕ_k, the hydrodynamic mass and damping (coupling) coefficients are defined by:

$$M_{jk} = -\Re\left\{\rho \iint\limits_S \phi_k \frac{\partial \phi_j}{\partial n} \cdot dS\right\} \qquad \text{and} \qquad N_{jk} = -\Im\left\{\rho\omega \iint\limits_S \phi_k \frac{\partial \phi_j}{\partial n} \cdot dS\right\} \tag{2.78}$$

Green's Second Theorem

Green's second theorem transforms a large volume-integral into a much easier to handle surface-integral. Its mathematical background is beyond the scope of this text. It is valid for any potential function, regardless the fact if it fulfills the Laplace condition or not.
Consider two separate velocity potentials ϕ_j and ϕ_k. Green's second theorem, applied to these potentials, is then:

$$\iiint\limits_{V^*} \left(\phi_j \cdot \nabla^2\phi_k - \phi_k \cdot \nabla^2\phi_j\right) \cdot dV^* = \iint\limits_{S^*} \left(\phi_j \frac{\partial \phi_k}{\partial n} - \phi_k \frac{\partial \phi_j}{\partial n}\right) \cdot dS^* \tag{2.79}$$

This theorem is generally valid for all kinds of potentials; it is not necessary that they fullfil the Laplace equation.
In Green's theorem, S^* is a closed surface with a volume V^*. This volume is bounded by the wall of an imaginary vertical circular cylinder with a very large radius R, the sea bottom at $z = -h$, the water surface at $z = \zeta$ and the wetted surface of the floating body, S; see figure 2.6.
Both of the above radiation potentials ϕ_j and ϕ_k must fulfill the Laplace equation ($\nabla^2\phi_j = \nabla^2\phi_k = 0$). So the left hand side of the above equation becomes zero which yields for the right hand side of this equation:

$$\iint\limits_{S^*} \phi_j \frac{\partial \phi_k}{\partial n} \cdot dS^* = \iint\limits_{S^*} \phi_k \frac{\partial \phi_j}{\partial n} \cdot dS^* \tag{2.80}$$

The boundary condition at the free surface becomes for $\Phi = \phi \cdot e^{-i\omega t}$:

$$-\omega^2\phi + g\frac{\partial \phi}{\partial z} = 0 \qquad \text{for: } z = 0 \tag{2.81}$$

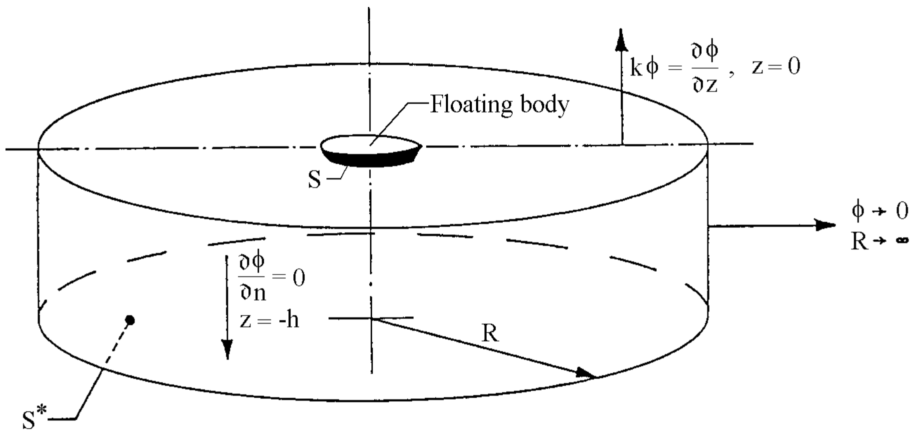

Figure 2.6: Boundary Conditions

or with the dispersion relation, $\omega^2/g = k \tanh kh$:

$$k \tanh kh \cdot \phi = \frac{\partial \phi}{\partial z} \qquad \text{for: } z = 0 \tag{2.82}$$

This implies that at the free surface of the fluid one can write:

$$\left.\begin{array}{lll} k \tanh kh \cdot \phi_k = \frac{\partial \phi_k}{\partial z} = \frac{\partial \phi_k}{\partial n} & \longrightarrow & \phi_k = \frac{1}{k \tanh kh} \cdot \frac{\partial \phi_k}{\partial n} \\ k \tanh kh \cdot \phi_j = \frac{\partial \phi_j}{\partial z} = \frac{\partial \phi_j}{\partial n} & \longrightarrow & \phi_j = \frac{1}{k \tanh kh} \cdot \frac{\partial \phi_j}{\partial n} \end{array}\right\} \text{ at the free surface}$$

When taking also the boundary condition at the sea bed and the radiation condition on the wall of the cylinder in figure 2.6:

$$\frac{\partial \phi}{\partial n} = 0 \text{ (for: } z = -h) \quad \text{and} \quad \lim_{R \to \infty} \phi = 0 \tag{2.83}$$

into account, the integral equation over the surface S^* reduces to:

$$\iint_S \phi_j \frac{\partial \phi_k}{\partial n} \cdot dS = \iint_S \phi_k \frac{\partial \phi_j}{\partial n} \cdot dS \tag{2.84}$$

in which S is the wetted surface of the body only.
Note that the ϕ_j and ϕ_k still have to be evaluated.

Potential Coefficients

The previous section provides - for the zero forward ship speed case - symmetry in the coefficients matrices with respect to their diagonals so that:

$$M_{jk} = M_{kj} \qquad \text{and} \qquad N_{jk} = N_{kj} \tag{2.85}$$

Because of the symmetry of a ship some coefficients are zero and the two matrices with hydrodynamic coefficients for a ship become:

$$\text{Hydrodynamic mass matrix:} \quad \begin{pmatrix} M_{11} & 0 & M_{13} & 0 & M_{15} & 0 \\ 0 & M_{22} & 0 & M_{24} & 0 & M_{26} \\ M_{31} & 0 & M_{33} & 0 & M_{35} & 0 \\ 0 & M_{42} & 0 & M_{44} & 0 & M_{46} \\ M_{51} & 0 & M_{53} & 0 & M_{55} & 0 \\ 0 & M_{62} & 0 & M_{64} & 0 & M_{66} \end{pmatrix} \tag{2.86}$$

$$\text{Hydrodynamic damping matrix:} \quad \begin{pmatrix} N_{11} & 0 & N_{13} & 0 & N_{15} & 0 \\ 0 & N_{22} & 0 & N_{24} & 0 & N_{26} \\ N_{31} & 0 & N_{33} & 0 & N_{35} & 0 \\ 0 & N_{42} & 0 & N_{44} & 0 & N_{46} \\ N_{51} & 0 & N_{53} & 0 & N_{55} & 0 \\ 0 & N_{62} & 0 & N_{64} & 0 & N_{66} \end{pmatrix} \tag{2.87}$$

For clarity, the symmetry of terms about the diagonal in these matrices (for example that $M_{13} = M_{31}$ for zero forward speed) has not been included here. The terms on the diagonals (such as M_{nn} for example) are the primary coefficients relating properties such as hydrodynamic mass in one direction to the inertia forces in that same direction. Off-diagonal terms (such as M_{13}) represent hydrodynamic mass only which is associated with an inertia dependent force in one direction caused by a motion component in another.

Forward speed has an effect on the velocity potentials itself, but is not discussed here. This effect is quite completely explained by [Timman and Newman, 1962].

2.3.4 Wave and Diffraction Loads

The wave and diffraction terms in the hydrodynamic force and moment are:

$$\vec{F}_w + \vec{F}_d = \rho \iint_S \left(\frac{\partial \Phi_w}{\partial t} + \frac{\partial \Phi_d}{\partial t} \right) \vec{n} dS \tag{2.88}$$

and:

$$\vec{M}_w + \vec{M}_d = \rho \iint_S \left(\frac{\partial \Phi_w}{\partial t} + \frac{\partial \Phi_d}{\partial t} \right) (\vec{r} \times \vec{n}) dS \tag{2.89}$$

The principles of linear superposition allow the determination of these forces on a restrained body with zero forward speed; $\partial \Phi / \partial n = 0$. This simplifies the boundary condition on the surface of the body to:

$$\frac{\partial \Phi}{\partial n} = \frac{\partial \Phi_w}{\partial n} + \frac{\partial \Phi_d}{\partial n} = 0 \tag{2.90}$$

The space and time dependent potentials, $\Phi_w(x, y, z, t)$ and $\Phi_d(x, y, z, t)$, are written now in terms of isolated space dependent potentials, $\phi_w(x, y, z)$ and $\phi_d(x, y, z)$, multiplied by a normalized oscillatory velocity, $v(t) = 1 \cdot e^{-i\omega t}$:

$$\begin{aligned} \Phi_w(x, y, z, t) &= \phi_w(x, y, z) \cdot e^{-i\omega t} \\ \Phi_d(x, y, z, t) &= \phi_d(x, y, z) \cdot e^{-i\omega t} \end{aligned} \tag{2.91}$$

This results into:

$$\frac{\partial \phi_w}{\partial n} = -\frac{\partial \phi_d}{\partial n} \tag{2.92}$$

With this and the expressions for the generalized direction-cosines it is found for the wave forces and moments on the restrained body in waves:

$$\begin{aligned} X_{w_k} &= -i\rho e^{-i\omega t} \iint_S (\phi_w + \phi_d) f_k dS \\ &= -i\rho e^{-i\omega t} \iint_S (\phi_w + \phi_d) \frac{\partial \phi_k}{\partial n} dS \qquad \text{for: } k = 1, ...6 \end{aligned} \tag{2.93}$$

in which ϕ_k is the radiation potential.

The potential of the incident waves, ϕ_w, is known, but the diffraction potential, ϕ_d, has to be determined. Green's second theorem provides a relation between this diffraction potential, ϕ_d, and a radiation potential, ϕ_k:

$$\iint_S \phi_d \frac{\partial \Phi_k}{\partial n} dS = \iint_S \phi_k \frac{\partial \phi_d}{\partial n} dS \tag{2.94}$$

and with $\partial \phi_w / \partial n = -\partial \phi_d / \partial n$ one finds:

$$\iint_S \phi_d \frac{\partial \Phi_k}{\partial n} dS = -\iint_S \phi_k \frac{\partial \phi_w}{\partial n} dS \tag{2.95}$$

This elimination of the diffraction potential results into the so-called Haskind relations:

$$X_{w_k} = -i\rho e^{-i\omega t} \iint_S \left(\phi_w \frac{\partial \phi_k}{\partial n} + \phi_k \frac{\partial \phi_w}{\partial n} \right) dS \qquad \text{for: } k = 1, ...6 \tag{2.96}$$

This limiters the problem of the diffraction potential because the expression for X_{w_k} depends only on the wave potential ϕ_w and the radiation potential ϕ_k.

These relations, found by [Haskind, 1957], are very important; they underlie the relative motion (displacement - velocity - acceleration) hypothesis, as used in strip theory. These relations are valid only for a floating body with a zero time-averaged speed in all directions. [Newman, 1962] however, has generalized the Haskind relations for a body with a constant forward speed. He derived equations which differ only slightly from those found by Haskind. According to Newman's approach the wave potential has to be defined in the moving $O(x, y, z)$ system. The radiation potential has to be determined for the constant forward speed case, taking an opposite sign into account.

The corresponding wave potential for deep water, as given in the previous section, now becomes:

$$\begin{aligned} \Phi_w &= \frac{-\zeta_a g}{\omega} \cdot e^{kz} \cdot \sin(\omega t - kx \cos \mu - ky \sin \mu) \\ &= \frac{-i\zeta_a g}{\omega} \cdot e^{kz} \cdot e^{ik(x \cos \mu + y \sin \mu)} e^{-i\omega t} \end{aligned} \tag{2.97}$$

so that the isolated space dependent term is given by:

$$\phi_w = \frac{-i\zeta_a g}{\omega} \cdot e^{kz} \cdot e^{ik(x \cos \mu + y \sin \mu)} \tag{2.98}$$

In these equations is μ the wave direction.
The velocity of the water particles in the direction of the outward normal n on the surface of the body is:

$$\begin{aligned} \frac{\partial \phi_w}{\partial n} &= \phi_w k \left\{ \frac{\partial z}{\partial n} + i \left(\frac{\partial x}{\partial n} \cos \mu + \frac{\partial y}{\partial n} \sin \mu \right) \right\} \\ &= \phi_w k \left\{ f_3 + i(f_1 \cos \mu + f_2 \sin \mu) \right\} \end{aligned} \tag{2.99}$$

With this, the wave loads are given by:

$$\begin{aligned} X_{w_k} = - & \quad i\rho e^{-i\omega t} \int_S \int \phi_w f_k dS \\ + & \quad i\rho e^{-i\omega t} k \int_S \int \phi_w \phi_k \left\{ f_3 + i(f_1 \cos \mu + f_2 \sin \mu) \right\} dS \qquad \text{for:} k = 1, ..6 \end{aligned} \tag{2.100}$$

The first term in this expression for the wave forces and moments is the so-called Froude-Krilov force or moment, which is the wave load caused by the undisturbed incident wave. The second term is caused by the disturbance caused by the presence of the (fixed) body.

2.3.5 Hydrostatic Loads

In the notations used here, the buoyancy forces and moments are:

$$\vec{F}_s = \rho g \iint_S z \vec{n} \cdot dS \quad \text{and} \quad \vec{M}_s = \rho g \iint_S z \left(\vec{r} \text{ x } \vec{n} \right) \cdot dS \tag{2.101}$$

or more generally:

$$X_{s_k} = \rho g \iint_S z f_k \cdot dS \qquad \text{for: } k = 1, ...6 \tag{2.102}$$

in which the X_{s_k} are the components of these hydrostatic forces and moments.

2.4 Equations of Motion

The total mass as well as its distribution over the body is considered to be constant with time. For ships and other floating structures, this assumption is normally valid during a time which is large relative to the period of the motions. This holds that small effects - such as for instance a decreasing mass due to fuel consumption - can be ignored.
The solid mass matrix of a floating structure is given below.

$$\text{Solid mass matrix: } m = \begin{pmatrix} \rho\nabla & 0 & 0 & 0 & 0 & 0 \\ 0 & \rho\nabla & 0 & 0 & 0 & 0 \\ 0 & 0 & \rho\nabla & 0 & 0 & 0 \\ 0 & 0 & 0 & I_{xx} & 0 & -I_{xz} \\ 0 & 0 & 0 & 0 & I_{yy} & 0 \\ 0 & 0 & 0 & -I_{zx} & 0 & I_{zz} \end{pmatrix} \tag{2.103}$$

The moments of inertia here are often expressed in terms of the radii of inertia and the solid mass of the structure. Since Archimedes law is valid for a floating structure:

$$\begin{aligned} I_{xx} &= {k_{xx}}^2 \cdot \rho\nabla \\ I_{yy} &= {k_{yy}}^2 \cdot \rho\nabla \\ I_{zz} &= {k_{zz}}^2 \cdot \rho\nabla \end{aligned} \tag{2.104}$$

When the distribution of the solid mass of a ship is unknown, the radii of inertia can be approximated by:

$$\text{for ships:} \left\{ \begin{array}{lcl} k_{xx} & \approx & 0.30 \cdot B \text{ to } 0.40 \cdot B \\ k_{yy} & \approx & 0.22 \cdot L \text{ to } 0.28 \cdot L \\ k_{zz} & \approx & 0.22 \cdot L \text{ to } 0.28 \cdot L \end{array} \right. \tag{2.105}$$

in which L is the length and B is the breadth of the ship. Often, the (generally small) coupling terms, $I_{xz} = I_{zx}$, are simply neglected.
Bureau Veritas proposes for the radius of inertia for roll of the ship's solid mass:

$$k_{xx} = 0.289 \cdot B \cdot \left(1.0 + \left(\frac{2 \cdot \overline{KG}}{B} \right)^2 \right) \tag{2.106}$$

in which $\overline{KG}$ is the height of the center of gravity, G, above the keel.
For many ships without cargo on board (ballast condition), the mass is concentrated at the ends (engine room aft and ballast water forward to avoid a large trim), while for ships with cargo on board (full load condition) the - more or less amidships laden - cargo plays an important role. Thus, the radii of inertia, k_{yy} and k_{zz}, are usually smaller in the full load condition than in the ballast condition for normal ships. Note that the longitudinal radius of gyration of a long homogeneous rectangular beam with a length L is equal to about $0.29 \cdot L \left(= \sqrt{1/12} \cdot L \right)$.
The equations of motions of a rigid body in a space fixed coordinate system follow from Newton's second law. The vector equations for the translations of and the rotations about the center of gravity are given respectively by:

$$\vec{F} = \frac{d}{dt}\left(m\vec{U}\right) \qquad \text{and} \qquad \vec{M} = \frac{d}{dt}\left(\vec{H}\right) \tag{2.107}$$

in which:

$\vec{F}$	=	resulting external force acting in the center of gravity
m	=	mass of the rigid body
$\vec{U}$	=	instantaneous velocity of the center of gravity
$\vec{M}$	=	resulting external moment acting about the center of gravity
$\vec{H}$	=	instantaneous angular momentum about the center of gravity
t	=	time

Two important assumptions are made for the loads in the right hand side of these equations:

a. The so-called **hydromechanical forces and moments** are induced by the harmonic oscillations of the rigid body, moving in the undisturbed surface of the fluid.

b. The so-called **wave exciting forces and moments** are produced by waves coming in on the restrained body.

Since the system is linear, these loads are added to obtain the total loads. Thus, after assuming small motions, symmetry of the body and that the x-, y- and z-axes are principal axes, one can write the following six motion equations for the ship:

$$\begin{array}{rlll}
\text{Surge:} & \frac{d}{dt}\left(\rho\nabla\cdot\dot{x}\right) & =\rho\nabla\cdot\ddot{x} & =X_{h_1}+X_{w_1} \\
\text{Sway:} & \frac{d}{dt}\left(\rho\nabla\cdot\dot{y}\right) & =\rho\nabla\cdot\ddot{y} & =X_{h_2}+X_{w_2} \\
\text{Heave:} & \frac{d}{dt}\left(\rho\nabla\cdot\dot{z}\right) & =\rho\nabla\cdot\ddot{z} & =X_{h_3}+X_{w_3} \\
\text{Roll:} & \frac{d}{dt}\left(I_{xx}\cdot\dot{\phi}-I_{xz}\cdot\dot{\psi}\right) & =I_{xx}\cdot\ddot{\phi}-I_{xz}\cdot\ddot{\psi} & =X_{h_4}+X_{w_4} \\
\text{Pitch:} & \frac{d}{dt}\left(I_{yy}\cdot\dot{\theta}\right) & =I_{xx}\cdot\ddot{\theta} & =X_{h_5}+X_{w_5} \\
\text{Yaw:} & \frac{d}{dt}\left(I_{zz}\cdot\dot{\psi}-I_{zx}\cdot\dot{\phi}\right) & =I_{zz}\cdot\ddot{\psi}-I_{zx}\cdot\ddot{\phi} & =X_{h_6}+X_{w_6}
\end{array} \tag{2.108}$$

in which:

ρ	=	density of water
∇	=	volume of displacement of the ship
I_{ij}	=	solid mass moment of inertia of the ship
X_{h_1}, X_{h_2}, X_{h_3}	=	hydromechanical forces in the x-, y- and z-directions respectively
X_{h_4}, X_{h_5}, X_{h_6}	=	hydromechanical moments about the x-, y- and z-axes respectively
X_{w_1}, X_{w_2}, X_{w_3}	=	exciting wave forces in the x-, y- and z-directions respectively
X_{w_4}, X_{w_5}, X_{w_6}	=	exciting wave moments about the x-, y- and z-axes respectively

Generally, a ship has a vertical-longitudinal plane of symmetry, so that its motions can be split into symmetric and anti-symmetric components. Surge, heave and pitch motions are symmetric motions, that is to say that a point to starboard has the same motion as the mirrored point to port side. It is obvious that the remaining motions sway, roll and yaw are anti-symmetric motions. Symmetric and anti-symmetric motions are not coupled; they don't have any effect on each other. For instance, a vertical force acting at the center of gravity can cause surge, heave and pitch motions, but will not result in sway, roll or yaw motions.
Because of this symmetry and anti-symmetry, two sets of three coupled equations of motion can be distinguished for ships:

$$\left.\begin{array}{rllll}
\text{Surge}: & \rho\nabla\cdot\ddot{x} & & -X_{h_1} & =X_{w_1} \\
\text{Heave}: & \rho\nabla\cdot\ddot{z} & & -X_{h_3} & =X_{w_3} \\
\text{Pitch}: & I_{xx}\cdot\ddot{\theta} & & -X_{h_5} & =X_{w_5}
\end{array}\right\}\quad\text{symmetric motions} \tag{2.109}$$

$$\left.\begin{array}{rllll}
\text{Sway}: & \rho\nabla\cdot\ddot{y} & -I_{xz}\cdot\ddot{\psi} & -X_{h_2} & =X_{w_2} \\
\text{Roll}: & I_{xx}\cdot\ddot{\phi} & & -X_{h_4} & =X_{w_4} \\
\text{Yaw}: & I_{zz}\cdot\ddot{\psi} & -I_{zx}\cdot\ddot{\phi} & -X_{h_6} & =X_{w_6}
\end{array}\right\}\quad\text{anti-symmetric motions} \tag{2.110}$$

Note that this distinction between symmetric and anti-symmetric motions disappears when the ship is anchored. Then, for instance, the pitch motions can generate roll motions via the anchor lines.
The coupled surge, heave and pitch equations of motion are:

$$\left.\begin{array}{rlllll}
(\rho\nabla + a_{11}) & \cdot \ddot{x} & +b_{11} \cdot \dot{x} & +c_{11} \cdot x & & \\
+a_{13} & \cdot \ddot{z} & +b_{13} \cdot \dot{z} & +c_{13} \cdot z & & \\
+a_{15} & \cdot \ddot{\theta} & +b_{15} \cdot \dot{\theta} & +c_{15} \cdot \theta & = X_{w_1} & \text{(surge)} \\
 & & & & & \\
a_{31} & \cdot \ddot{x} & +b_{31} \cdot \dot{x} & +c_{31} \cdot x & & \\
+(\rho\nabla + a_{33}) & \cdot \ddot{z} & +b_{33} \cdot \dot{z} & +c_{33} \cdot z & & \\
+a_{35} & \cdot \ddot{\theta} & +b_{35} \cdot \dot{\theta} & +c_{35} \cdot \theta & = X_{w_3} & \text{(heave)} \\
 & & & & & \\
a_{51} & \cdot \ddot{x} & +b_{51} \cdot \dot{x} & +c_{51} \cdot x & & \\
+a_{53} & \cdot \ddot{z} & +b_{53} \cdot \dot{z} & +c_{53} \cdot z & & \\
+(+I_{yy} + a_{55}) & \cdot \ddot{\theta} & +b_{55} \cdot \dot{\theta} & +c_{55} \cdot \theta & = X_{w_5} & \text{(pitch)}
\end{array}\right\} \begin{array}{c}\text{symmetric}\\ \text{motions}\end{array} \tag{2.111}$$

The coupled sway, roll and yaw equations of motion are:

$$\left.\begin{array}{rlllll}
(\rho\nabla + a_{22}) & \cdot \ddot{y} & +b_{22} \cdot \dot{y} & +c_{22} \cdot y & & \\
+a_{24} & \cdot \ddot{\phi} & +b_{24} \cdot \dot{\phi} & +c_{24} \cdot \phi & & \\
+a_{26} & \cdot \ddot{\psi} & +b_{26} \cdot \dot{\psi} & +c_{26} \cdot \psi & = X_{w_2} & \text{(sway)} \\
 & & & & & \\
a_{42} & \cdot \ddot{y} & +b_{42} \cdot \dot{y} & +c_{42} \cdot y & & \\
+(+I_{xx} + a_{44}) & \cdot \ddot{\phi} & +b_{44} \cdot \dot{\phi} & +c_{44} \cdot \phi & & \\
+(-I_{xz} + a_{46}) & \cdot \ddot{\psi} & +b_{46} \cdot \dot{\psi} & +c_{46} \cdot \psi & = X_{w_4} & \text{(roll)} \\
 & & & & & \\
a_{62} & \cdot \ddot{y} & +b_{62} \cdot \dot{y} & +c_{62} \cdot y & & \\
+(-I_{zx} + a_{64}) & \cdot \ddot{\phi} & +b_{64} \cdot \dot{\phi} & +c_{64} \cdot \phi & & \\
+(+I_{zz} + a_{66}) & \cdot \ddot{\psi} & +b_{66} \cdot \dot{\psi} & +c_{66} \cdot \psi & = X_{w_6} & \text{(yaw)}
\end{array}\right\} \begin{array}{c}\text{anti-symmetric}\\ \text{motions}\end{array} \tag{2.112}$$

In many applications, $I_{xz} = I_{zx}$ is not known or small; hence their terms are often omitted. After the determination of the in and out of phase terms of the hydromechanical and the wave loads, these equations can be solved with a numerical method.

2.5 Strip Theory Approaches

Strip theory is a computational method by which the forces on and motions of a three-dimensional floating body can be determined using results from two-dimensional potential theory. Strip theory considers a ship to be made up of a finite number of transverse two-dimensional slices which are rigidly connected to each other. Each of these slices will have a form which closely resembles the segment of the ship which it represents. Each slice is treated hydrodynamically as if it is a segment of an infinitely long floating cylinder; see figure 2.7.
This means that all waves which are produced by the oscillating ship (hydromechanical loads) and the diffracted waves (wave loads) are assumed to travel perpendicular to the middle line plane (thus parallel to the y-z plane) of the ship. This holds too that the

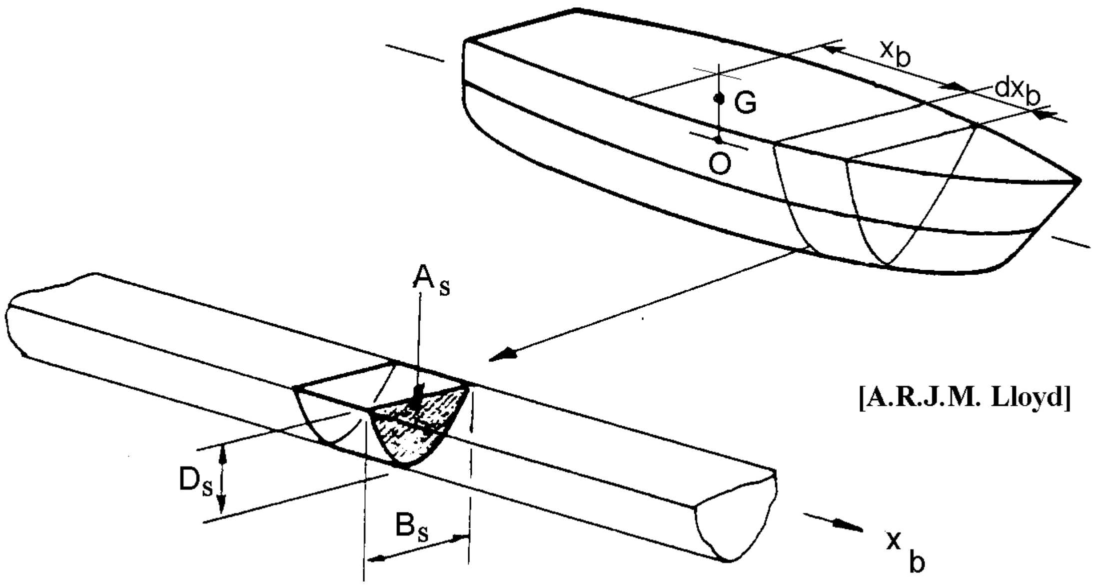

Figure 2.7: Strip Theory Representation by Cross Sections

strip theory supposes that the fore and aft side of the body (such as a pontoon) does not produce waves in the x direction. For the zero forward speed case, interactions between the cross sections are ignored as well.

Fundamentally, strip theory is valid for long and slender bodies only. In spite of this restriction, experiments have shown that strip theory can be applied successfully for floating bodies with a length to breadth ratio larger than three, ($L/B \geqq 3$), at least from a practical point of view.

2.5.1 Zero Forward Speed

When applying the strip theory, the loads on the body are found by an integration of the 2-D loads:

$$
\begin{array}{lrclrcl}
\text{surge:} & X_{h_1} & = & \int\limits_L X'_{h_1} \cdot dx_b & X_{w_1} & = & \int\limits_L X'_{w_1} \cdot dx_b \\
\text{sway:} & X_{h_2} & = & \int\limits_L X'_{h_2} \cdot dx_b & X_{w_2} & = & \int\limits_L X'_{w_2} \cdot dx_b \\
\text{heave:} & X_{h_3} & = & \int\limits_L X'_{h_3} \cdot dx_b & X_{w_3} & = & \int\limits_L X'_{w_3} \cdot dx_b \\
\text{roll:} & X_{h_4} & = & \int\limits_L X'_{h_4} \cdot dx_b & X_{w_4} & = & \int\limits_L X'_{w_4} \cdot dx_b \\
\text{pitch:} & X_{h_5} & = & -\int\limits_L X'_{h_3} \cdot x_b \cdot dx_b & X_{w_5} & = & -\int\limits_L X'_{w_3} \cdot x_b \cdot dx_b
\end{array}
$$

$$\text{yaw:} \qquad X_{h_6} = \int_L X'_{h_2} \cdot x_b \cdot dx_b \qquad X_{w_6} = \int_L X'_{w_2} \cdot x_b \cdot dx_b \tag{2.113}$$

in which:

X'_{h_j} = sectional hydromechanical force or moment in direction j per unit ship length
X'_{w_j} = sectional exciting wave force or moment in direction j per unit ship length

The appearance of two-dimensional surge forces seems strange here. It is strange! A more or less empirical method is used in SEAWAY for the surge motion, by defining an equivalent longitudinal cross section which is swaying. Then, the 2-D hydrodynamic sway coefficients of this equivalent cross section are translated to 2-D hydrodynamic surge coefficients by an empirical method based on theoretical results from three-dimensional calculations and these coefficients are used to determine 2-D loads. In this way, all sets of six surge loads can be treated in the same numerical way in SEAWAY for the determination of the 3-D loads. Inaccuracies of the hydromechanical coefficients of (slender) ships are of minor importance, because these coefficients are relatively small.
Note how in the strip theory the pitch and yaw moments are derived from the 2-D heave and sway forces, respectively, while the roll moments are obtained directly.
The equations of motions are defined in the moving axis system with the origin at the average position of the center of gravity, G. All two-dimensional potential coefficients have been defined so far in an axis system with the origin, O, in the water plane; the hydromechanical and exciting wave moments have to be corrected for the distance $\overline{OG}$.

2.5.2 Forward Ship Speed

Relative to an oscillating ship moving forward in the undisturbed surface of the fluid, the equivalent displacements, $\zeta^*_{h_j}$, velocities, $\dot{\zeta}^*_{h_j}$, and accelerations, $\ddot{\zeta}^*_{h_j}$, at forward ship speed V in the arbitrary direction j of a water particle in a cross section are defined by:

$$\zeta^*_{h_j}, \qquad \dot{\zeta}^*_{h_j} = \frac{D}{Dt}\left\{\zeta^*_{h_j}\right\} \qquad \text{and} \qquad \ddot{\zeta}^*_{h_j} = \frac{D}{Dt}\left\{\dot{\zeta}^*_{h_j}\right\} \tag{2.114}$$

in which:

$$\frac{D}{Dt} = \left\{\frac{\partial}{\partial t} - V \cdot \frac{\partial}{\partial x}\right\} \tag{2.115}$$

is an operator which transforms the potentials $\Phi(x_0, y_0, z_0, t)$, defined in the earth bounded (fixed) coordinate system, to the potentials $\Phi(x, y, z, t)$, defined in the ship's steadily translating coordinate system with speed V.
Relative to a restrained ship, moving forward with speed V in waves, the equivalent j components of water particle displacements, $\zeta^*_{w_j}$, velocities, $\dot{\zeta}^*_{w_j}$, and accelerations, $\ddot{\zeta}^*_{w_j}$, in a cross section are defined in a similar way by:

$$\zeta^*_{w_j}, \qquad \dot{\zeta}^*_{w_j} = \frac{D}{Dt}\left\{\zeta^*_{w_j}\right\} \qquad \text{and} \qquad \ddot{\zeta}^*_{w_j} = \frac{D}{Dt}\left\{\dot{\zeta}^*_{w_j}\right\} \tag{2.116}$$

The effect of this operator can be understood easily when one realizes that in that earth-bound coordinate system the sailing ship penetrates through a "virtual vertical disk". When a ship sails with speed V and a trim angle, θ, through still water, the relative vertical velocity of a water particle with respect to the bottom of the sailing ship becomes $V \cdot \theta$.

Two different types of strip theory methods are discussed here:

1. **Ordinary Strip Theory Method**
According to this classic method, the uncoupled two-dimensional potential hydromechanical loads and wave loads in an arbitrary direction j are defined by:

$$
\begin{aligned}
X^*_{h_j} &= \frac{D}{Dt}\left\{M'_{jj} \cdot \dot{\zeta}^*_{h_j}\right\} + N'_{jj} \cdot \dot{\zeta}^*_{h_j} + X'_{rs_j} \\
X^*_{w_j} &= \frac{D}{Dt}\left\{M'_{jj} \cdot \dot{\zeta}^*_{w_j}\right\} + N'_{jj} \cdot \dot{\zeta}^*_{w_j} + X'_{fk_j}
\end{aligned} \tag{2.117}
$$

This in the first formulation of the strip theory that can be found in the literature. It contains a more or less intuitive approach to the forward speed problem, as published in detail by [Korvin-Kroukovsky and Jacobs, 1957] and others.

2. **Modified Strip Theory Method**
According to this modified method, these loads become:

$$
\begin{aligned}
X^*_{h_j} &= \frac{D}{Dt}\left\{\left(M'_{jj} - \frac{i}{\omega_e}N'_{jj}\right) \cdot \dot{\zeta}^*_{h_j}\right\} + X'_{rs_j} \\
X^*_{w_j} &= \frac{D}{Dt}\left\{\left(M'_{jj} - \frac{i}{\omega_e}N'_{jj}\right) \cdot \dot{\zeta}^*_{w_j}\right\} + X'_{fk_j}
\end{aligned} \tag{2.118}
$$

This formulation is a more fundamental approach of the forward speed problem, as published in detail by [Tasai, 1969] and others.

In the equations above, M'_{jj} and N'_{jj} are the 2-D potential mass and damping coefficients. X'_{rs_j} is the two-dimensional quasi-static restoring spring term, generally present for heave, roll and pitch only. X'_{fk_j} is the two-dimensional Froude-Krilov force or moment which is calculated by an integration of the directional pressure gradient in the undisturbed wave over the cross sectional area of the hull.
Equivalent directional components of the orbital acceleration and velocity, derived from these Froude-Krilov loads, are used to calculate the diffraction parts of the total wave forces and moments.
From a theoretical point of view, one should prefer the use of the modified method, but it appeared from user's experience that for ships with moderate forward speed ($Fn \leqq 0.30$), the ordinary method provides a better fit with experimental data. Thus from a practical point of view, the use of the ordinary method is advised generally.

2.5.3 End-Terms

From the previous, it is obvious that in the equations of motion longitudinal derivatives of the two-dimensional potential mass M'_{ij} and damping N'_{ij} will appear. These derivatives have to be determined numerically over the whole ship length in such a manner that the following relation is fulfilled:

$$\begin{aligned}
\int\limits_{x_b(0)-\varepsilon}^{x_b(L)+\varepsilon} \frac{df(x_b)}{dx_b} dx_b &= \int\limits_{x_b(0)-\varepsilon}^{x_b(0)} \frac{df(x_b)}{dx_b} dx_b + \int\limits_{x_b(0)}^{x_b(L)} \frac{df(x_b)}{dx_b} dx_b + \int\limits_{x_b(L)}^{x_b(L)+\varepsilon} \frac{df(x_b)}{dx_b} dx_b \\
&= f(0) + \int\limits_{x_b(0)}^{x_b(L)} \frac{df(x_b)}{dx_b} dx_b - f(L) \\
&= 0
\end{aligned} \qquad (2.119)$$

with $\varepsilon \ll L$, while $f(x_b)$ is equal to the local values of $M'_{jj}(x_b)$ or $N'_{jj}(x_b)$; see figure 2.8.

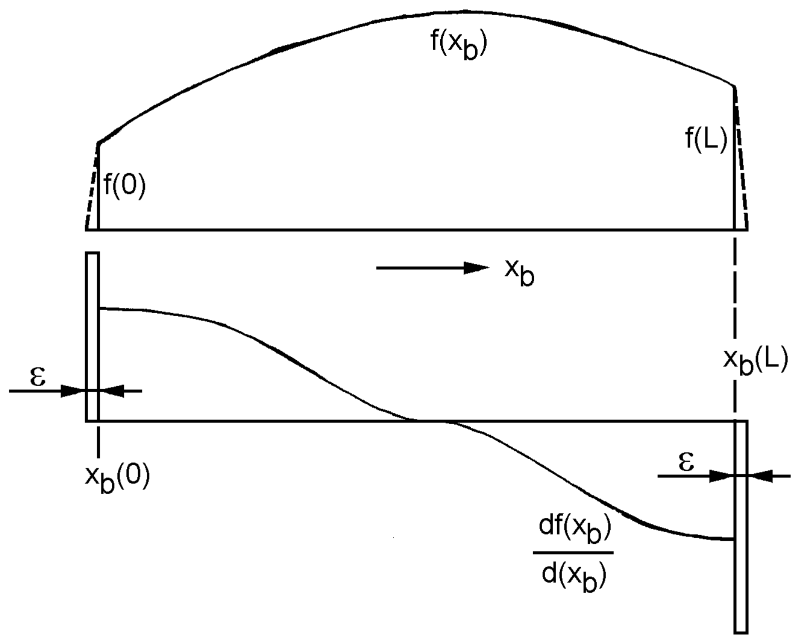

Figure 2.8: Integration of Derivatives

The numerical integrations of the derivatives are carried out in the region $x_b(0) \leq x_b \leq x_b(L)$ only. So, the additional so-called "end terms" are defined by $f(0)$ and $f(L)$. Because the integration of the derivatives has to be carried out in the region $x_b(0) - \varepsilon \leq x_b \leq x_b(L) + \varepsilon$, some algebra provides the integral and the first and second order moments (with respect to G) over the whole ship length:

$$\int\limits_{x_b(0)-\varepsilon}^{x_b(L)+\varepsilon} \frac{df(x_b)}{dx_b} \cdot dx_b = 0$$

$$\int\limits_{x_b(0)-\varepsilon}^{x_b(L)+\varepsilon} \frac{df(x_b)}{dx_b} \cdot x_b \cdot dx_b = - \int\limits_{x_b(0)}^{x_b(L)} f(x_b) \cdot dx_b$$

$$\int_{x_b(0)-\varepsilon}^{x_b(L)+\varepsilon} \frac{df(x_b)}{dx_b} \cdot x_b^2 \cdot dx_b = -2 \int_{x_b(0)}^{x_b(L)} f(x_b) \cdot x_b \cdot dx_b \tag{2.120}$$

Note that these expressions are valid for the integrations of the potential coefficients over the full ship length only. They can not be used for calculating local hydromechanical loads. Also for the wave loads, these expressions can not be used, because there these derivatives are multiplied with x_b-depending orbital motions.

2.6 Hydrodynamic Coefficients

The two-dimensional hydrodynamic sway, heave and roll coefficients can be calculated by several methods:

1. Methods based on Ursell's Theory and Conformal Mapping
   [Ursell, 1949] derived an analytical solution for solving the problem of calculating the hydrodynamic coefficients of an oscillating circular cylinder in the surface of a fluid.

   (a) Deep Water Coefficients by Lewis Conformal Mapping
   [Tasai, 1959] and [Tasai, 1961] added the so-called Lewis transformation - which is a very simple and in a lot of cases also more or less realistic method to transform ship-like cross sections to this unit circle - to Ursell's solution . This transformation is carried out by using a scale factor and two mapping coefficients. Only the breadth, the draft and the area of the mapped cross section will be equal to that of the actual cross section.

   (b) Deep Water Coefficients by Close-Fit Conformal Mapping
   A more accurate mapping has been added by [Tasai, 1960] too, by using more than only two mapping coefficients. The accuracy obtained depends on the number of mapping coefficients. Generally, a maximum of 10 coefficients are used for defining the cross section. These coefficients are determined in such a way that the Root Mean Square of the differences between the offsets of the mapped and the actual cross section is minimal.

   (c) Shallow Water Coefficients by Lewis Conformal Mapping
   For shallow water, the theory of [Keil, 1974] - based on a variation of Ursell's potential theory for circular cylinders at deep water - and Lewis conformal mapping can be used.

2. Frank's Pulsating Source Theory for Deep Water
   Mapping methods require an intersection of the cross section with the water plane and, as a consequence of this, they are not suitable for submerged cross sections, like at a bulbous bow. Also, conformal mapping can fail for cross sections with very low sectional area coefficients, such as are sometimes present in the aft body of a ship. [Frank, 1967] considered a cylinder of constant cross sections with an arbitrarily symmetrical shape, of which the cross sections are simply a region of connected line elements. This vertical cross section can be fully or partly immersed in a previously undisturbed fluid of infinite depth. He developed an integral equation method utilizing the Green's function which represents a complex potential of a pulsating

point source of unit strength at the midpoint of each line element. Wave systems were defined in such a way that all required boundary conditions were fulfilled. The linearized Bernoulli equation provides the pressures after which the potential coefficients were obtained from the in-phase and out-of-phase components of the resultant hydrodynamic loads.

The 2-D potential pitch and yaw (moment) coefficients follow from the previous heave and sway coefficients and the arm, i.e., the distance of the cross section to the center of gravity G.

A more or less empiric procedure has followed by the author for the surge motion. An equivalent longitudinal cross section has been defined. For each frequency, the two-dimensional potential hydrodynamic sway coefficient of this equivalent cross section is translated to two-dimensional potential hydrodynamic surge coefficients by an empiric method, which is based on theoretical results of three-dimensional calculations.

The 3-D coefficients follow from an integration of these 2-D coefficients over the ship's length. Viscous terms can be added for surge and roll.

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Chapter 3

Conformal Mapping

Ursell's derivation of potential coefficients is valid for circular cross sections only. For the determination of the two-dimensional added mass and damping in the sway, heave and roll mode of the motions of ship-like cross sections by Ursell's method, the cross sections have to be mapped conformally to the unit circle. The advantage of conformal mapping is that the velocity potential of the fluid around an arbitrarily shape of a cross section in a complex plane can be derived from the more convenient circular section in another complex plane. In this manner hydrodynamic problems can be solved directly with the coefficients of the mapping function.
The general transformation formula is given by:

$$z = M_s \sum_{n=0}^{N} \left(a_{-(2n-1} \zeta^{2n-1} \right)$$

with:

$z = x + iy$	= plane of the ship's cross section
$\zeta = ie^{\alpha} e^{-i\theta}$	= plane of the unit circle
M_s	= scale factor
a_{-1}	= +1
a_{2n-1}	= conformal mapping coefficients ($n = 1, ..., N$)
N	= number of parameters

From this follows the relation between the coordinates in the z-plane (the ship's cross section) and the variables in the ζ-plane (the circular cross section):

$$\begin{aligned} x &= -M_s \sum_{n=0}^{N} \left\{ (-1)^n a_{2n-1} e^{-(2n-1)\alpha} \sin\left((2n-1)\theta\right) \right\} \\ y &= +M_s \sum_{n=0}^{N} \left\{ (-1)^n a_{2n-1} e^{-(2n-1)\alpha} \cos\left((2n-1)\theta\right) \right\} \end{aligned}$$

[0] J.M.J. Journée, *"Theoretical Manual of SEAWAY, Release 4.19"*, Report 1216a, February 2001, Ship Hydromechanics Laboratory, Delft University of Technology, Mekelweg 2, 2628 CD Delft, The Netherlands. For updates see web site: http://dutw189.wbmt.tudelft.nl/~johan or http://www.shipmotions.nl.

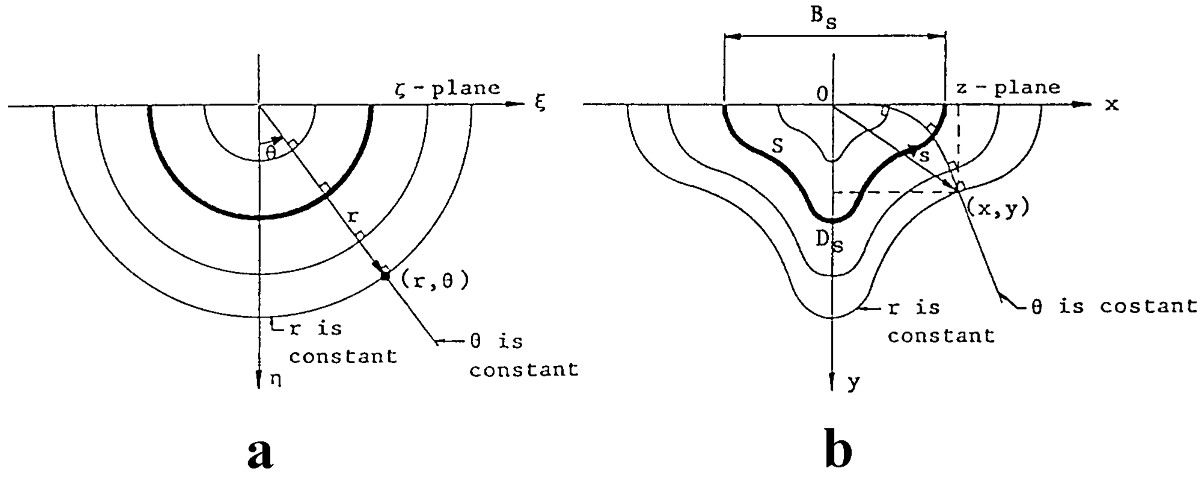

Figure 3.1: Mapping Relation between Two Planes

The contour of the - by conformal mapping approximated - ship's cross section follows from putting $\alpha = 0$ in the previous relations:

$$\begin{aligned} x_0 &= -M_s \sum_{n=0}^{N} \{(-1)^n a_{2n-1} \sin((2n-1)\theta)\} \\ y_0 &= +M_s \sum_{n=0}^{N} \{(-1)^n a_{2n-1} \cos((2n-1)\theta)\} \end{aligned}$$

The breadth on the waterline of the approximate ship's cross section is defined by:

$$b_0 = 2M_s\lambda_a \qquad \text{with:} \qquad \lambda_a = \sum_{n=0}^{N} a_{2n-1}$$

with scale factor:

$$M_s = \frac{b_0}{2\lambda_a}$$

and the draft is defined by:

$$d_0 = M_s\lambda_b \qquad \text{with:} \qquad \lambda_b = \sum_{n=0}^{N} \{(-1)^n a_{2n-1}\}$$

3.1 Lewis Conformal Mapping

A very simple and in a lot of cases also a more or less realistic transformation of the cross sectional hull form will be obtained with $N = 2$ in the transformation formula, the well known Lewis transformation ([Lewis, 1929]). A extended description of the representation of ship hull forms by Lewis two-parameter conformal mapping is given by [Kerczek and Tuck, 1969].

This two-parameter Lewis transformation of a cross section is defined by:

$$z = M_s \cdot \left(a_{-1} \cdot \zeta + a_1 \cdot \zeta^{-1} + a_3 \cdot \zeta^{-3}\right)$$

In here a_{-1}=+1 and the conformal mapping coefficients a_1 and a_3 are called Lewis coefficients, while M_s is the scale factor.
Then:

$$\begin{aligned} x &= M_s \cdot \left(e^{\alpha} \sin\theta + a_1 e^{-\alpha} \sin\theta - a_3 e^{-3\alpha} \sin 3\theta\right) \\ y &= M_s \cdot \left(e^{\alpha} \cos\theta - a_1 e^{-\alpha} \cos\theta + a_3 e^{-3\alpha} \cos 3\theta\right) \end{aligned}$$

By putting $\alpha = 0$ is the contour of this so-called Lewis form expressed as:

$$\begin{aligned} x_0 &= M_s \cdot ((1 + a_1) \sin\theta - a_3 \sin 3\theta) \\ y_0 &= M_s \cdot ((1 - a_1) \cos\theta + a_3 \cos 3\theta) \end{aligned}$$

with the scale factor:

$$M_s = \frac{B_s/2}{1 + a_1 + a_3} \qquad \text{or:} \qquad M_s = \frac{D_s}{1 - a_1 + a_3}$$

in which:

B_s = sectional breadth on the load water line
D_s = sectional draft

Now the coefficients a_1 and a_3 and the scale factor M_s will be determined in such a manner that the sectional breadth, draft and area of the approximate cross section and of the actual cross section are identical.
The half breadth to draft ratio H_0 is given by:

$$H_0 = \frac{B_s/2}{D_s} = \frac{1 + a_1 + a_3}{1 - a_1 + a_3}$$

An integration of the Lewis form delivers the sectional area coefficient σ_s:

$$\sigma_s = \frac{A_s}{B_s D_s} = \frac{\pi}{4} \cdot \frac{1 - a_1^2 - 3a_3^2}{(1 + a_3)^2 - a_1^2}$$

in which A_s is the area of the cross section.
Putting a_1, derived from the expression for H_0, into the expression for σ_s yields a quadratic equation in a_3:

$$c_1 a_3^2 + c_2 a_3 + c_3 = 0$$

in which:

$$\begin{aligned} c_1 &= 3 + \frac{4\sigma_s}{\pi} + \left(1 - \frac{4\sigma_s}{\pi}\right) \cdot \left(\frac{H_0 - 1}{H_0 + 1}\right)^2 \\ c_2 &= 2c_1 - 6 \\ c_3 &= c_1 - 4 \end{aligned}$$

The (valid) solutions for a_3 and a_1 will become:

$$\begin{aligned} a_3 &= \frac{-c_1 + 3 + \sqrt{9 - 2c_1}}{c_1} \\ a_1 &= \frac{H_0 - 1}{H_0 + 1} \cdot (a_3 + 1) \end{aligned}$$

Lewis forms with the other solution of a_3 in the quadratic equation, with a minus sign before the square root:

$$a_3 = \frac{-c_1 + 3 - \sqrt{9 - 2c_1}}{c_1}$$

are looped; they intersect themselves at a point within the fourth quadrant. Since ships are "better behaved", these solutions are not considered.
It is obvious that a transformation of a half immersed circle with radius R will result in $M_s = R$, $a_1 = 0$ and $a_3 = 0$.
Some typical and realistic Lewis forms are presented in figure 3.2.

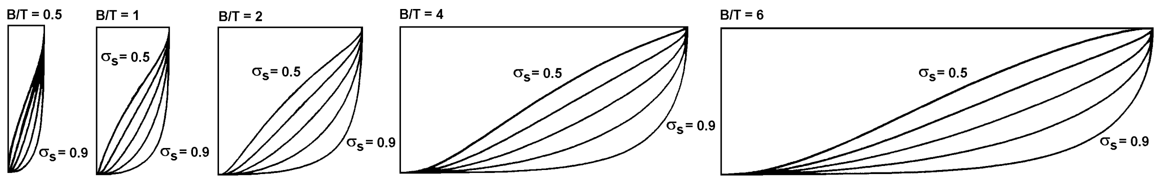

Figure 3.2: Typical Lewis Forms

3.1.1 Boundaries of Lewis Forms

In some cases the Lewis transformation can give more or less ridiculous results. The following typical Lewis hull forms, with the regions of the half breadth to draft ratio H_0 and the area coefficient σ_s to match as presented before, can be distinguished:

- re-entrant forms, bounded by:

$$\begin{aligned} \text{for} \quad H_0 \leq 1.0: &\quad \sigma_s < \frac{3\pi}{32}(2 - H_0) \\ \text{for} \quad H_0 \geq 1.0: &\quad \sigma_s < \frac{3\pi}{32}\left(2 - \frac{1}{H_0}\right) \end{aligned}$$

- conventional forms, bounded by:

$$\begin{aligned} \text{for} \quad H_0 \leq 1.0: &\quad \frac{3\pi}{32}(2 - H_0) < \sigma_s < \frac{3\pi}{32}\left(3 + \frac{H_0}{4}\right) \\ \text{for} \quad H_0 \geq 1.0: &\quad \frac{3\pi}{32}\left(2 - \frac{1}{H_0}\right) < \sigma_s < \frac{3\pi}{32}\left(3 + \frac{1}{4H_0}\right) \end{aligned}$$

- bulbous and not-tunneled forms, bounded by:

$$H_0 \leq 1.0 \quad \text{and} \quad \frac{3\pi}{32}\left(3 + \frac{H_0}{4}\right) < \sigma_s < \frac{3\pi}{32}\left(3 + \frac{1}{4H_0}\right)$$

- tunneled and not-bulbous forms, bounded by:

$$H_0 \geq 1.0 \quad \text{and} \quad \frac{3\pi}{32}\left(3+\frac{1}{4H_0}\right) \quad < \quad \sigma_s \quad < \quad \frac{3\pi}{32}\left(3+\frac{H_0}{4}\right)$$

- combined bulbous and tunneled forms, bounded by:

$$\text{for} \quad H_0 \quad \leq \quad 1.0: \qquad \frac{3\pi}{32}\left(3+\frac{1}{4H_0}\right) \quad < \quad \sigma_s \quad < \quad \frac{\pi}{32}\left(10+H_0+\frac{1}{H_0}\right)$$

$$\text{for} \quad H_0 \quad \geq \quad 1.0: \qquad \frac{3\pi}{32}\left(3+\frac{H_0}{4}\right) \quad < \quad \sigma_s \quad < \quad \frac{\pi}{32}\left(10+H_0+\frac{1}{H_0}\right)$$

- non-symmetric forms, bounded by:

$$0 < H_0 < \infty \quad \text{and} \qquad \sigma_s \quad > \quad \frac{\pi}{32}\left(1+H_0+\frac{1}{H_0}\right)$$

These ranges of the half breadth to draft ratio H_0 and the area coefficient σ_s for the different typical Lewis forms are shown in figure 3.3.

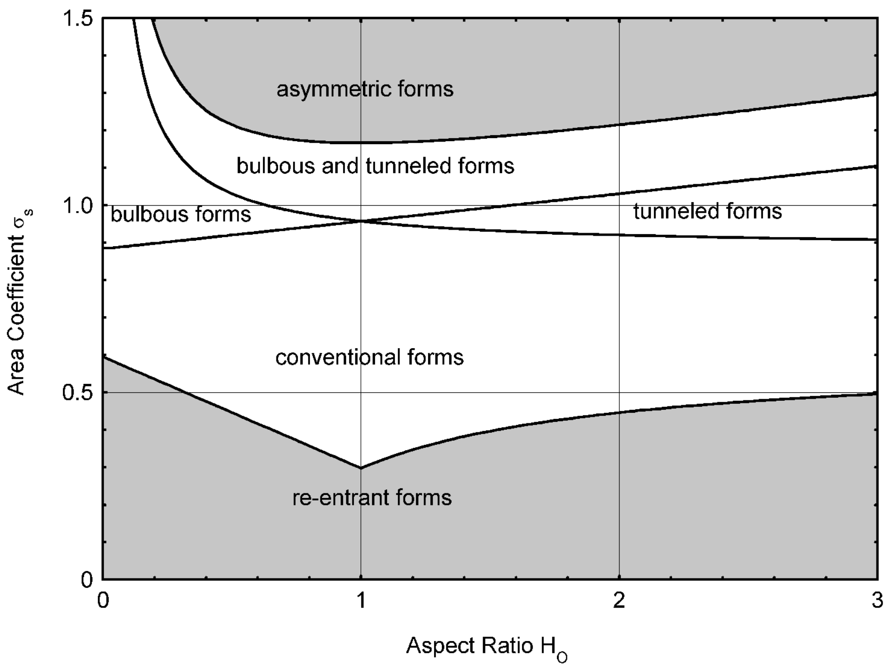

Figure 3.3: Ranges of H_0 and σ_s of Lewis Forms

3.1.2 Acceptable Lewis Forms

Not-acceptable forms of ships are supposed to be the re-entrant forms and the asymmetric forms. So conventional forms, bulbous forms and tunneled forms are considered to be valid forms here, see figure 3.3. To obtain ship-like Lewis forms, the area coefficient σ_s is bounded by a lower limit to omit re-entrant Lewis forms and by an upper limit to omit non-symmetric Lewis forms:

$$\begin{array}{llcccc}
\text{for} \quad H_0 \leq 1.0: & \frac{3\pi}{32}(2 - H_0) & < & \sigma_s & < & \frac{\pi}{32}\left(10 + H_0 + \frac{1}{H_0}\right) \\
\text{for} \quad H_0 \geq 1.0: & \frac{3\pi}{32}\left(2 - \frac{1}{H_0}\right) & < & \sigma_s & < & \frac{\pi}{32}\left(10 + H_0 + \frac{1}{H_0}\right)
\end{array}$$

If a value of σ_s is outside of this range it has to be set to the value of the nearest border of this range, to calculate the Lewis coefficients.
Numerical problems, for instance with bulbous or aft cross sections of a ship, are avoided when the following requirements are fulfilled:

$$\frac{B_s}{2} > \gamma D_s \qquad \text{and} \qquad Ds > \gamma \frac{B_s}{2}$$

with for instance $\gamma = 0.01$.

3.2 Extended Lewis Conformal Mapping

Somewhat better approximations will be obtained by taking into account also the first order moments of half the cross section about the x_0- and y_0-axes. These two additions to the Lewis formulation were proposed by [Reed and Nowacki, 1974] and have been simplified by [Athanassoulis and Loukakis, 1985] by taking into account the vertical position of the centroid of the cross section. This has been done by extending the Lewis transformation to $N = 3$ in the general transformation formula.
The three-parameter Extended-Lewis transformation is defined by:

$$Z = M_s(a_{-1}\zeta + a_1\zeta^{-1} + a_3\zeta^{-3} + a_5\zeta^{-5})$$

in which $a_{-1} = +1$.
So:

$$\begin{array}{rcl}
x &=& M_s(e^{\alpha}\sin\theta + a_1 e^{-\alpha}\sin\theta - a_3 e^{-3\alpha}\sin 3\theta + a_5 e^{-5\alpha}\sin 5\theta) \\
y &=& M_s(e^{\alpha}\cos\theta - a_1 e^{-\alpha}\cos\theta + a_3 e^{-3\alpha}\cos 3\theta - a_5 e^{-5\alpha}\cos 5\theta)
\end{array}$$

By putting $\alpha = 0$, the contour of this approximate form is expressed as:

$$\begin{array}{rcl}
x_0 &=& M_s\left((1 + a_1)\sin\theta - a_3\sin 3\theta + a_5\sin 5\theta\right) \\
y_0 &=& M_s\left((1 - a_1)\cos\theta + a_3\cos 3\theta - a_5\cos 5\theta\right)
\end{array}$$

with the scale factor:

$$M_s = \frac{B_s/2}{1 + a_1 + a_3 + a_5} \qquad \text{or:} \qquad M_s = \frac{D_s}{1 - a_1 + a_3 - a_5}$$

and:

$$\begin{array}{ll}
B_s & = \text{sectional breadth on the waterline} \\
D_s & = \text{sectional draft}
\end{array}$$

Now the coefficients a_1, a_3 and a_5 and the scale factor M_s can be determined in such a manner that, except the sectional breadth, draft and area, also the centroids of the approximate cross section and the actual cross section of the ship have an equal position. The half beam to draft ratio is given by:

$$H_0 = \frac{B_s/2}{D_s} = \frac{1 + a_1 + a_3 + a_5}{1 - a_1 + a_3 - a_5}$$

An integration of the approximate form results into the sectional area coefficient:

$$\sigma_s = \frac{A_s}{B_s D_s} = \frac{\pi}{4} \cdot \frac{1 - a_1^2 - 3a_3^2 - 5a_5^2}{(1 + a_3)^2 - (a_1 + a_5)^2}$$

For the relative distance of the centroid to the keel point a more complex expression has been obtained by [Athanassoulis and Loukakis, 1985]:

$$\kappa = \frac{\overline{KB}}{D_s} = 1 - \frac{\sum\limits_{i=0}^{3} \sum\limits_{j=0}^{3} \sum\limits_{k=0}^{3} \left(A_{ijk} a_{2i-1} a_{2j-1} a_{2k-1} \right)}{H_0 \sigma_s \sum\limits_{i=0}^{3} a_{2i-1}^3}$$

in which:

$$A_{ijk} = \frac{1}{4} \left(\frac{1 - 2k}{3 - 2(i + j + k)} - \frac{1 - 2k}{1 - 2(i - j + k)} + \frac{1 - 2k}{1 - 2(i + j - k)} + \frac{1 - 2k}{1 - 2(-i + j + k)} \right)$$

The following requirements should be fulfilled when also bulbous cross sections are allowed:

- re-entrant forms are avoided when both the following requirements are fulfilled:

$$\begin{aligned} 1 - a_1 - 3a_3 - 5a_5 &> 0 \\ 1 + a_1 - 3a_3 + 5a_5 &> 0 \end{aligned}$$

- existence of a point of self-intersection is avoided when both the following requirements are fulfilled:

$$\begin{aligned} 9a_3^2 + 145a_5^2 + 10a_3 a_5 + 20H_0 a_5 &> 0 \\ 9a_3^2 + 145a_5^2 - 10a_3 a_5 - 20H_0 a_5 &> 0 \end{aligned}$$

Taking these restrictions into account, the equations above can be solved in an iterative manner.

3.3 Close-Fit Conformal Mapping

A more accurate transformation of the cross sectional hull form can be obtained by using a greater number of parameters N. A very simple and straight on iterative least squares method of the author to determine the Close-Fit conformal mapping coefficients will be described here shortly.

The scale factor M_s and the conformal mapping coefficients a_{2n-1}, with a maximum value of n varying from $N = 2$ until $N = 10$, have been determined successfully from the offsets of various cross sections in such a manner that the mean squares of the deviations of the actual cross section from the approximate described cross section is minimized.
The general transformation formula is given by:

$$Z = M_s \sum_{n=0}^{N} \left\{ a_{2n-1} \zeta^{-(2n-1)} \right\}$$

in which: $a_{-1} = +1$.
Then the contour of the approximate cross section is given by:

$$\begin{aligned} x_0 &= -M_s \sum_{n=0}^{N} \{(-1)^n a_{2n-1} \sin((2n-1)\theta)\} \\ y_0 &= +M_s \sum_{n=0}^{N} \{(-1)^n a_{2n-1} \cos((2n-1)\theta)\} \end{aligned}$$

with the scale factor:

$$M_s = \frac{B_s/2}{\sum\limits_{n=0}^{N} \{a_{2n-1}\}} \qquad \text{or:} \qquad M_s = \frac{D_s}{\sum\limits_{n=0}^{N} \{(-1)^n a_{2n-1}\}}$$

The procedure starts with initial values for $[M_s \cdot a_{2n-1}]$. The initial values of M_s, a_1 and a_3 are obtained with the Lewis method as has been described before, while the initial values of a_5 until a_{2N-1} are set to zero. With these $[M_s \cdot a_{2n-1}]$ values, a θ_i-value is determined for each offset in such a manner that the actual offset (x_i, y_i) lies on the normal of the approximate contour of the cross section in (x_{0i}, y_{0i}). Now θ_i has to be determined. Therefore a function $F(\theta_i)$, will be defined by the distance from the offset (x_i, y_i) to the normal of the contour to the actual cross section through (x_{0i}, y_{0i}), see figure 3.4.

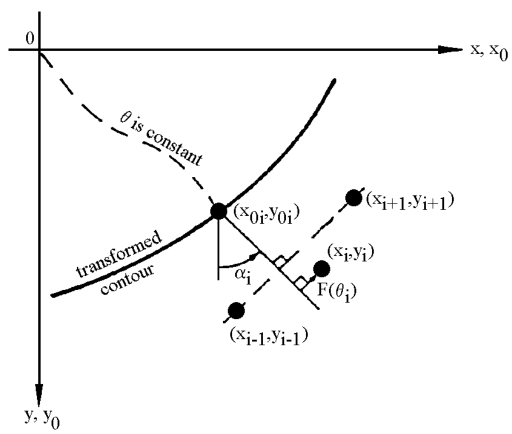

Figure 3.4: Close-Fit Conformal Mapping

These offsets have to be selected at approximately equal mutual circumferential lengths, eventually with somewhat more dense offsets near sharp corners. Then α_i is defined by:

$$\begin{aligned} \cos\alpha_i &= \frac{+x_{i+1} - x_{i-1}}{\sqrt{(x_{i+1} - x_{i-1})^2 + (y_{i+1} - y_{i-1})^2}} \\ \sin\alpha_i &= \frac{-y_{i+1} + y_{i-1}}{\sqrt{(x_{i+1} - x_{i-1})^2 + (y_{i+1} - y_{i-1})^2}} \end{aligned}$$

With this θ_i-value, the numerical value of the square of the deviation of (x_i, y_i) from (x_{0i}, y_{0i}) is calculated:

$$e_i = (x_i - x_{0i})2 + (y_i - y_{0i})2$$

After doing this for all $I + 1$ offsets, the numerical value of he sum of the squares of deviations is known:

$$E = \sum_{i=0}^{I} \{e_i\}$$

The sum of the squares of these deviations can also be expressed as:

$$\begin{aligned} E = & \sum_{i=0}^{I} \left\{ \left(x_i + \sum_{n=0}^{N} \{(-1)^n [M_s \cdot a_{2n-1}] \sin((2n-1)\theta_i)\} \right)^2 \right. \\ & \left. +(\left(y_i - \sum_{n=0}^{N} \{(-1)^n [M_s \cdot a_{2n-1}] \cos((2n-1)\theta_i)\} \right)^2 \right\} \end{aligned}$$

Then new values of $[M_s \cdot a_{2n-1}]$ have to be obtained in such a manner that E is minimized. This means that each of the derivatives of this equation with respect to each coefficients $[M_s \cdot a_{2n-1}]$ is zero, so:

$$\frac{\partial E}{\partial\{M_s a_{2j-1}\}} = 0 \qquad \text{for: } j = 0, ...N$$

This yields N+1 equations:

$$\begin{aligned} & \sum_{i=0}^{I} \left\{ -\sin((2j-1)\theta_i) \sum_{n=0}^{N} \{(-1)^n [M_s \cdot a_{2n-1}] \sin((2n-1)\theta_i)\} + \right. \\ & \left. -\cos((2j-1)\theta_i) \sum_{n=0}^{N} \{(-1)^n [M_s \cdot a_{2n-1}] \cos((2n-1)\theta_i)\} \right\} = \\ = & \sum_{i=0}^{I} \{x_i \sin((2j-1)\theta_i) - y_i \cos((2j-1)\theta_i)\} \qquad \text{for: } j = 0, ...N \end{aligned}$$

which are rewritten as:

$$\begin{aligned} & \sum_{n=0}^{N} \left\{ (-1)^n [M_s \cdot a_{2n-1}] \sum_{i=0}^{I} \{\cos((2j-2n)\theta_i)\} \right\} = \\ = & \sum_{i=0}^{I} \{-x_i \sin((2j-1)\theta_i) + y_i \cos((2j-1)\theta_i)\} \\ & \qquad \text{for: } j = 0, ...N \end{aligned}$$

To obtain the exact breadth and draft, the last two equations are replaced by the equations for the breadth at the water line and the draft:

$$\left\{ \sum_{n=0}^{N} \{(-1)^n [M_s \cdot a_{2n-1}]\} \sum_{i=0}^{I} \{\cos((2j-2n)\theta_i)\} \right\} =$$

$$= \sum_{i=0}^{I} \{-x_i \sin((2j-1)\theta_i) + y_i \cos((2j-1)\theta_i)\} \qquad \text{for: } j = 0, ...N-2$$

$$\sum_{n=0}^{N} \{[M_s \cdot a_{2n-1}]\} = B_s/2 \quad \text{for: } j = N-1$$

$$\sum_{n=0}^{N} \{(-1)^n [M_s \cdot a_{2n-1}]\} = D_s \qquad \text{for: } j = N$$

These $N+1$ equations can be solved numerically, so that new values for $[M_s \cdot a_{2n-1}]$ will be obtained. These new values are used instead of the initial values to obtain new θ_i-values of the $I+1$ offsets again, etc. This procedure will be repeated several times and stops when the difference between the numerical E-values of two subsequent calculations becomes less than a certain threshold value ΔE, depending on the dimensions of the cross section; for instance:

$$\Delta E = (I+1) \left(0.00005 \sqrt{b_{max}^2 + d_{max}^2} \right)^2$$

in which:

b_{max} = maximum half breadth of the cross section
d_{max} = maximum height of the cross section

Because $a_{-1} = +1$ the scale factor M_s is equal to the final solution of the first coefficient ($n = 0$). The N other coefficients a_{2n-1} can be found by dividing the final solutions of $[M_s \cdot a_{2n-1}]$ by this M_s-value.
Reference is also given here to a report of [Jong, 1973]. In that report several other, suitable but more complex, methods are described to determine the scale factor M_s and the conformal mapping coefficients a_{2n-1} from the offsets of a cross section.
Attention has to be paid to divergence in the calculation routines and re-entrant forms. In these cases the number N will be increased until the divergence or re-entrance vanish. In the most worse case a "maximum" value of N will be attained without success.. One can then switch to Lewis coefficients with an area coefficient of the cross section set to the nearest border of the valid Lewis form area.

3.4 Comparisons

A first example has been given here for the amidships cross section of a container vessel, with a breadth of 25.40 meter and a draft of 9.00 meter, with offsets as tabled below:

i (−)	$D_s - y_i$ (m)	x_i (m)
0	0.000	0.000
1	0.135	4.950
2	0.270	9.900
3	0.500	10.960
4	1.000	11.740
5	2.000	12.440
6	3.050	12.700
7	6.000	12.700
8	9.000	12.700

For the least squares method in the conformal mapping method, 33 new offsets at equidistant length intervals on the contour of this cross section can be determined by a second degree interpolation routine. The calculated data of the two-parameter Lewis and the N-parameter Close-Fit conformal mapping of this amidships cross section are tabled below. The last line lists the RMS-values for the deviations of the 33 equidistant points on the approximate contour of this cross section..

		Lewis Conformal Mapping	N-Parameter Close Fit Conformal Mapping								
N	$(-)$	2	2	3	4	5	6	7	8	9	10
$2N$-1	$(-)$	3	3	5	7	9	11	13	15	17	19
M_s	(m)	12.2400	12.2457	12.2841	12.3193	12.3186	12.3183	12.3191	12.3190	12.3195	12.3194
a_{-1}	$(-)$	+1.0000	+1.0000	+1.0000	+1.0000	+1.0000	+1.0000	+1.0000	+1.0000	+1.0000	+1.0000
a_1	$(-)$	+0.1511	+0.1511	+0.1640	+0.1634	+0.1631	+0.1633	+0.1633	+0.1632	+0.1632	+0.1632
a_3	$(-)$	-0.1136	-0.1140	-0.1167	-0.1245	-0.1246	-0.1243	-0.1244	-0.1245	-0.1245	-0.1245
a_5	$(-)$			-0.0134	-0.0133	-0.0105	-0.0108	-0.0108	-0.0108	-0.0107	-0.0107
a_7	$(-)$				+0.0053	+0.0054	+0.0031	+0.0030	+0.0032	+0.0031	+0.0030
a_9	$(-)$					-0.0024	-0.0023	-0.0024	-0.0026	-0.0029	-0.0029
a_{11}	$(-)$						+0.0021	+0.0022	+0.0012	+0.0014	+0.0015
a_{13}	$(-)$							+0.0002	+0.0002	+0.0021	+0.0020
a_{15}	$(-)$								+0.0009	+0.0007	+0.0000
a_{17}	$(-)$									-0.0016	-0.0015
a_{19}	$(-)$										+0.0006
RMS	(m)	0.181	0.180	0.076	0.039	0.027	0.019	0.018	0.017	0.009	0.008

Another example is given in figure 3.5, which shows the differences between a Lewis transformation and a 10-parameter close-fit conformal mapping of a rectangular cross section with a breadth of 20.00 meter and a draft of 10.00 meter.

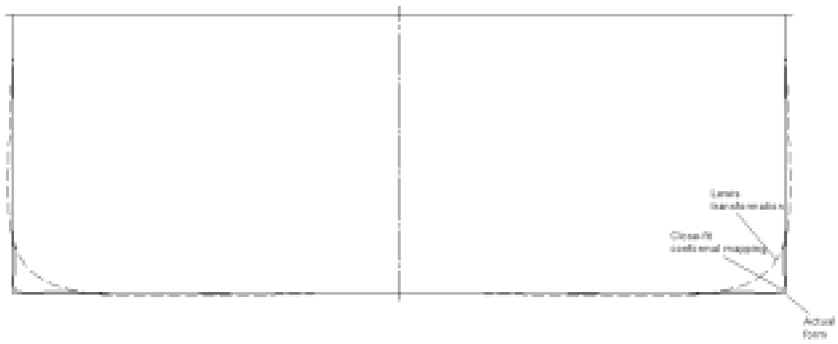

Figure 3.5: Lewis and Close-Fit Conformal Mapping of a Rectangle

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Chapter 4

2-D Potential Coefficients

This chapter described the various methods, used in the SEAWAY program, to obtain the 2-D potential coefficients:

- the theory of Tasai for deep water, based on Ursell's potential theory for circular cylinders and N-parameter conformal mapping
- the theory of Keil for shallow and deep water, based on a variation of Ursell's potential theory for circular cylinders and Lewis conformal mapping
- the theory of Frank for deep water, using pulsating sources on the cross sectional contour.

During the ship motions calculations different coordinate systems, as shown in figure 2.2, will be used. The two-dimensional hydrodynamic potential coefficients have been defined here with respect to the $O(x, y, z)$ coordinate system for the moving ship in still water.
However, in this section deviating axes systems are used for the determination of the two-dimensional hydrodynamic potential coefficients for sway, heave and roll motions. This holds for the sway and roll coupling coefficients a change of sign. The signs of the uncoupled sway, heave and roll coefficients do not change.
For each cross section, the following two-dimensional hydrodynamic coefficients have to be obtained:

M'_{22} and N'_{22}	= 2-D potential mass and damping coefficients of sway
M'_{24} and N'_{24}	= 2-D potential mass and damping coupling coefficients of roll into sway
M'_{33} and N'_{33}	= 2-D potential mass and damping coefficients of heave
M'_{44} and N'_{44}	= 2-D potential mass and damping coefficients of roll
M'_{42} and N'_{42}	= 2-D potential mass and damping coupling coefficients of sway into roll

The 2-D potential pitch and yaw (moment) coefficients follow from the previous heave and sway coefficients and the arm, i.e., the distance of the cross section to the center of gravity G.
Finally, an approximation is given for the determination of the surge coefficients M'_{11} and N'_{11}.

[0] J.M.J. Journée, *"Theoretical Manual of SEAWAY, Release 4.19"*, Report 1216a, February 2001, Ship Hydromechanics Laboratory, Delft University of Technology, Mekelweg 2, 2628 CD Delft, The Netherlands. For updates see web site: http://dutw189.wbmt.tudelft.nl/~johan or http://www.shipmotions.nl.

4.1 Theory of Tasai

In this section, the determination of the hydrodynamic coefficients of a heaving, swaying and rolling cross section of a ship in shallow at zero forward speed is based on work published by [Ursell, 1949], [Tasai, 1959], [Tasai, 1960], [Tasai, 1961] and [Jong, 1973]. Tasai's notations have been maintained here as far as possible.

4.1.1 Heave Motions

The determination of the hydrodynamic coefficients of a heaving cross section of a ship in deep and still water at zero forward speed, as described here, is based on work published by [Ursell, 1949], [Tasai, 1959] and [Tasai, 1960]. Starting points for the derivation these coefficients here are the velocity potentials and the conjugate stream functions of the fluid as they have been derived by Tasai and also by [Jong, 1973].
Suppose an infinite long cylinder in the surface of a fluid, of which a cross section is given in the next figure.

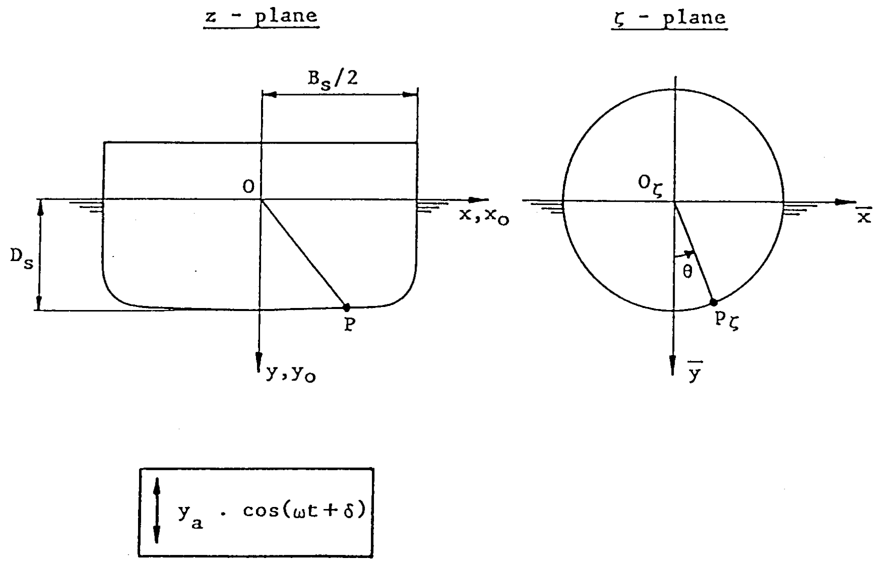

Figure 4.1: Axes System for Heave

The cylinder is forced to carry out a simple harmonic vertical motion about its initial position with a frequency of oscillation ω and a small amplitude of displacement y_a:

$$y = y_a \cos(\omega t + \delta)$$

in which δ is a phase angle.
Respectively, the vertical velocity and acceleration of the cylinder are:

$$\begin{aligned} \dot{y} &= -\omega y_a \sin(\omega t + \delta) \\ \ddot{y} &= -\omega^2 y_a \sin(\omega t + \delta) \end{aligned}$$

This forced vertical oscillation of the cylinder causes a surface disturbance of the fluid.

Because the cylinder is supposed to be infinitely long, the generated waves will be two-dimensional. These waves travel away from the cylinder and a stationary state is rapidly attained.
Two kinds of waves will be produced:

- A standing wave system, denoted here by subscript A. The amplitudes of these waves decrease strongly with the distance to the cylinder.
- A regular progressive wave system, denoted here by subscript B. These waves dissipate energy. At a distance of a few wave lengths from the cylinder, the waves on each side can be described by a single regular wave train.. The wave amplitude at infinity η_a is proportional to the amplitude of oscillation of the cylinder y_a, provided that this amplitude is sufficiently small compared with the radius of the cylinder and the wave length is not much smaller than the diameter of the cylinder.

The two-dimensional velocity potential of the fluid has to fulfil the following requirements:

1. The velocity potential must satisfy to the equation of Laplace:
$$\nabla^2\Phi = \frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} = 0$$
2. Because the heave motion of the fluid is symmetrical about the y-axis, this velocity potential has the following relation:
$$\Phi(-x, y) = \Phi(+x, y)$$
from which follows:
$$\frac{\partial\Phi}{\partial\theta} = 0 \qquad \text{for:} \quad \theta = 0$$
3. The linearized free surface condition in deep water is expressed as follows:
$$\frac{\omega^2}{g}\Phi + \frac{\partial\Phi}{\partial y} = 0 \qquad \text{for:} \quad |x| \geq \frac{B_s}{2} \quad \text{and} \quad y = 0$$

In consequence of the conformal mapping, this free surface condition can be written as:

$$\frac{\xi_b}{\sigma_a}\Phi\sum_{n=0}^{N}\left\{(2n-1)a_{2n-1}e^{-(2n-1)\alpha}\right\} \pm \frac{\partial\Phi}{\partial\theta} = 0 \quad \text{for:} \quad \alpha \geq 0 \quad \text{and} \quad \theta = \pm\frac{\pi}{2}$$

in which:

$$\frac{\xi_b}{\sigma_a} = \frac{\omega^2}{g}M_s \quad \text{or:} \quad \xi_b = \frac{\omega^2 b_0}{2g} \text{ (non-dimensional frequency squared)}$$

From the definition of the velocity potential follows the boundary condition on the surface of the cylinder for $\alpha = 0$:

$$\frac{\partial\Phi_0(\theta)}{\partial n} = \dot{y}\frac{\partial y_0}{\partial n}$$

in which n is the outward normal of the cylinder surface.

Using the stream function Ψ, this boundary condition on the surface of the cylinder (α=0) reduces to:

$$\begin{aligned}
\frac{-\partial \Psi_0(\theta)}{\partial \theta} &= \dot{y}\frac{\partial y_0}{\partial \alpha} \\
&= -\dot{y}M_s \sum_{n=0}^{N} \{(-1)^n(2n-1)a_{2n-1}\cos((2n-1)\theta)\}
\end{aligned}$$

Integration results into the following requirement for the stream function on the surface of the cylinder:

$$\Psi_0(\theta) = \dot{y}M_s \sum_{n=0}^{N} \{(-1)^n a_{2n-1}\sin((2n-1)\theta)\}$$

in which $C(t)$ is a function of the time only.
When defining:

$$h(\theta) \quad = \quad \frac{2x_0}{b_0} \quad = \quad -\frac{1}{\sigma_a}\sum_{n=0}^{N}\{(-1)^n a_{2n-1}\sin((2n-1)\theta)\}$$

the stream function on the surface of the cylinder is given by:

$$\Psi_0(\theta) = -\dot{y}\frac{b_0}{2}h(\theta) + C(t)$$

Because of the symmetry of the fluid about the y-axis, it is clear that:

$$C(t) = 0$$

So:

$$\Psi_0(\theta) = -\dot{y}\frac{b_0}{2}h(\theta)$$

For the standing wave system a velocity potential and a stream function satisfying to the equation of Laplace, the symmetrical motion of the fluid and the free surface condition has to be found.
The following set of velocity potentials, as they are given by [Tasai, 1959], [Tasai, 1960] and [Jong, 1973], fulfil these requirements:

$$\Phi_a = \frac{g\eta_a}{\pi\omega}\left(\sum_{m=1}^{\infty}\left\{P_{2m}\phi_{A_{2m}}(\alpha,\theta)\cos(\omega t)\right\} + \sum_{m=1}^{\infty}\left\{Q_{2m}\phi_{A_{2m}}(\alpha,\theta)\sin(\omega t)\right\}\right)$$

in which:

$$\begin{aligned}
\phi_{A_{2m}}(\alpha,\theta) &= e^{-2m\alpha}\cos(2m\theta) \\
&\quad -\frac{\xi_b}{\sigma_a}\sum_{n=0}^{N}\left\{(-1)^n\frac{2n-1}{2m+2n-1}a_{2n-1}e^{-(2m+2n-1)\alpha}\cos((2m+2n-1)\theta)\right\}
\end{aligned}$$

The set of conjugate stream functions is expressed as:

$$\Psi_A = \frac{g\eta_a}{\pi\omega}\left(\sum_{m=1}^{\infty}\left\{P_{2m}\psi_{A_{2m}}(\alpha,\theta)\cos(\omega t)\right\} + \sum_{m=1}^{\infty}\left\{Q_{2m}\psi_{A_{2m}}(\alpha,\theta)\sin(\omega t)\right\}\right)$$

in which:

$$\begin{aligned}\psi_{A_{2m}}(\alpha,\theta) &= e^{-2m\alpha}\sin(2m\theta) \\ &\quad -\frac{\xi_b}{\sigma_a}\sum_{n=0}^{N}\left\{(-1)^n\frac{2n-1}{2m+2n-1}a_{2n-1}e^{-(2m+2n-1)\alpha}\sin\left((2m+2n-1)\theta\right)\right\}\end{aligned}$$

These sets of functions tend to zero as α tends to infinity.
In these expressions the magnitudes of the P_{2m} and the Q_{2m} series follow from the boundary conditions as will be explained further on.
Another requirement is a diverging wave train for α goes to infinity. It is therefore necessary to add a stream function, satisfying the free surface condition and the symmetry about the y-axis, representing such a train of waves at infinity. For this, a function describing a source at the origin O is chosen.
[Tasai, 1959], [Tasai, 1960] and [Jong, 1973] gave the velocity potential of the progressive wave system as:

$$\Phi_B = \frac{g\eta_a}{\pi\omega}\left(\phi_{B_c}(x,y)\cos(\omega t)+\phi_{B_s}(x,y)\sin(\omega t)\right)$$

in which:

$$\begin{aligned}\phi_{B_c} &= \pi e^{-\nu y}\cos(\nu x) \\ \phi_{B_s} &= \pi e^{-\nu y}\sin(\nu|x|)+\int_0^\infty \frac{\nu\sin(ky)-k\cos(ky)}{k^2+\nu^2}e^{-k|x|}dk\end{aligned}$$

while:

$$\nu = \frac{\omega^2}{g} \quad \text{(wave number for deep water)}$$

Changing the parameters delivers:

$$\Phi_B = \frac{g\eta_a}{\pi\omega}\left(\phi_{B_c}(\alpha,\theta)\cos(\omega t)+\phi_{B_s}(\alpha,\theta)\sin(\omega t)\right)$$

The conjugate stream function is given by:

$$\Psi_B = \frac{g\eta_a}{\pi\omega}\left(\psi_{B_c}(x,y)\cos(\omega t)+\psi_{B_s}(x,y)\sin(\omega t)\right)$$

in which:

$$\begin{aligned}\psi_{B_c} &= \pi e^{-\nu y}\sin(\nu|x|) \\ \psi_{B_s} &= -\pi e^{-\nu y}\cos(\nu x)+\int_0^\infty \frac{\nu\cos(ky)+k\sin(ky)}{k^2+\nu^2}e^{-k|x|}dk\end{aligned}$$

Changing the parameters delivers:

$$\Psi_B = \frac{g\eta_a}{\pi\omega}\left(\psi_{B_c}(\alpha,\theta)\cos(\omega t)+\psi_{B_s}(\alpha,\theta)\sin(\omega t)\right)$$

When calculating the integrals in the expressions for ϕ_{B_s} and ϕ_{B_c} numerically, the convergence is very slowly.

Power series expansions, as given by [Porter, 1960], can be used instead of these last integrals over k. The summation in these expansions converge much faster than the numeric integration procedure. This has been shown before for the sway case.
The total velocity potential and stream function to describe the waves generated by a heaving cylinder are:

$$\begin{aligned}\Phi &= \Phi_A + \Phi_B \\ \Psi &= \Psi_A + \Psi_B\end{aligned}$$

So the velocity potential and the conjugate stream function are expressed by:

$$\begin{aligned}\Phi(\alpha,\theta) &= \frac{g\eta_a}{\pi\omega}\left(\left(\phi_{B_c}(\alpha,\theta) + \sum_{m=1}^{\infty}\left\{P_{2m}\phi_{A_{2m}}(\alpha,\theta)\right\}\right)\cos(\omega t)\right.\\ &\qquad \left. + \left(\phi_{B_s}(\alpha,\theta) + \sum_{m=1}^{\infty}\left\{Q_{2m}\phi_{A_{2m}}(\alpha,\theta)\right\}\right)\sin(\omega t)\right)\\ \Psi(\alpha,\theta) &= \frac{g\eta_a}{\pi\omega}\left(\left(\psi_{B_c}(\alpha,\theta) + \sum_{m=1}^{\infty}\left\{P_{2m}\psi_{A_{2m}}(\alpha,\theta)\right\}\right)\cos(\omega t)\right.\\ &\qquad \left. + \left(\psi_{B_s}(\alpha,\theta) + \sum_{m=1}^{\infty}\left\{Q_{2m}\psi_{A_{2m}}(\alpha,\theta)\right\}\right)\sin(\omega t)\right)\end{aligned}$$

When putting $\alpha = 0$, the stream function is equal to the expression found before from the boundary condition on the surface of the cylinder:

$$\begin{aligned}\Psi_0(\theta) &= \frac{g\eta_a}{\pi\omega}\left(\left(\psi_{B0_c}(\theta) + \sum_{m=1}^{\infty}\left\{P_{2m}\psi_{A0_{2m}}(\theta)\right\}\right)\cos(\omega t)\right.\\ &\qquad \left. + \left(\psi_{B0_s}(\theta) + \sum_{m=1}^{\infty}\left\{Q_{2m}\psi_{A0_{2m}}(\theta)\right\}\right)\sin(\omega t)\right)\\ &= -\dot{y}\frac{b_0}{2}h(\theta)\end{aligned}$$

in which:

$$\begin{aligned}\psi_{A0_{2m}}(\theta) &= \sin(2m\theta)\\ &\quad -\frac{\xi_b}{\sigma_a}\sum_{n=0}^{N}\left\{(-1)^n\frac{2n-1}{2m+2n-1}a_{2n-1}\sin\left((2m+2n-1)\theta\right)\right\}\end{aligned}$$

In this expression $\psi_{B0_c}(\theta)$ and $\psi_{B0_s}(\theta)$ are the values of $\psi B_c(\alpha,\theta)$ and $\psi B_s(\alpha,\theta)$ at the surface of the cylinder, so for $\alpha = 0$.
So for each θ, the following equation has been obtained:

$$\begin{aligned}&\left(\psi_{B0_c}(\theta) + \sum_{m=1}^{\infty}\left\{P_{2m}\psi_{A0_{2m}}(\theta)\right\}\right)\cos(\omega t)\\ &+\left(\psi_{B0_s}(\theta) + \sum_{m=1}^{\infty}\left\{Q_{2m}\psi_{A0_{2m}}(\theta)\right\}\right)\sin(\omega t) = -\dot{y}\frac{\pi\omega b_0}{2g\eta_a}h(\theta)\end{aligned}$$

The right hand side of this equation can be written as:

$$\begin{aligned}-\dot{y}\frac{\pi\omega b_0}{2g\eta_a}h(\theta) &= h(\theta)\frac{y_a}{\eta_a}\pi\xi_b\sin(\omega t+\delta)\\ &= h(\theta)\left(A_0\cos(\omega t)+B_0\sin(\omega t)\right)\end{aligned}$$

in which:

$$A_0 = \frac{y_a}{\eta_a}\pi\xi_b\sin\delta \qquad \text{and} \qquad B_0 = \frac{y_a}{\eta_a}\pi\xi_b\cos\delta$$

This results for each θ into a set of two equations:

$$\begin{aligned}\psi_{B0_c}(\theta)+\sum_{m=1}^{\infty}\left\{P_{2m}\psi_{A0_{2m}}(\theta)\right\} &= h(\theta)A_0\\ \psi_{B0_s}(\theta)+\sum_{m=1}^{\infty}\left\{Q_{2m}\psi_{A0_{2m}}\theta)\right\} &= h(\theta)B_0\end{aligned}$$

When putting $\theta=\pi/2$, so at the intersection of the surface of the cylinder with the free surface of the fluid where $h(\theta)$=1, we obtain the coefficients A_0 and B_0:

$$\begin{aligned}A_0 &= \psi_{B0_c}(\pi/2)+\sum_{m=1}^{\infty}\left\{P_{2m}\psi_{A0_{2m}}(\pi/2)\right\}\\ B_0 &= \psi_{B0_s}(\pi/2)+\sum_{m=1}^{\infty}\left\{Q_{2m}\psi_{A0_{2m}}(\pi/2)\right\}\end{aligned}$$

in which:

$$\psi_{A0_{2m}}(\pi/2)=\frac{\xi_b}{\sigma_a}(-1)^m\sum_{n=0}^{N}\left\{\frac{2n-1}{2m+2n-1}a_{2n-1}\right\}$$

A substitution of A_0 and B_0 into the set of two equations for each θ, results for any θ-value less than $\pi/2$ into a set of two equations with the unknown parameters P_{2m} and Q_{2m}.
So:

$$\begin{aligned}\psi_{B0_c}(\theta)-h(\theta)\psi_{B0_c}(\pi/2) &= \sum_{m=1}^{\infty}\left\{f_{2m}(\theta)P_{2m}\right\}\\ \psi_{B0_s}(\theta)-h(\theta)\psi_{B0_s}(\pi/2) &= \sum_{m=1}^{\infty}\left\{f_{2m}(\theta)Q_{2m}\right\}\end{aligned}$$

in which:

$$f_{2m}(\theta)=-\psi_{A0_{2m}}(\theta)+h(\theta)\psi_{A0_{2m}}(\pi/2)$$

The series in these two sets of equations converges uniformly with an increasing value of m. For practical reasons the maximum value of m is limited to M.
Each θ-value less than $\pi/2$ will deliver an equation for the P_{2m} and Q_{2m} series. For a lot of θ-values, the best fit values of P_{2m} and Q_{2m} are supposed to be those found by means of a least squares method. Note that at least M values of θ, less than $\pi/2$, are required to solve these equations.
Another favorable method is to multiply both sides of the equations with $\Delta\theta$. Then the summations over θ can be replaced by integrations.

Herewith, two sets of M equations have been obtained, one set for P_{2m} and one set for Q_{2m}:

$$\sum_{m=1}^{M}\left\{P_{2m}\int_{0}^{\pi/2} f_{2m}(\theta)f_{2n}(\theta)d\theta\right\} = \int_{0}^{\pi/2}\left(\psi_{B0_c}(\theta)-h(\theta)\psi_{B0_c}(\pi/2)\right)f_{2n}(\theta)d\theta \qquad n=1,...M$$

$$\sum_{m=1}^{M}\left\{Q_{2m}\int_{0}^{\pi/2} f_{2m}(\theta)f_{2n}(\theta)d\theta\right\} = \int_{0}^{\pi/2}\left(\psi_{B0_s}(\theta)-h(\theta)\psi_{B0_s}(\pi/2)\right)f_{2n}(\theta)d\theta \qquad n=1,...M$$

Now the P_{2m} and Q_{2m} series can be solved by a numerical method.
With these P_{2m} and Q_{2m} values, the coefficients A_0 and B_0 are known too.
From the definition of these coefficients follows the amplitude ratio of the radiated waves and the forced heave oscillation:

$$\frac{\eta_a}{y_a} = \frac{\pi\xi_b}{\sqrt{A_0^2+B_0^2}}$$

With the solved P_{2m} and Q_{2m} values, the velocity potential on the surface of the cylinder (α=0) is known too:

$$\begin{aligned}\Phi_0(\theta) &= \frac{g\eta_a}{\pi\omega}\left(\left(\phi_{B0_c}(\theta)+\sum_{m=1}^{M}\left\{P_{2m}\phi_{A0_{2m}}(\theta)\right\}\right)\cos(\omega t)\right.\\ &\left.\quad+\left(\phi_{B0_s}(\theta)+\sum_{m=1}^{M}\left\{Q_{2m}\phi_{A0_{2m}}(\theta)\right\}\right)\sin(\omega t)\right)\end{aligned}$$

in which:

$$\begin{aligned}\phi_{A0_{2m}}(\theta) &= \cos(2m\theta)\\ &\quad-\frac{\xi_b}{\sigma_a}\sum_{n=0}^{N}\left\{(-1)^n\frac{2n-1}{2m+2n-1}a_{2n-1}\cos\left((2m+2n-1)\theta\right)\right\}\end{aligned}$$

In this expression $\phi_{B0_c}(\theta)$ and $\phi_{B0_s}(\theta)$ are the values of $\phi_{B_c}(\alpha,\theta)$ and $\phi_{B_s}(\alpha,\theta)$ at the surface of the cylinder, so for $\alpha=0$.
Now the hydrodynamic pressure on the surface of the cylinder can be obtained from the linearized equation of Bernoulli:

$$\begin{aligned}p(\theta) &= -\rho\frac{\partial\Phi_0(\theta)}{\partial t}\\ &= \frac{-\rho g\eta_a}{\pi}\left(\left(\phi_{B0_s}(\theta)+\sum_{m=1}^{M}\left\{Q_{2m}\phi_{A0_{2m}}(\theta)\right\}\right)\cos(\omega t)\right.\\ &\left.\quad-\left(\phi_{B0_c}(\theta)+\sum_{m=1}^{M}\left\{P_{2m}\phi_{A0_{2m}}(\theta)\right\}\right)\sin(\omega t)\right)\end{aligned}$$

It is obvious that this pressure is symmetric in θ.

Heave Coefficients

The two-dimensional hydrodynamic vertical force, acting on the cylinder in the direction of the y-axis, can be found by integrating the vertical component of the hydrodynamic pressure on the surface of the cylinder:

$$\begin{aligned} F'_y &= -\int_{-\pi/2}^{+\pi/2} p(\theta)\frac{dx_0}{ds}ds \\ &= -2\int_0^{\pi/2} p(\theta)\frac{dx_0}{d\theta}d\theta \end{aligned}$$

With this the two-dimensional hydrodynamic vertical force due to heave oscillations can be written as follows:

$$F'_y = \frac{\rho g b_0 \eta_a}{\pi}\left(M_0\cos(\omega t) - N_0\sin(\omega t)\right)$$

in which:

$$\begin{aligned} M_0 &= -\frac{1}{\sigma_a}\int_0^{\pi/2}\phi_{B0_s}(\theta)\sum_{n=0}^{N}\left\{(-1)^n(2n-1)a_{2n-1}\cos\left((2n-1)\theta\right)\right\}d\theta \\ &\quad -\frac{1}{\sigma_a}\sum_{m=1}^{M}\left\{(-1)^m Q_{2m}\sum_{n=0}^{N}\left\{\frac{(2n-1)^2}{(2m)^2-(2n-1)^2}a_{2n-1}\right\}\right\} \\ &\quad +\frac{\pi\xi_b}{4\sigma_a^2}\left(Q_2+\sum_{m=1}^{N}\left\{(-1)^m Q_{2m}\sum_{n=0}^{N-m}\left\{(2n-1)a_{2n-1}a_{2m+2n-1}\right\}\right\}\right) \end{aligned}$$

and N_0 as obtained from this expression for M_0, by replacing there $\phi_{B0_s}(\theta)$ by $\phi_{B0_c}(\theta)$ and Q_{2m} by P_{2m}.
For the determination of M_0 and N_0 it is advised: $M \geqq N$.
These expressions coincide with those given by [Tasai, 1960].
With:

$$F'_y = \frac{\rho g b_0 \eta_a}{\pi}\left(M_0\cos(\omega t+\delta-\delta) - N_0\sin(\omega t+\delta-\delta)\right)$$

and:

$$\sin\delta = \frac{\eta_a}{y_a\pi\xi_b}A_0 \qquad\qquad \cos\delta = \frac{\eta_a}{y_a\pi\xi_b}B_0$$

the two-dimensional hydrodynamic vertical force can be resolved into components in phase and out phase with the vertical displacement of the cylinder:

$$F'_y = \frac{\rho g b_0 \eta_a^2}{\pi^2\xi_b y_a}\left((M_0B_0+N_0A_0)\cos(\omega t+\delta) + (M_0A_0-N_0B_0)\sin(\omega t+\delta)\right)$$

This hydrodynamic vertical force can also be written as:

$$\begin{aligned} F'_y &= -M'_{33}\ddot{y} - N'_{33}\dot{y} \\ &= M'_{33}\omega^2 y_a\cos(\omega t+\delta) + N'_{33}\omega y_a\sin(\omega t+\delta) \end{aligned}$$

in which:

M'_{33} = 2-D hydrodynamic mass coefficient of heave
N'_{33} = 2-D hydrodynamic damping coefficient of heave

When using also the amplitude ratio of the radiated waves and the forced heave oscillation, found before, the two-dimensional hydrodynamic mass and damping coefficients of heave are given by:

$$M'_{33} = \frac{\rho b_0^2}{2} \cdot \frac{M_0 B_0 + N_0 A_0}{A_0^2 + B_0^2} \qquad \text{and} \qquad N'_{33} = \frac{\rho b_0^2}{2} \cdot \frac{M_0 A_0 - N_0 B_0}{A_0^2 + B_0^2} \cdot \omega$$

The signs of these two coefficients are proper in both, the axes system of Tasai and the ship motions $O(x, y, z)$ coordinate system.
The energy delivered by the exciting forces should be equal to the energy radiated by the waves.
So:

$$\frac{1}{T_{osc}} \int_0^{T_{osc}} N'_{33}\dot{y} \cdot \dot{y} dt = \frac{\rho g \eta_a^2 c}{2}$$

in which T_{osc} is the period of oscillation.
With the relation for the wave speed $c = g/\omega$, follows the relation between the two-dimensional heave damping coefficient and the amplitude ratio of the radiated waves and the forced heave oscillation:

$$N'_{33} = \frac{\rho g^2}{\omega^3} \left(\frac{\eta_a}{y_a} \right)^2$$

With this amplitude ratio the two-dimensional hydrodynamic damping coefficient of heave is also given by:

$$N'_{33} = \frac{\rho \pi^2 b_0^2}{4} \cdot \frac{1}{A_0^2 + B_0^2} \cdot \omega$$

When comparing this expression for N'_{33} with the expression found before, the following energy balance relation is found:

$$M_0 A_0 - N_0 B_0 = \frac{\pi^2}{2}$$

4.1.2 Sway Motions

The determination of the hydrodynamic coefficients of a swaying cross section of a ship in deep and still water at zero forward speed, is based here on work published by [Tasai, 1961] for the Lewis method. Starting points for the derivation these coefficients here are the velocity potentials and the conjugate stream functions of the fluid as they have been derived by Tasai and also by [Jong, 1973].
Suppose an infinite long cylinder in the surface of a fluid, of which a cross section is given in figure 4.2.

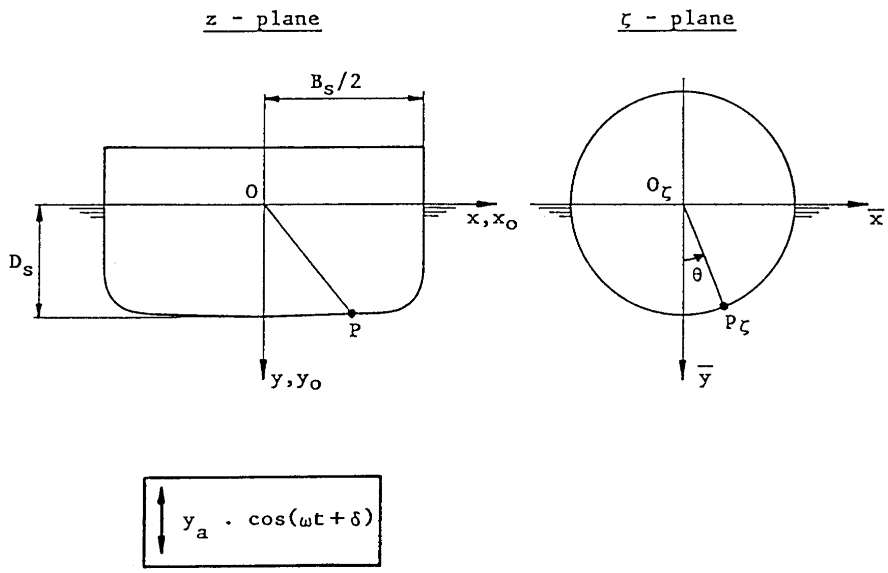

Figure 4.2: Axes System for Sway

The cylinder is forced to carry out a simple harmonic lateral motion about its initial position with a frequency of oscillation ω and a small amplitude of displacement x_a:

$$x = x_a \cos(\omega t + \varepsilon)$$

in which ε is a phase angle.
Respectively, the lateral velocity and acceleration of the cylinder are:

$$\dot{x} = -\omega x_a \sin(\omega t + \varepsilon) \qquad \text{and} \qquad \ddot{x} = -\omega^2 x_a \cos(\omega t + \varepsilon)$$

This forced lateral oscillation of the cylinder causes a surface disturbance of the fluid. Because the cylinder is supposed to be infinitely long, the generated waves will be two-dimensional. These waves travel away from the cylinder and a stationary state is rapidly attained.
Two kinds of waves will be produced:

- A standing wave system, denoted here by subscript A. The amplitudes of these waves decrease strongly with the distance to the cylinder.

- A regular progressive wave system, denoted here by subscript B. These waves dissipate energy. At a distance of a few wave lengths from the cylinder, the waves on each side can be described by a single regular wave train.. The wave amplitude at infinity η_a is proportional to the amplitude of oscillation of the cylinder x_a, provided that this amplitude is sufficiently small compared with the radius of the cylinder and the wave length is not much smaller than the diameter of the cylinder.

The two-dimensional velocity potential of the fluid has to fulfil the following three requirements:

1. The velocity potential must satisfy to the equation of Laplace:

$$\nabla^2 \Phi = \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = 0$$

2. Because the sway motion of the fluid is not symmetrical about the y-axis, this velocity potential has the following anti-symmetric relation:

$$\Phi(-x, y) = -\Phi(+x, y)$$

3. The linearized free surface condition in deep water is expressed as follows:

$$\frac{\omega^2}{g}\Phi + \frac{\partial \Phi}{\partial y} = 0 \qquad \text{for: } |x| \geq \frac{B_s}{2} \quad \text{and} \quad y = 0$$

In consequence of the conformal mapping, this last equation results into:

$$\frac{\xi_b}{\sigma_a}\Phi \sum_{n=0}^{N} \left\{(2n-1)a_{2n-1}e^{-(2n-1)\alpha}\right\} \pm \frac{\partial \Phi}{\partial \theta} = 0 \quad \text{for:} \quad \alpha \geq 0 \quad \text{and} \quad \theta = \pm\frac{\pi}{2}$$

in which:

$$\frac{\xi_b}{\sigma_a} = \frac{\omega^2}{g} M_s \quad \text{or:} \quad \xi_b = \frac{\omega^2 b_0}{2g} \text{ (non-demensional frequency squared)}$$

From the definition of the velocity potential follows the boundary condition on the surface S of the cylinder for $\alpha = 0$:

$$\frac{\partial \Phi_0(\theta)}{\partial n} = \dot{x}\frac{\partial x_0}{\partial n}$$

in which n is the outward normal of the cylinder surface S.
Using the stream function Ψ this boundary condition on the surface of the cylinder ($\alpha = 0$) reduces to:

$$\begin{aligned} \frac{\partial \Psi_0(\theta)}{\partial \theta} &= -\dot{x}\frac{\partial x_0}{\partial \alpha} \\ &= -\dot{x}M_s \sum_{n=0}^{N} \left\{(-1)^n (2n-1)a_{2n-1} \sin\left((2n-1)\theta\right)\right\} \end{aligned}$$

Integration results into the following requirement for the stream function on the surface of the cylinder:

$$\Psi_0(\theta) = \dot{x} M_s \sum_{n=0}^{N} \{(-1)^n a_{2n-1} \cos((2n-1)\theta)\} + C(t)$$

in which $C(t)$ is a function of the time only.
When defining:

$$g(\theta) \quad = \quad \frac{2y_0}{b_0} \quad = \quad \frac{1}{\sigma_a} \sum_{n=0}^{N} \{(-1)^n a_{2n-1} \cos((2n-1)\theta)\}$$

the stream function on the surface of the cylinder is given by:

$$\Psi_0(\theta) = \dot{x} \frac{b_0}{2} g(\theta) + C(t)$$

For the standing wave system a velocity potential and a stream function satisfying to the equation of Laplace, the non-symmetrical motion of the fluid and the free surface condition has to be found.
The following set of velocity potentials, as they are given by [Tasai, 1961] and [Jong, 1973], fulfil these requirements:

$$\Phi_A = \frac{g\eta_a}{\pi\omega} \left(\sum_{m=1}^{\infty} \left\{ P_{2m} \phi_{A_{2m}}(\alpha, \theta) \cos(\omega t) \right\} + \sum_{m=1}^{\infty} \left\{ Q_{2m} \phi_{A_{2m}}(\alpha, \theta) \sin(\omega t) \right\} \right)$$

in which:

$$\begin{aligned} \phi_{A_{2m}}(\alpha, \theta) \quad &= \quad e^{-(2m+1)\alpha} \sin((2m+1)\theta) \\ &\quad -\frac{\xi_b}{\sigma_a} \sum_{n=0}^{N} \left\{ (-1)^n \frac{2n-1}{2m+2n} a_{2n-1} e^{-(2m+2n)\alpha} \sin((2m+2n)\theta) \right\} \end{aligned}$$

The set of conjugate stream functions is expressed as:

$$\Psi_A = \frac{g\eta_a}{\pi\omega} \left(\sum_{m=1}^{\infty} \left\{ P_{2m} \psi_{A_{2m}}(\alpha, \theta) \cos(\omega t) \right\} + \sum_{m=1}^{\infty} \left\{ Q_{2m} \psi_{A_{2m}}(\alpha, \theta) \sin(\omega t) \right\} \right)$$

in which:

$$\begin{aligned} \psi_{A_{2m}}(\alpha, \theta) \quad &= \quad -e^{-(2m+1)\alpha} \cos((2m+1)\theta) \\ &\quad +\frac{\xi_b}{\sigma_a} \sum_{n=0}^{N} \left\{ (-1)^n \frac{2n-1}{2m+2n} a_{2n-1} e^{-(2m+2n)\alpha} \cos((2m+2n)\theta) \right\} \end{aligned}$$

These sets of functions tend to zero as α tends to infinity.
In these expressions the magnitudes of the P_{2m} and the Q_{2m} series follow from the boundary conditions, as will be explained further on.
Another requirement is a diverging wave train for α goes to infinity. Therefore it is necessary to add a stream function, satisfying the equation of Laplace, the non-symmetrical motion of the fluid and the free surface condition, representing such a train of waves at

infinity. For this, a function describing a two-dimensional horizontal doublet at the origin O is chosen.
[Tasai, 1961] and [Jong, 1973] gave the velocity potential of the progressive wave system as:

$$\Phi_B = \frac{g\eta_a}{\pi\omega}\left(\phi_{B_c}(x,y)\cos(\omega t) + \phi_{B_s}(x,y)\sin(\omega t)\right)$$

in which:

$$\begin{aligned}
\phi_{B_c} \cdot j &= -\pi e^{-\nu y}\sin(\nu|x|) \\
\phi_{B_s} \cdot j &= \pi e^{-\nu y}\cos(\nu x) - \int_0^\infty \frac{\nu\cos(ky) + k\sin(ky)}{k^2+\nu^2} e^{-k|x|} dk + \frac{|x|}{\nu(x^2+y^2)}
\end{aligned}$$

while:

$$\begin{aligned}
j &= +1 \quad \text{for } x > 0 \\
j &= -1 \quad \text{for } x < 0 \\
\nu &= \frac{\omega^2}{g} \quad \text{(wave number for deep water)}
\end{aligned}$$

Changing the parameters delivers:

$$\Phi_B = \frac{g\eta_a}{\pi\omega}\left(\phi_{B_c}(\alpha,\theta)\cos(\omega t) + \phi_{B_s}(\alpha,\theta)\sin(\omega t)\right)$$

The conjugate stream function is given by:

$$\Psi_B = \frac{g\eta_a}{\pi\omega}\left(\psi_{B_c}(x,y)\cos(\omega t) + \psi_{B_s}(x,y)\sin(\omega t)\right)$$

in which:

$$\begin{aligned}
\psi_{B_c} &= \pi e^{-\nu y}\cos(\nu x) \\
\psi_{B_s} &= \pi e^{-\nu y}\sin(\nu|x|) + \int_0^\infty \frac{\nu\sin(ky) - k\cos(ky)}{k^2+\nu^2} e^{-k|x|} dk - \frac{y}{\nu(x^2+y^2)}
\end{aligned}$$

Changing the parameters delivers:

$$\Psi_B = \frac{g\eta_a}{\pi\omega}\left(\psi_{B_c}(\alpha,\theta)\cos(\omega t) + \psi_{B_s}(\alpha,\theta)\sin(\omega t)\right)$$

When calculating the integrals in the expressions for ψ_{B_s} and ϕ_{B_s} numerically, the convergence is very slowly.
Power series expansions, as given by [Porter, 1960], can be used instead of these last integrals over k:

$$\begin{aligned}
\int_0^\infty \frac{\nu\cos(ky) + k\sin(ky)}{k^2+\nu^2} e^{-kx} dk &= \left(Q\sin(\nu x) - (S-\pi)\cos(\nu x)\right) e^{-\nu y} \\
\int_0^\infty \frac{\nu\sin(ky) - k\cos(ky)}{k^2+\nu^2} e^{-kx} dk &= \left(Q\cos(\nu x) + (S-\pi)\sin(\nu x)\right) e^{-\nu y}
\end{aligned}$$

in which:

$$\begin{aligned}
Q &= \gamma + \ln\left(\nu\sqrt{x^2+y^2}\right) + \sum_{n=1}^{\infty}\{p_n\cos(n\beta)\} \\
S &= \beta + \sum_{n=1}^{\infty}\{p_n\sin(n\beta)\} \\
\beta &= \arctan\left(\frac{x}{y}\right) \\
p_n &= \frac{\left(\nu\sqrt{x^2+y^2}\right)^n}{n\cdot n!} \\
\gamma &= 0.5772156649\text{...... (Euler constant)}
\end{aligned}$$

The summation in these expansions converge much faster than the numeric integration procedure. A suitable maximum value of n should be chosen.
The total velocity potential and stream function to describe the waves generated by a swaying cylinder are:

$$\begin{aligned}
\Phi &= \Phi_A + \Phi_B \\
\Psi &= \Psi_A + \Psi_B
\end{aligned}$$

So the velocity potential and the conjugate stream function are expressed by:

$$\begin{aligned}
\Phi(\alpha,\theta) &= \frac{g\eta_a}{\pi\omega}\left(\left(\phi_{B_c}(\alpha,\theta) + \sum_{m=1}^{\infty}\left\{P_{2m}\phi_{A_{2m}}(\alpha,\theta)\right\}\right)\cos(\omega t)\right. \\
&\quad \left. + \left(\phi_{B_s}(\alpha,\theta) + \sum_{m=1}^{\infty}\left\{Q_{2m}\phi_{A_{2m}}(\alpha,\theta)\right\}\right)\sin(\omega t)\right) \\
\Psi(\alpha,\theta) &= \frac{g\eta_a}{\pi\omega}\left(\left(\psi_{B_c}(\alpha,\theta) + \sum_{m=1}^{\infty}\left\{P_{2m}\psi_{A_{2m}}(\alpha,\theta)\right\}\right)\cos(\omega t)\right. \\
&\quad \left. + \left(\psi_{B_s}(\alpha,\theta) + \sum_{m=1}^{\infty}\left\{Q_{2m}\psi_{A_{2m}}(\alpha,\theta)\right\}\right)\sin(\omega t)\right)
\end{aligned}$$

When putting $\alpha = 0$, the stream function is equal to the expression found before from the boundary condition on the surface of the cylinder:

$$\begin{aligned}
\Psi_0(\theta) &= \frac{g\eta_a}{\pi\omega}\left(\left(\psi_{B0_c}(\theta) + \sum_{m=1}^{\infty}\left\{P_{2m}\psi_{A0_{2m}}(\theta)\right\}\right)\cos(\omega t)\right. \\
&\quad \left. + \left(\psi_{B0_s}(\theta) + \sum_{m=1}^{\infty}\left\{Q_{2m}\psi_{A0_{2m}}(\theta)\right\}\right)\sin(\omega t)\right) \\
&= \dot{x}\frac{b_0}{2}g(\theta) + C(t)
\end{aligned}$$

in which:

$$\begin{aligned}
\psi_{A0_{2m}}(\theta) &= -\cos\left((2m+1)\theta\right) \\
&\quad + \frac{\xi_b}{\sigma_a}\sum_{n=0}^{N}\left\{(-1)^n\frac{2n-1}{2m+2n}a_{2n-1}\cos\left((2m+2n)\theta\right)\right\}
\end{aligned}$$

In this expression $\Psi_{B0_c}(\theta)$ and $\Psi_{B0_s}(\theta)$ are the values of $\Psi_{B_c}(\alpha,\theta)$ and $\Psi_{B_s}(\alpha,\theta)$ at the surface of the cylinder, so for $\alpha = 0$.
So for each θ-value, the following equation has been obtained:

$$\begin{aligned}\left(\psi_{B0_c}(\theta) + \sum_{m=1}^{\infty}\left\{P_{2m}\psi_{A0_{2m}}(\theta)\right\}\right)\cos(\omega t) + & \\ \left(\psi_{B0_s}(\theta) + \sum_{m=1}^{\infty}\left\{Q_{2m}\psi_{A0_{2m}}(\theta)\right\}\right)\sin(\omega t) &= \dot{x}\frac{\pi\omega b_0}{2g\eta_a}g(\theta) + C^*(t)\end{aligned}$$

When putting $\theta = \pi/2$, so at the intersection of the surface of the cylinder with the free surface of the fluid where $g(\theta) = 0$, we obtain the constant $C^*(t)$:

$$\begin{aligned}C^*(t) &= \left(\psi_{B0_c}(\pi/2) + \sum_{m=1}^{\infty}\left\{P_{2m}\psi_{A0_{2m}}(\pi/2)\right\}\right)\cos(\omega t) + \\ &\quad \left(\psi_{B0_s}(\pi/2) + \sum_{m=1}^{\infty}\left\{Q_{2m}\psi_{A0_{2m}}(\pi/2)\right\}\right)\sin(\omega t)\end{aligned}$$

in which:

$$\Psi_{A0_{2m}}(\pi/2) = \frac{\xi_b}{\sigma_a}(-1)^m\sum_{n=0}^{N}\left\{\frac{2n-1}{2m+2n}a_{2n-1}\right\}$$

A substitution of $C^*(t)$ in the equation for each θ-value, results for any θ-value less than $\pi/2$ into the following equation:

$$\begin{aligned}\left(\psi_{B0_c}(\theta) - \psi_{B0_c}(\pi/2) + \sum_{m=1}^{\infty}\left\{P_{2m}\left(\psi_{A0_{2m}}(\theta) - \psi_{A0_{2m}}(\pi/2)\right)\right\}\right)\cos(\omega t) + & \\ \left(\psi_{B0_s}(\theta) - \psi_{B0_s}(\pi/2) + \sum_{m=1}^{\infty}\left\{Q_{2m}\left(\psi_{A0_{2m}}(\theta) - \psi_{A0_{2m}}(\pi/2)\right)\right\}\right)\sin(\omega t) &= \dot{x}\frac{\pi\omega}{g\eta_a}\frac{b_0}{2}g(\theta)\end{aligned}$$

The right hand side of this equation can be written as:

$$\begin{aligned}\dot{x}\frac{\pi\omega b_0}{2g\eta_a}g(\theta) &= g(\theta)\left(-\frac{x_a}{\eta_a}\pi\xi_b\sin(\omega t + \varepsilon)\right) \\ &= g(\theta)\left(P_0\cos(\omega t) + Q_0\sin(\omega t)\right)\end{aligned}$$

in which:

$$P_0 = -\frac{x_a}{\eta_a}\pi\xi_b\sin\varepsilon \qquad \text{and} \qquad Q_0 = -\frac{x_a}{\eta_a}\pi\xi_b\cos\varepsilon$$

This delivers for any θ-value less than $\pi/2$ two sets of equations with the unknown parameters P_{2m} and Q_{2m}.
So:

$$\begin{aligned}\psi_{B0_c}(\theta) - \psi_{B0_c}(\pi/2) &= g(\theta)P_0 + \sum_{m=1}^{\infty}\left\{f_{2m}(\theta)P_{2m}\right\} \\ \psi_{B0_s}(\theta) - \psi_{B0_s}(\pi/2) &= g(\theta)Q_0 + \sum_{m=1}^{\infty}\left\{f_{2m}(\theta)Q_{2m}\right\}\end{aligned}$$

in which:

$$f_{2m}(\theta) = -\psi_{A0_{2m}}(\theta) + \psi_{A0_{2m}}(\pi/2)$$

These equations can also be written as:

$$\begin{aligned}
\psi_{B0_c}(\theta) - \psi_{B0_c}(\pi/2) &= \sum_{m=0}^{\infty} \{f_{2m}(\theta) P_{2m}\} \\
\psi_{B0_s}(\theta) - \psi_{B0_s}(\pi/2) &= \sum_{m=0}^{\infty} \{f_{2m}(\theta) Q_{2m}\}
\end{aligned}$$

in which:

$$\begin{aligned}
\text{for } m &= 0: \qquad f_0(\theta) = g(\theta) \\
\text{for } m &> 0: \qquad f_{2m}(\theta) = -\psi_{A0_{2m}}(\theta) + \psi_{A0_{2m}}(\pi/2)
\end{aligned}$$

The series in these two sets of equations converges uniformly with an increasing value of m. For practical reasons the maximum value of m is limited to M.
Each θ-value less than $\pi/2$ will deliver an equation for the P_{2m} and Q_{2m} series. The best fit values of P_{2m} and Q_{2m} are supposed to be those found by means of a least squares method. Note that at least $M+1$ values of θ, less than $\pi/2$, are required to solve these equations. Another favorable method is to multiply both sides of the equations with $\Delta\theta$. Then the summations over θ can be replaced by integrations.
Herewith, two sets of $M+1$ equations have been obtained, one set for P_{2m} and one set for Q_{2m}:

$$\begin{aligned}
\sum_{m=0}^{M} \left\{ P_{2m} \int_0^{\pi/2} f_{2m}(\theta) f_{2n}(\theta) d\theta \right\} &= \int_0^{\pi/2} \left(\psi_{B0_c}(\theta) - \psi_{B0_c}(\pi/2)\right) f_{2n}(\theta) d\theta \qquad n = 0, ...M \\
\sum_{m=0}^{M} \left\{ Q_{2m} \int_0^{\pi/2} f_{2m}(\theta) f_{2n}(\theta) d\theta \right\} &= \int_0^{\pi/2} \left(\psi_{B0_s}(\theta) - \psi_{B0_s}(\pi/2)\right) f_{2n}(\theta) d\theta \qquad n = 0, ...M
\end{aligned}$$

Now the P_{2m} and Q_{2m} series can be solved with a numerical method. Then P_0 and Q_0 are known now and from the definition of these coefficients follows the amplitude ratio of the radiated waves and the forced sway oscillation:

$$\frac{\eta_a}{x_a} = \frac{\pi \xi_b}{\sqrt{P_0^2 + Q_0^2}}$$

With the solved P_{2m} and Q_{2m} values, the velocity potential on the surface of the cylinder ($\alpha = 0$) is known too:

$$\begin{aligned}
\Phi_0(\theta) = \frac{g\eta_a}{\pi\omega} (&\left(\left(\phi_{B0_c}(\theta) + \sum_{m=1}^{\infty} \left\{ P_{2m} \phi_{A0_{2m}}(\theta) \right\} \right) \cos(\omega t) \right. \\
&\left. + \left(\phi_{B0_s}(\theta) + \sum_{m=1}^{\infty} \left\{ Q_{2m} \phi_{A0_{2m}}(\theta) \right\} \right) \sin(\omega t) \right)
\end{aligned}$$

in which:

$$\begin{aligned}\phi_{A0_{2m}}(\theta) &= \sin\left((2m+1)\theta\right) \\ &\quad -\frac{\xi_b}{\sigma_a}\sum_{n=0}^{N}\left\{(-1)^n\frac{2n-1}{2m+2n}a_{2n-1}\sin\left((2m+2n)\theta\right)\right\}\end{aligned}$$

In this expression $\phi_{B0_c}(\theta)$ and $\phi_{B0_s}(\theta)$ are the values of $\phi_{B_c}(\alpha,\theta)$ and $\phi_{B_s}(\alpha,\theta)$ at the surface of the cylinder, so for $\alpha = 0$.
Now the hydrodynamic pressure on the surface of the cylinder can be obtained from the linearized equation of Bernoulli:

$$\begin{aligned}p(\theta) &= -\rho\frac{\partial\Phi_0(\theta)}{\partial t} \\ &= \frac{-\rho g\eta_a}{\pi}\left(\left(\phi_{B0_s}(\theta)+\sum_{m=1}^{\infty}\left\{Q_{2m}\phi_{A0_{2m}}(\theta)\right\}\right)\cos(\omega t)\right. \\ &\quad \left. -\left(\phi_{B0_c}(\theta)+\sum_{m=1}^{\infty}\left\{P_{2m}\phi_{A0_{2m}}(\theta)\right\}\right)\sin(\omega t)\right)\end{aligned}$$

It is obvious that this pressure is skew-symmetric in θ.

Sway Coefficients

The two-dimensional hydrodynamic lateral force, acting on the cylinder in the direction of the $x-$axis, can be found by integrating the lateral component of the hydrodynamic pressure on the surface S of the cylinder:

$$\begin{aligned}F'_x &= -\int_0^{\pi/2}\left(p(+\theta)-p(-\theta)\right)\frac{-dy_0}{ds}ds \\ &= 2\int_0^{\pi/2}p(\theta)\frac{dy_0}{d\theta}d\theta\end{aligned}$$

With this the two-dimensional hydrodynamic lateral force due to sway oscillations can be written as follows:

$$F'_x = \frac{-\rho g b_0\eta_a}{\pi}\left(M_0\cos(\omega t) - N_0\sin(\omega t)\right)$$

in which:

$$\begin{aligned}M_0 &= -\frac{1}{\sigma_a}\int_0^{\pi/2}\phi_{B0_s}(\theta)\sum_{n=0}^{N}\left\{(-1)^n(2n-1)a_{2n-1}\sin\left((2n-1)\theta\right)\right\}d\theta \\ &\quad +\frac{\pi}{4\sigma_a}\sum_{m=1}^{N-1}\left\{(-1)^m Q_{2m}(2m+1)a_{2m+1}\right\} \\ &\quad +\frac{\xi_b}{\sigma_a^2}\sum_{m=1}^{M}\left\{(-1)^m Q_{2m}\sum_{n=0}^{N}\sum_{i=0}^{N}\left\{\frac{(2n-1)(2i-1)}{(2m+2i)^2-(2n-1)^2}a_{2n-1}a_{2i-1}\right\}\right\}\end{aligned}$$

and N_0 as obtained from this expression for M_0, by replacing there $\phi_{B0_s}(\theta)$ by $\phi_{B0_c}(\theta)$ and Q_{2m} by P_{2m}.
For the determination of M_0 and N_0 it is required: $M \geqq N$.
With:

$$F'_x = \frac{-\rho g b_0 \eta_a}{\pi}\left(M_0 \cos(\omega t + \varepsilon - \varepsilon) - N_0 \sin(\omega t + \varepsilon - \varepsilon)\right)$$

and:

$$\sin\varepsilon = \frac{-\eta_a P_0}{x_a \pi \xi_b} \qquad \cos\varepsilon = \frac{-\eta_a Q_0}{x_a \pi \xi_b}$$

the two-dimensional hydrodynamic lateral force can be resolved into components in phase and out phase with the lateral displacement of the cylinder:

$$F'_x = \frac{\rho g b_0 \eta_a^2}{\pi^2 \xi_b x_a}\left((M_0 Q_0 + N_0 P_0)\cos(\omega t + \varepsilon) + (M_0 P_0 - N_0 Q_0)\sin(\omega t + \varepsilon)\right)$$

This hydrodynamic lateral force can also be written as:

$$\begin{aligned} F'_x &= -M'_{22}\ddot{x} - N'_{22}\dot{x} \\ &= M'_{22}\omega^2 x_a \cos(\omega t + \varepsilon) + N'_{22}\omega x_a \sin(\omega t + \varepsilon) \end{aligned}$$

in which:

M'_{22} = 2-D hydrodynamic mass coefficient of sway
N'_{22} = 2-D hydrodynamic damping coefficient of sway

When using also the amplitude ratio of the radiated waves and the forced sway oscillation, found before, the two-dimensional hydrodynamic mass and damping coefficients of sway are given by:

$$M'_{22} = \frac{\rho b_0^2}{2}\cdot\frac{M_0 Q_0 + N_0 P_0}{P_0^2 + Q_0^2} \qquad \text{and} \qquad N'_{22} = \frac{\rho b_0^2}{2}\cdot\frac{M_0 P_0 - N_0 Q_0}{P_0^2 + Q_0^2}\cdot\omega$$

The signs of these two coefficients are proper in both, the axes system of Tasai and the ship motions $O(x, y, z)$ coordinate system.
The energy delivered by the exciting forces should be equal to the energy radiated by the waves.
So:

$$\frac{1}{T_{osc}}\int_0^{T_{osc}} N'_{22}\dot{x}\cdot\dot{x}dt = \frac{\rho g \eta_a^2 c}{2}$$

in which T_{osc} is the period of oscillation.
With the relation for the wave speed $c = g/\omega$, follows the relation between the two-dimensional sway damping coefficient and the amplitude ratio of the radiated waves and the forced sway oscillation:

$$N'_{22} = \frac{\rho g^2}{\omega^3}\left(\frac{\eta_a}{x_a}\right)^2$$

With this amplitude ratio the two-dimensional hydrodynamic damping coefficient of sway is also given by:

$$N'_{22} = \frac{\rho \pi^2 b_0^2}{4}\cdot\frac{1}{P_0^2 + Q_0^2}\cdot\omega$$

When comparing this expression for N'_{22} with the expression found before, the following energy balance relation is found:

$$M_0 P_0 - N_0 Q_0 = \frac{\pi^2}{2}$$

Coupling of Sway into Roll

In the case of a sway oscillation generally a roll moment is produced. The hydrodynamic pressure is skew-symmetric in θ.
The two-dimensional hydrodynamic moment acting on the cylinder in the clockwise direction can be found by integrating the roll component of the hydrodynamic pressure on the surface S of the cylinder:

$$\begin{aligned} M'_R &= \int_0^{\pi/2} (p(+\theta) - p(-\theta)) \left(-x_0 \frac{+dx_0}{ds} + y_0 \frac{-dy_0}{ds} \right) ds \\ &= -2 \int_0^{\pi/2} p(\theta) \left(x_0 \frac{dx_0}{d\theta} + y_0 \frac{dy_0}{d\theta} \right) d\theta \end{aligned}$$

With this the two-dimensional hydrodynamic roll moment due to sway oscillations can be written as follows:

$$M'_R = \frac{\rho g b_0^2 \eta_a}{\pi} \left(Y_R \cos(\omega t) - X_R \sin(\omega t) \right)$$

in which:

$$\begin{aligned} Y_R = & \frac{1}{2\sigma_a^2} \int_0^{\pi/2} \phi_{B0_s}(\theta) \sum_{n=0}^{N} \sum_{i=0}^{N} \left\{ (-1)^{n+i} (2i-1) \cdot a_{2n-1} a_{2i-1} \sin\left((2n-2i)\theta\right) \right\} d\theta \\ & + \frac{1}{2\sigma_a^2} \sum_{m=1}^{M} \left\{ (-1)^m Q_{2m} \sum_{n=0}^{N} \sum_{i=0}^{N} \left\{ \frac{(2i-1)(2n-2i)}{(2m+1)^2 - (2n-2i)^2} \cdot a_{2n-1} a_{2i-1} \right\} \right\} \\ & - \frac{\pi \xi_b}{8\sigma_a^3} \sum_{m=1}^{N} \{ (-1)^m Q_{2m} \cdot \\ & \left(\sum_{n=m}^{N} \sum_{i=0}^{n-m} \left\{ \frac{(-2m+2n-2i-1)(2i-1)}{2n-2i} \cdot a_{2n-1} a_{2i-1} a_{-2m+2n-2i-1} \right\} \right. \\ & \left. + \sum_{n=0}^{N-m} \sum_{i=m+n}^{N} \left\{ \frac{(-2m-2n+2i-1)(2i-1)}{2n-2i} \cdot a_{2n-1} a_{2i-1} a_{-2m-2n+2i-1} \right\} \right) \Bigg\} \end{aligned}$$

and X_R as obtained from this expression for Y_R, by replacing there $\phi_{B0_s}(\theta)$ by $\phi_{B0_c}(\theta)$ and Q_{2m} by P_{2m}.
With:

$$M'_R = \frac{\rho g b_0^2 \eta_a}{\pi} \left(Y_R \cos(\omega t + \varepsilon - \varepsilon) - X_R \sin(\omega t + \varepsilon - \varepsilon) \right)$$

and:

$$\sin \varepsilon = \frac{-\eta_a}{x_a \pi \xi_b} P_0 \qquad \cos \varepsilon = \frac{-\eta_a}{x_a \pi \xi_b} Q_0$$

the two-dimensional hydrodynamic roll moment can be resolved into components in phase and out phase with the lateral displacement of the cylinder:

$$M'_R = \frac{-\rho g b_0^2 \eta_a^2}{\pi^2 \xi_b x_a} \left((Y_R Q_0 + X_R P_0) \cos(\omega t + \varepsilon) + (Y_R P_0 - X_R Q_0) \sin(\omega t + \varepsilon) \right)$$

This hydrodynamic roll moment can also be written as:

$$\begin{aligned} M'_R &= -M'_{42}\ddot{x} - N'_{42}\dot{x} \\ &= M'_{42}\omega^2 x_a \cos(\omega t + \varepsilon) + N'_{42}\omega x_a \sin(\omega t + \varepsilon) \end{aligned}$$

in which:

M'_{22} = 2-D hydrodynamic mass coupling coefficient of sway into roll
N'_{22} = 2-D hydrodynamic damping coupling coefficient of sway into roll

When using also the amplitude ratio of the radiated waves and the forced sway oscillation, found before, the two-dimensional hydrodynamic mass and damping coupling coefficients of sway into roll in Tasai's axes system are given by:

$$M'_{42} = \frac{-\rho b_0^3}{2} \cdot \frac{Y_R Q_0 + X_R P_0}{P_0^2 + Q_0^2} \qquad \text{and} \qquad N'_{42} = \frac{-\rho b_0^3}{2} \cdot \frac{Y_R P_0 - X_R Q_0}{P_0^2 + Q_0^2} \cdot \omega$$

In the ship motions $O(x, y, z)$ coordinate system these two coupling coefficients will change sign.

4.1.3 Roll Motions

The determination of the hydrodynamic coefficients of a rolling cross section of a ship in deep and still water at zero forward speed, is based here on work published by [Tasai, 1961] for the Lewis method. Starting points for the derivation these coefficients here are the velocity potentials and the conjugate stream functions of the fluid as they have been derived by Tasai and also by [Jong, 1973].
Suppose an infinite long cylinder in the surface of a fluid, of which a cross section is given in figure 4.3.

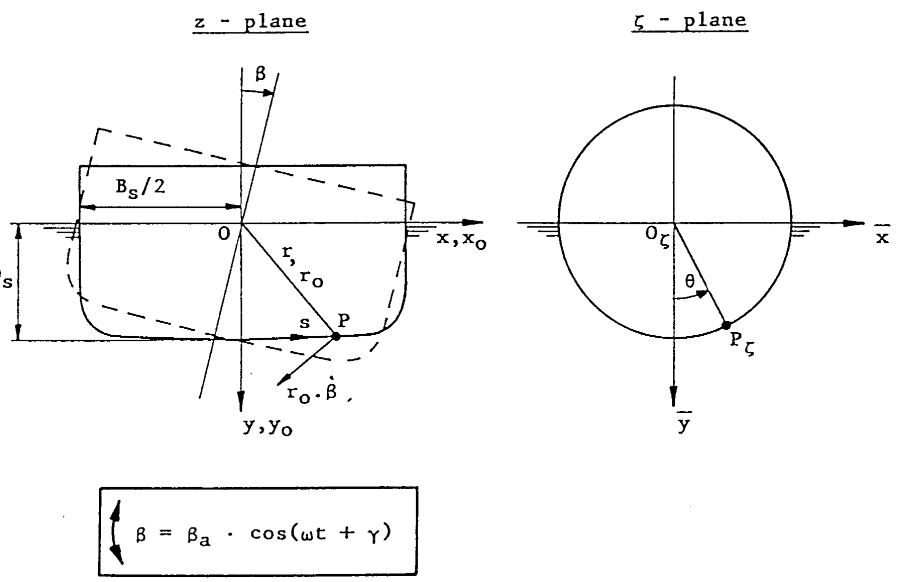

Figure 4.3: Axes System for Sway Oscillations

The cylinder is forced to carry out a simple harmonic roll motion about the origin O with a frequency of oscillation ω and a small amplitude of displacement β_a:

$$\beta = \beta_a \cos(\omega t + \gamma)$$

in which γ is a phase angle.
Respectively, the angular velocity and acceleration of the cylinder are:

$$\dot{\beta} = -\omega\beta_a \sin(\omega t + \gamma) \qquad \text{and} \qquad \ddot{\beta} = -\omega^2\beta_a \cos(\omega t + \gamma)$$

This forced angular oscillation of the cylinder causes a surface disturbance of the fluid. Because the cylinder is supposed to be infinitely long, the generated waves will be two-dimensional. These waves travel away from the cylinder and a stationary state is rapidly attained.
Two kinds of waves will be produced:

- A standing wave system, denoted here by subscript A. The amplitudes of these waves decrease strongly with the distance to the cylinder.

- A regular progressive wave system, denoted here by subscript B. These waves dissipate energy. At a distance of a few wave lengths from the cylinder, the waves on each side can be described by a single regular wave train.. The wave amplitude at infinity η_a is proportional to the amplitude of oscillation of the cylinder β_a, provided that this amplitude is sufficiently small compared with the radius of the cylinder and the wave length is not much smaller than the diameter of the cylinder.

The two-dimensional velocity potential of the fluid has to fulfil the following three requirements:

1. The velocity potential must satisfy to the equation of Laplace:

$$\nabla^2 \Phi = \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = 0$$

2. Because the roll motion of the fluid is not symmetrical about the y-axis, this velocity potential has the following relation:

$$\Phi(-x, y) = -\Phi(+x, y)$$

3. The linearized free surface condition in deep water is expressed as follows:

$$\frac{\omega^2}{g}\Phi + \frac{\partial \Phi}{\partial y} = 0 \qquad \text{for:} \quad |x| \geq \frac{B_s}{2} \quad \text{and} \quad y = 0$$

In consequence of the conformal mapping, this last equation results into:

$$\frac{\xi_b}{\sigma_a}\Phi \sum_{n=0}^{N} \left\{(2n-1)a_{2n-1}e^{-(2n-1)\alpha}\right\} \pm \frac{\partial \Phi}{\partial \theta} = 0 \quad \text{for:} \quad \alpha \geq 0 \quad \text{and} \quad \theta = \pm\frac{\pi}{2}$$

in which:

$$\frac{\xi_b}{\sigma_a} = \frac{\omega^2}{g} M_s \quad \text{or:} \quad \xi_b = \frac{\omega^2 b_0}{2g} \text{ (non-dimensional frequency squared)}$$

From the definition of the velocity potential follows the boundary condition on the surface S of the cylinder for $\alpha = 0$:

$$\frac{\partial \Phi_0(\theta)}{\partial n} = r_0 \dot{\beta} \frac{\partial r_0}{\partial s}$$

in which n is the outward normal of the cylinder surface S and r_0 is the radius from the origin to the surface of the cylinder.
Using the stream function Ψ this boundary condition on the surface of the cylinder ($\alpha = 0$) reduces to:

$$\frac{-\partial \Psi_0(\theta)}{\partial s} = \dot{\beta} \frac{\partial}{\partial s}\left(\frac{x_0^2 + y_0^2}{2}\right)$$

Integration results into the following requirement for the stream function on the surface of the cylinder:

$$\Psi_0(\theta) = -\frac{\dot{\beta}}{2}(x_0^2 + y_0^2) + C(t)$$

in which $C(t)$ is a function of the time only.
The vertical oscillation at the intersection of the surface of the cylinder and the waterline is defined by:

$$\chi = \frac{b_0}{2}\beta = \chi_a \sin(\omega t + \gamma)$$

When defining:

$$\begin{aligned}
\mu(\theta) &= \frac{x_0^2 + y_0^2}{\left(\frac{b_0}{2}\right)^2} \\
&= \left(-\frac{1}{\sigma_a}\sum_{n=0}^{N}\{(-1)^n a_{2n-1}\sin((2n-1)\theta)\}\right)^2 \\
&\quad + \left(+\frac{1}{\sigma_a}\sum_{n=0}^{N}\{(-1)^n a_{2n-1}\cos((2n-1)\theta)\}\right)^2
\end{aligned}$$

the stream function on the surface of the cylinder is given by:

$$\Psi_0(\theta) = \dot{\chi}\frac{b_0}{4}\mu(\theta) + C(t)$$

For the standing wave system a velocity potential and a stream function satisfying to the equation of Laplace, the non-symmetrical motion of the fluid and the free surface condition has to be found.
The following set of velocity potentials, as they are given by [Tasai, 1961] and [Jong, 1973], fulfil these requirements:

$$\Phi_A = \frac{g\eta_a}{\pi\omega}\left(\sum_{m=1}^{\infty}\{P_{2m}\phi_{A_{2m}}(\alpha,\theta)\cos(\omega t)\} + \sum_{m=1}^{\infty}\{Q_{2m}\phi_{A_{2m}}(\alpha,\theta)\sin(\omega t)\}\right)$$

in which:

$$\begin{aligned}
\phi_{A_{2m}}(\alpha,\theta) &= e^{-(2m+1)\alpha}\sin((2m+1)\theta) \\
&\quad -\frac{\xi_b}{\sigma_a}\sum_{n=0}^{N}\left\{(-1)^n\frac{2n-1}{2m+2n}a_{2n-1}e^{-(2m+2n)\alpha}\sin((2m+2n)\theta)\right\}
\end{aligned}$$

The set of conjugate stream functions is expressed as:

$$\Psi_A = \frac{g\eta_a}{\pi\omega}\left(\sum_{m=1}^{\infty}\{P_{2m}\psi_{A_{2m}}(\alpha,\theta)\cos(\omega t)\} + \sum_{m=1}^{\infty}\{Q_{2m}\psi_{A_{2m}}(\alpha,\theta)\sin(\omega t)\}\right)$$

in which:

$$\begin{aligned}
\psi_{A_{2m}}(\alpha,\theta) &= -e^{-(2m+1)\alpha}\cos((2m+1)\theta) \\
&\quad +\frac{\xi_b}{\sigma_a}\sum_{n=0}^{N}\left\{(-1)^n\frac{2n-1}{2m+2n}a_{2n-1}e^{-(2m+2n)\alpha}\sin((2m+2n)\theta)\right\}
\end{aligned}$$

These sets of functions tend to zero as α tends to infinity.

In these expressions the magnitudes of the P_{2m} and the Q_{2m} series follow from the boundary conditions, as will be explained further on.
Another requirement is a diverging wave train for α goes to infinity. Therefore it is necessary to add a stream function, satisfying the equation of Laplace, the non-symmetrical motion of the fluid and the free surface condition, representing such a train of waves at infinity. For this, a function describing a two-dimensional horizontal doublet at the origin O is chosen.
[Tasai, 1961] and [Jong, 1973] gave the velocity potential of the progressive wave system as:

$$\Phi_B = \frac{g\eta_a}{\pi\omega}\left(\phi_{B_c}(x,y)\cos(\omega t) + \phi_{B_s}(x,y)\sin(\omega t)\right)$$

in which:

$$\begin{aligned}\phi_{B_c}\cdot j &= -\pi e^{-\nu y}\sin(\nu|x|)\\ \phi_{B_s}\cdot j &= \pi e^{-\nu y}\cos(\nu x) - \int_0^\infty \frac{\nu\cos(ky) + k\sin(ky)}{k^2+\nu^2}e^{-k|x|}dk + \frac{|x|}{\nu(x^2+y^2)}\end{aligned}$$

while:

$$\begin{aligned} j &= +1 \quad \text{for } x > 0\\ j &= -1 \quad \text{for } x < 0\\ \nu &= \frac{\omega^2}{g} \quad \text{(wave number for deep water)}\end{aligned}$$

Changing the parameters delivers:

$$\Phi_B = \frac{g\eta_a}{\pi\omega}\left(\phi_{B_c}(\alpha,\theta)\cos(\omega t) + \phi_{B_s}(\alpha,\theta)\sin(\omega t)\right)$$

The conjugate stream function is given by:

$$\Psi_B = \frac{g\eta_a}{\pi\omega}\left(\psi_{B_c}(x,y)\cos(\omega t) + \psi_{B_s}(x,y)\sin(\omega t)\right)$$

in which:

$$\begin{aligned}\psi_{B_c} &= \pi e^{-\nu y}\cos(\nu x)\\ \psi_{B_s} &= \pi e^{-\nu y}\sin(\nu|x|) + \int_0^\infty \frac{\nu\sin(ky) - k\cos(ky)}{k^2+\nu^2}e^{-k|x|}dk - \frac{y}{\nu(x^2+y^2)}\end{aligned}$$

Changing the parameters delivers:

$$\Psi_B = \frac{g\eta_a}{\pi\omega}\left(\psi_{B_c}(\alpha,\theta)\cos(\omega t) + \psi_{B_s}(\alpha,\theta)\sin(\omega t)\right)$$

When calculating the integrals in the expressions for ψ_{B_s} and ϕ_{B_s} numerically, the convergence is very slowly.
Power series expansions, as given by [Porter, 1960], can be used instead of these last integrals over k:
The summation in these expansions converge much faster than the numeric integration procedure. This has been shown before for the sway case.

The total velocity potential and stream function to describe the waves generated by a swaying cylinder are:

$$\begin{aligned}\Phi &= \Phi_A + \Phi_B \\ \Psi &= \Psi_A + \Psi_B\end{aligned}$$

So the velocity potential and the conjugate stream function are expressed by:

$$\begin{aligned}\Phi(\alpha,\theta) &= \frac{g\eta_a}{\pi\omega}\left(\left(\phi_{B_c}(\alpha,\theta) + \sum_{m=1}^{\infty}\left\{P_{2m}\phi_{A_{2m}}(\alpha,\theta)\right\}\right)\cos(\omega t)\right. \\ &\quad \left. + \left(\phi_{B_s}(\alpha,\theta) + \sum_{m=1}^{\infty}\left\{Q_{2m}\phi_{A_{2m}}(\alpha,\theta)\right\}\right)\sin(\omega t)\right) \\ \Psi(\alpha,\theta) &= \frac{g\eta_a}{\pi\omega}\left(\left(\psi_{B_c}(\alpha,\theta) + \sum_{m=1}^{\infty}\left\{P_{2m}\psi_{A_{2m}}(\alpha,\theta)\right\}\right)\cos(\omega t)\right. \\ &\quad \left. + \left(\psi_{B_s}(\alpha,\theta) + \sum_{m=1}^{\infty}\left\{Q_{2m}\psi_{A_{2m}}(\alpha,\theta)\right\}\right)\sin(\omega t)\right)\end{aligned}$$

When putting $\alpha = 0$, the stream function is equal to the expression found before from the boundary condition on the surface of the cylinder:

$$\begin{aligned}\Psi_0(\theta) &= \frac{g\eta_a}{\pi\omega}\left(\left(\psi_{B0_c}(\theta) + \sum_{m=1}^{\infty}\left\{P_{2m}\psi_{A0_{2m}}(\theta)\right\}\right)\cos(\omega t)\right. \\ &\quad \left. + \left(\psi_{B0_s}(\theta) + \sum_{m=1}^{\infty}\left\{Q_{2m}\psi_{A0_{2m}}(\theta)\right\}\right)\sin(\omega t)\right) \\ &= -\dot{\chi}\frac{b_0}{4}\mu(\theta) + C(t)\end{aligned}$$

in which:

$$\begin{aligned}\psi_{A0_{2m}}(\theta) &= -\cos\left((2m+1)\theta\right) \\ &\quad + \frac{\xi_b}{\sigma_a}\sum_{n=0}^{N}\left\{(-1)^n\frac{2n-1}{2m+2n}a_{2n-1}\cos\left((2m+2n)\theta\right)\right\}\end{aligned}$$

In this expression $\Psi_{B0_c}(\theta)$ and $\Psi_{B0_s}(\theta)$ are the values of $\Psi_{B_c}(\alpha,\theta)$ and $\Psi_{B_s}(\alpha,\theta)$ at the surface of the cylinder, so for $\alpha = 0$.
So for each θ-value, the following equation has been obtained:

$$\begin{aligned}&\left(\psi_{B0_c}(\theta) + \sum_{m=1}^{\infty}\left\{P_{2m}\psi_{A0_{2m}}(\theta)\right\}\right)\cos(\omega t) + \\ &\left(\psi_{B0_s}(\theta) + \sum_{m=1}^{\infty}\left\{Q_{2m}\psi_{A0_{2m}}(\theta)\right\}\right)\sin(\omega t) \quad = -\dot{\chi}\frac{\pi\omega b_0}{4g\eta_a}\mu(\theta) + C^*(t)\end{aligned}$$

When putting $\theta = \pi/2$, so at the intersection of the surface of the cylinder with the free surface of the fluid where $\mu(\theta) = 1$, we obtain the constant $C^*(t)$:

$$C^*(t) = \psi_{B0_c}(\pi/2) + \sum_{m=1}^{\infty}\left\{P_{2m}\psi_{A0_{2m}}(\pi/2)\right\}\cos(\omega t)$$

$$+\left(\psi_{B0_s}(\pi/2)+\sum_{m=1}^{\infty}\left\{Q_{2m}\psi_{A0_{2m}}(\pi/2)\right\}\right)\sin(\omega t)$$
$$+\dot{\chi}\frac{\pi\omega b_0}{4g\eta_a}$$

in which:

$$\Psi_{A0_{2m}}(\pi/2)=\frac{\xi_b}{\sigma_a}(-1)^m\sum_{n=0}^{N}\frac{2n-1}{2m+2n}a_{2n-1}$$

A substitution of $C^*(t)$ in the equation for each θ-value, results for any θ-value less than $\pi/2$ into the following equation:

$$\begin{aligned}
&\left(\psi_{B0_c}(\theta)-\psi_{B0_c}(\pi/2)+\sum_{m=1}^{\infty}\left\{P_{2m}\left(\psi_{A0_{2m}}(\theta)-\psi_{A0_{2m}}(\pi/2)\right)\right\}\right) && \cos(\omega t)\\
+&\left(\psi_{B0_s}(\theta)-\psi_{B0_s}(\pi/2)+\sum_{m=1}^{\infty}\left\{Q_{2m}\left(\psi_{A0_{2m}}(\theta)-\psi_{A0_{2m}}(\pi/2)\right)\right\}\right) && \sin(\omega t)\\
&= && -\dot{\chi}\frac{\pi\omega b_0}{4g\eta_a}\left(\mu(\theta)-1\right)
\end{aligned}$$

The right hand side of this equation can be written as:

$$\begin{aligned}
-\dot{\chi}\frac{\pi\omega b_0}{4g\eta_a}\left(\mu(\theta)-1\right) &= \left(\mu(\theta)-1\right)\left(-\frac{\pi\chi_a}{2\eta_a}\xi_b\sin(\omega t+\gamma)\right)\\
&= \left(\mu(\theta)-1\right)\left(P_0\cos(\omega t)+Q_0\sin(\omega t)\right)
\end{aligned}$$

in which:

$$P_0=\frac{\pi\chi_a}{2\eta_a}\xi_b\sin\gamma \qquad \text{and} \qquad Q_0=\frac{\pi\chi_a}{2\eta_a}\xi_b\cos\gamma$$

This delivers for any θ-value less than $\pi/2$ two sets of equations with the unknown parameters P_{2m} and Q_{2m}.
So:

$$\begin{aligned}
\psi_{B0_c}(\theta)-\psi_{B0_c}(\pi/2) &= \left(\mu(\theta)-1\right)P_0+\sum_{m=1}^{\infty}\left\{f_{2m}(\theta)P_{2m}\right\}\\
\psi_{B0_s}(\theta)-\psi_{B0_s}(\pi/2) &= \left(\mu(\theta)-1\right)Q_0+\sum_{m=1}^{\infty}\left\{f_{2m}(\theta)Q_{2m}\right\}
\end{aligned}$$

in which:

$$f_{2m}(\theta)=-\psi_{A0_c}(\theta)+\psi_{A0_c}(\pi/2)$$

These equations can also be written as:

$$\begin{aligned}
\psi_{B0_c}(\theta)-\psi_{B0_c}(\pi/2) &= \sum_{m=0}^{\infty}\left\{f_{2m}(\theta)P_{2m}\right\}\\
\psi_{B0_s}(\theta)-\psi_{B0_s}(\pi/2) &= \sum_{m=0}^{\infty}\left\{f_{2m}(\theta)Q_{2m}\right\}
\end{aligned}$$

in which:

$$\begin{aligned} \text{for } m &= 0: \qquad f_0(\theta) = \mu(\theta) - 1 \\ \text{for } m &> 0: \qquad f_{2m}(\theta) = -\psi_{A0_{2m}}(\theta) + \psi_{A0_{2m}}(\pi/2) \end{aligned}$$

The series in these two sets of equations converges uniformly with an increasing value of m. For practical reasons the maximum value of m is limited to M.
Each θ-value less than $\pi/2$ will deliver an equation for the P_{2m} and Q_{2m} series. The best fit values of P_{2m} and Q_{2m} are supposed to be those found by means of a least squares method. Note that at least M+1 values of θ, less than $\pi/2$, are required to solve these equations.
Another favorable method is to multiply both sides of the equations with $\Delta\theta$. Then the summations over θ can be replaced by integrations.
Herewith, two sets of $M+1$ equations have been obtained, one set for P_{2m} and one set for Q_{2m}:

$$\begin{aligned} \sum_{m=0}^{M} \left\{ P_{2m} \int_0^{\pi/2} f_{2m}(\theta) f_{2n}(\theta) d\theta \right\} &= \int_0^{\pi/2} \left(\psi_{B0_c}(\theta) - \psi_{B0_c}(\pi/2) \right) f_{2n}(\theta) d\theta \qquad n = 0, ...M \\ \sum_{m=0}^{M} \left\{ Q_{2m} \int_0^{\pi/2} f_{2m}(\theta) f_{2n}(\theta) d\theta \right\} &= \int_0^{\pi/2} \left(\psi_{B0_s}(\theta) - \psi_{B0_s}(\pi/2) \right) f_{2n}(\theta) d\theta \qquad n = 0, ...M \end{aligned}$$

Now the P_{2m} and Q_{2m} series can be solved with a numerical method. Then P_0 and Q_0 are known now and from the definition of these coefficients follows the amplitude ratio of the radiated waves and the forced sway oscillation:

$$\frac{\eta_a}{\chi_a} = \frac{\pi \xi_b}{2\sqrt{P_0^2 + Q_0^2}}$$

With the solved P_{2m} and Q_{2m} values, the velocity potential on the surface of the cylinder (α=0) is known too:

$$\begin{aligned} \Phi_0(\theta) = \frac{g\eta_a}{\pi\omega} & \left(\left(\phi_{B0_c}(\theta) + \sum_{m=1}^{M} \left\{ P_{2m} \phi_{A0_{2m}}(\theta) \right\} \right) \cos(\omega t) \right. \\ & \left. + \left(\phi_{B0_s}(\theta) + \sum_{m=1}^{M} \left\{ Q_{2m} \phi_{A0_{2m}}(\theta) \right\} \right) \sin(\omega t) \right) \end{aligned}$$

in which:

$$\begin{aligned} \phi_{A0_{2m}}(\theta) = & \sin\left((2m+1)\theta\right) \\ & - \frac{\xi_b}{\sigma_a} \sum_{n=0}^{N} \left\{ (-1)^n \frac{2n-1}{2m+2n} a_{2n-1} \sin\left((2m+2n)\theta\right) \right\} \end{aligned}$$

In this expression $\phi_{B0_c}(\theta)$ and $\phi_{B0_s}(\theta)$ are the values of $\phi_{B_c}(\alpha,\theta)$ and $\phi_{B_s}(\alpha,\theta)$ at the surface of the cylinder, so for $\alpha = 0$.

Now the hydrodynamic pressure on the surface of the cylinder can be obtained from the linearized equation of Bernoulli:

$$\begin{aligned} p(\theta) &= -\rho \frac{\partial \Phi_0(\theta)}{\partial t} \\ &= \frac{-\rho g \eta_a}{\pi} \left(\left(\phi_{B0_s}(\theta) + \sum_{m=1}^{M} \left\{ Q_{2m} \phi_{A0_{2m}}(\theta) \right\} \right) \cos(\omega t) \right. \\ &\qquad \left. - \left(\phi_{B0_c}(\theta) + \sum_{m=1}^{M} \left\{ P_{2m} \phi_{A0_{2m}}(\theta) \right\} \right) \sin(\omega t) \right) \end{aligned}$$

It is obvious that this pressure is skew-symmetric in θ.

Roll Coefficients

The two-dimensional hydrodynamic moment acting on the cylinder in the clockwise direction can be found by integrating the roll component of the hydrodynamic pressure on the surface S of the cylinder:

$$\begin{aligned} M_R' &= \int_0^{\pi/2} (p(+\theta) - p(-\theta)) \left(-x_0 \frac{+dx_0}{ds} + y_0 \frac{-dy_0}{ds} \right) ds \\ &= -2 \int_0^{\pi/2} p(\theta) \left(x_0 \frac{dx_0}{d\theta} + y_0 \frac{dy_0}{d\theta} \right) d\theta \end{aligned}$$

With this the two-dimensional hydrodynamic moment due to roll oscillations can be written as follows:

$$M_R' = \frac{\rho g b_0^2 \eta_a}{\pi} \left(Y_R \cos(\omega t) - X_R \sin(\omega t) \right)$$

in which:

$$\begin{aligned} Y_R &= \frac{1}{2\sigma_a^2} \int_0^{\pi/2} \phi_{B0_s}(\theta) \sum_{n=0}^{N} \sum_{i=0}^{N} \left\{ (-1)^{n+i} (2i-1) \cdot a_{2n-1} a_{2i-1} \sin\left((2n-2i)\theta \right) \right\} d\theta \\ &+ \frac{1}{2\sigma_a^2} \sum_{m=1}^{M} \left\{ (-1)^m Q_{2m} \sum_{n=0}^{N} \sum_{i=0}^{N} \left\{ \frac{(2i-1)(2n-2i)}{(2m+1)^2 - (2n-2i)^2} \cdot a_{2n-1} a_{2i-1} \right\} \right\} \\ &- \frac{\pi \xi_b}{8\sigma_a^3} \sum_{m=1}^{N} \{ (-1)^m Q_{2m} \cdot \\ &\left(\sum_{n=m}^{N} \sum_{i=0}^{n-m} \left\{ \frac{(-2m+2n-2i-1)(2i-1)}{2n-2i} \cdot a_{2n-1} a_{2i-1} a_{-2m+2n-2i-1} \right\} \right. \\ &\left. \left. + \sum_{n=0}^{N-m} \sum_{i=m+n}^{N} \left\{ \frac{(-2m-2n+2i-1)(2i-1)}{2n-2i} \cdot a_{2n-1} a_{2i-1} a_{-2m-2n+2i-1} \right\} \right) \right\} \end{aligned}$$

and X_R as obtained from this expression for Y_R, by replacing there $\phi_{B0_s}(\theta)$ by $\phi_{B0_c}(\theta)$ and Q_{2m} by P_{2m}.

These expressions are similar to those found before for the hydrodynamic roll moment due to sway oscillations.
With:

$$M'_R = \frac{\rho g b_0^2 \eta_a}{\pi} \left(Y_R \cos(\omega t + \gamma - \gamma) - X_R \sin(\omega t + \gamma - \gamma) \right)$$

and:

$$\sin\gamma = \frac{2\eta_a}{\chi_a \pi \xi_b} P_0 \qquad\qquad \cos\gamma = \frac{2\eta_a}{\chi_a \pi \xi_b} Q_0$$

the two-dimensional hydrodynamic roll moment can be resolved into components in phase and out phase with the angular displacement of the cylinder:

$$M'_R = \frac{2\rho g b_0^2 \eta_a^2}{\pi^2 \xi_b \chi_a} \left((Y_R Q_0 + X_R P_0) \cos(\omega t + \gamma) + (Y_R P_0 - X_R Q_0) \sin(\omega t + \gamma) \right)$$

This hydrodynamic roll moment can also be written as:

$$\begin{aligned} M'_R &= -M'_{44}\ddot{\beta} - N'_{44}\dot{\beta} \\ &= M'_{44}\omega^2 \beta_a \cos(\omega t + \gamma) + N'_{44}\omega \beta_a \sin(\omega t + \gamma) \end{aligned}$$

in which:

M'_{44} = 2-D hydrodynamic mass moment of inertia coefficient of roll
N'_{44} = 2-D hydrodynamic damping coefficient of roll

When using also the amplitude ratio of the radiated waves and the forced roll oscillation, found before, the two-dimensional hydrodynamic mass and damping coefficients of roll in Tasai's axes system are given by:

$$M'_{44} = \frac{\rho b_0^4}{8} \cdot \frac{Y_R Q_0 + X_R P_0}{P_0^2 + Q_0^2} \qquad \text{and} \qquad N'_{44} = \frac{\rho b_0^4}{8} \cdot \frac{Y_R P_0 - X_R Q_0}{P_0^2 + Q_0^2} \cdot \omega$$

The signs of these two coefficients are proper in both, the axes system of Tasai and the ship motions $O(x, y, z)$ coordinate system.
The energy delivered by the exciting moments should be equal to the energy radiated by the waves.
So:

$$\frac{1}{T_{osc}} \int_0^{T_{osc}} N'_{44}\dot{\beta} \cdot \dot{\beta} dt = \frac{\rho g \eta_a^2 c}{2}$$

in which T_{osc} is the period of oscillation.
With the relation for the wave speed $c = g/\omega$, follows the relation between the two-dimensional roll damping coefficient and the amplitude ratio of the radiated waves and the forced roll oscillation:

$$N'_{44} = \frac{\rho g^2}{\omega^3} \left(\frac{\eta_a}{\beta_a} \right)^2$$

With this amplitude ratio the two-dimensional hydrodynamic damping coefficient of roll is also given by:

$$N'_{44} = \frac{\rho \pi^2 b_0^4}{64} \cdot \frac{1}{P_0^2 + Q_0^2} \cdot \omega$$

When comparing this expression for N'_{44} with the expression found before, the following energy balance relation is found:

$$Y_R P_0 - X_R Q_0 = \frac{\pi^2}{8}$$

Coupling of Roll into Sway

The two-dimensional hydrodynamic lateral force, acting on the cylinder in the direction of the x-axis, can be found by integrating the angular component of the hydrodynamic pressure on the surface S of the cylinder:

$$\begin{aligned} F'_x &= -\int_0^{\pi/2} \left(p(+\theta) - p(-\theta)\right) \frac{-dy_0}{ds} ds \\ &= 2 \int_0^{\pi/2} p(\theta) \frac{dy_0}{d\theta} d\theta \end{aligned}$$

With this the two-dimensional hydrodynamic angular force due to roll oscillations can be written as follows:

$$F'_x = \frac{-\rho g b_0 \eta_a}{\pi} \left(M_0 \cos(\omega t) - N_0 \sin(\omega t)\right)$$

in which:

$$\begin{aligned} M_0 &= -\frac{1}{\sigma_a} \int_0^{\pi/2} \phi_{B0_s}(\theta) \sum_{n=0}^{N} \left\{(-1)^n (2n-1) a_{2n-1} \sin\left((2n-1)\theta\right)\right\} d\theta \\ &\quad + \frac{\pi}{4\sigma_a} \sum_{m=1}^{N-1} \left\{(-1)^m Q_{2m} (2m+1) a_{2m+1}\right\} \\ &\quad + \frac{\xi_b}{\sigma_a^2} \sum_{m=1}^{M} \left\{(-1)^m Q_{2m} \sum_{n=0}^{N} \sum_{i=0}^{N} \left\{ \frac{(2n-1)(2i-1)}{(2m+2i)^2 - (2n-1)^2} a_{2n-1} a_{2i-1} \right\}\right\} \end{aligned}$$

and N_0 as obtained from this expression for M_0, by replacing there $\phi_{B0_s}(\theta)$ by $\phi_{B0_c}(\theta)$ and Q_{2m} by P_{2m}.
For the determination of M_0 and N_0 it is required: $M \geqq N$.
These expressions are similar to those found before for the hydrodynamic lateral force due to sway oscillations.
With:

$$F'_x = \frac{-\rho g b_0 \eta_a}{\pi} \left(M_0 \cos(\omega t + \gamma - \gamma) - N_0 \sin(\omega t + \gamma - \gamma)\right)$$

and:

$$\sin\gamma = \frac{2\eta_a}{\chi_a \pi \xi_b} P_0 \qquad\qquad \cos\gamma = \frac{2\eta_a}{\chi_a \pi \xi_b} Q_0$$

the two-dimensional hydrodynamic angular force can be resolved into components in phase and out phase with the angular displacement of the cylinder:

$$F'_x = \frac{-2\rho g b_0 \eta_a^2}{\pi^2 \xi_b \chi_a} \left((M_0 Q_0 + N_0 P_0) \cos(\omega t + \gamma) + (M_0 P_0 - N_0 Q_0) \sin(\omega t + \gamma)\right)$$

This hydrodynamic angular force can also be written as:

$$\begin{aligned} F'_x &= -M'_{24}\ddot{\beta} - N'_{24}\dot{\beta} \\ &= M'_{24}\omega^2 \beta_a \cos(\omega t + \gamma) + N'_{24}\omega \beta_a \sin(\omega t + \gamma) \end{aligned}$$

in which:

M'_{24} = 2-D hydrodynamic mass coupling coefficient of roll into sway
N'_{24} = 2-D hydrodynamic damping coupling coefficient of roll into sway

When using also the amplitude ratio of the radiated waves and the forced roll oscillation, found before, the two-dimensional hydrodynamic mass and damping coupling coefficients of roll into sway are given by:

$$M'_{24} = \frac{-\rho b_0^3}{8} \cdot \frac{M_0 Q_0 + N_0 P_0}{P_0^2 + Q_0^2} \qquad \text{and} \qquad N'_{24} = \frac{-\rho b_0^3}{8} \cdot \frac{M_0 P_0 - N_0 Q_0}{P_0^2 + Q_0^2} \cdot \omega$$

In the ship motions $O(x, y, z)$ coordinate system these two coupling coefficients will change sign.

4.1.4 Low and High Frequencies

The velocity potentials for very small and very large frequencies are derived and discussed in the next subsections.

Zero-Frequency Coefficients

The two-dimensional hydrodynamic mass coefficient of sway of a Lewis cross section is given by [Tasai, 1961]:

$$M'_{22}(\omega = 0) = \frac{\rho\pi}{2}\left(\frac{D_s}{1-a_1+a_3}\right)^2 \left((1-a_1)^2 + 3a_3^2\right)$$

The two-dimensional hydrodynamic mass coupling coefficient of sway into roll of a Lewis cross section is given by [Grim, 1955]:

$$\begin{aligned} M'_{42}(\omega &= 0) = -M'_{22}(\omega = 0) \cdot \frac{16}{3\pi} \cdot \frac{D_s}{1-a_1+a_3} \cdot \\ &\frac{a_1\left(1-a_1+\frac{4}{5}a_3 - a_1a_3 + \frac{3}{5}a_3^2\right) + \frac{4}{5}a_3 - \frac{12}{7}a_3^2}{(1-a_1)^2+3a_3^2} \end{aligned}$$

In Tasai's axes system M'_{42} will change sign.
The two-dimensional hydrodynamic mass moment of inertia coefficient of roll of a Lewis cross section is given by [Grim, 1955]:

$$\begin{aligned} M'_{44}(\omega &= 0) = \frac{16\rho}{\pi} \cdot \left(\frac{B_s}{2(1+a_1+a_3)}\right)^4 \cdot \\ &\left(a_1^2(1+a_3)^2 + \frac{8}{9}a_1a_3(1+a_3) + \frac{16}{9}a_3^2\right) \end{aligned}$$

The two-dimensional hydrodynamic mass coupling coefficient of roll into sway of a Lewis cross section is given by [Grim, 1955]:

$$\begin{aligned} M'_{24}(\omega &= 0) = -M'_{44}(\omega = 0) \cdot \frac{\pi}{6} \cdot \frac{1-a_1+a_3}{D_s} \cdot \\ &\frac{a_1(1-a_1)(1+a_3) + \frac{3}{5}a_1a_3(1+a_3) + \frac{4}{5}a_3(1-a_1) - \frac{12}{7}a_3^2}{a_1^2(1+a_3)^2 + \frac{8}{9}a_1a_3(1+a_3) + \frac{16}{9}a_3^2} \end{aligned}$$

In Tasai's axes system M'_{24} will change sign.

All potential damping values for zero frequency will be zero:

$$\begin{aligned} N'_{22}(\omega &= 0) = 0 \\ N'_{42}(\omega &= 0) = 0 \\ N'_{33}(\omega &= 0) = 0 \\ N'_{44}(\omega &= 0) = 0 \\ N'_{24}(\omega &= 0) = 0 \end{aligned}$$

Infinite Frequency Coefficients

The two-dimensional hydrodynamic mass coefficient of sway of a Lewis cross section is given by Landweber and de Macagno:

$$M'_{22}(\omega = \infty) = \frac{2\rho}{\pi}\left(\frac{D_s}{1 - a_1 + a_3}\right)^2 \left((1 - a_1 + a_3)^2 + \frac{16}{3}a_3^2\right)$$

The two-dimensional hydrodynamic mass coefficient of heave of a Lewis cross section is given by [Tasai, 1959]:

$$M'_{33}(\omega = \infty) = \frac{\rho\pi}{2}\left(\frac{B_s}{2(1 + a_1 + a_3)}\right)^2 \left((1 + a_1)^2 + 3a_3^2\right)$$

The two-dimensional hydrodynamic mass moment of inertia coefficient of roll of a Lewis cross section is given by [Kumai, 1959]:

$$M'_{44}(\omega = \infty) = \rho\pi\left(\frac{B_s}{2(1 + a_1 + a_3)}\right)^4 \left(a_1^2(1 + a_3)^2 + 2a_3^2\right)$$

All potential damping values for infinite frequency will be zero:

$$\begin{array}{rcl}
N'_{22}(\omega = \infty) & = & 0 \\
N'_{42}(\omega = \infty) & = & 0 \\
N'_{33}(\omega = \infty) & = & 0 \\
N'_{44}(\omega = \infty) & = & 0 \\
N'_{24}(\omega = \infty) & = & 0
\end{array}$$

4.2 Theory of Keil

In this section, the determination of the hydrodynamic coefficients of a heaving, swaying and rolling cross section of a ship in shallow at zero forward speed is based on work published by [Keil, 1974]. This method is based on a Lewis conformal mapping of the ships' cross section to the unit circle

4.2.1 Notation

His notations have been maintained here as far as possible and references are given here to the formula numbers in his report.

a	=	Lewis coefficient
A_{index}	=	source strength
$\bar{A}_{index}$	=	amplitude ratio
b	=	Lewis coefficient
B	=	breadth of body
c	=	wave velocity
C_{index}	=	non-dimensional force or moment
E_{index}	=	non-dimensional exciting force or moment
F_{index}	=	hydrodynamic force
g	=	acceleration of gravity
G	=	function (real part)
h	=	water depth
H	=	function (imaginary part)
$H_{indices}$	=	fictive moment arm
HT	=	water depth - draft ratio
I"	=	hydrodynamic moment of inertia
k_x	=	wave number in x-direction
k_y	=	wave number in y-direction
m"	=	hydrodynamic mass
$M_{indices}$	=	hydrodynamic moment
N_{index}	=	hydrodynamic damping coefficient
p	=	pressure
$r = \sqrt{y^2 + z^2}$	=	polar coordinates
t	=	time or integer value
T	=	draft
$\bar{U}$	=	velocity amplitude of horizontal oscillation
$\bar{V}$	=	velocity amplitude of vertical oscillation
A_x	=	cross sectional area
x, y, z	=	space-bound coordinates
$Y_{indices}$	=	transfer functions
γ	=	Euler constant (= 0.57722)
ε	=	phase lag
$\bar{\zeta}$	=	wave amplitude
θ	=	polar coordinate or pitch amplitude

λ	=	wave length
μ	=	wave direction
$\nu = \omega^2/g$	=	wave number at deep water
$\nu_0 = 2\pi/\lambda$	=	wave number
ρ	=	density of water
φ	=	roll angle
$\varphi_{indices}$	=	part of potential
$\Phi_{indices}$	=	time-dependent potential
$\psi_{indices}$	=	part of stream function
$\Psi_{indices}$	=	time-dependent stream function
ω	=	circular frequency of oscillation

In here, the indices - that he used - are:

E	=	related to excitation
H	=	horizontal or related to horizontal motions
j	=	imaginary part
n	=	ordering of potential parts
Q	=	related to transverse motions
r	=	real part
R	=	related to roll motions
V	=	vertical or related to vertical motions
W	=	related to waves

4.2.2 Basic Assumptions

The following figure shows the 2-D coordinate system as used by Keil and maintained here. The potentials of the incoming waves are described in Appendix 1.
The wave number $\nu_0 = 2\pi/\lambda$, follows from:

$$\nu = \frac{\omega^2}{g} = \frac{2\pi}{\lambda} \tanh \frac{2\pi h}{\lambda} = \nu_0 \tanh \left[\nu_0 h\right]$$

The fluid is supposed to be incompressible and inviscid. The flow caused by the oscillating body in the surface of this fluid can be described by a potential flow. The problem will be linearized, i.e., contributions of second and higher order in the definition of the boundary conditions will be ignored. Physically, this yields an assumption of small amplitude motions.
The space-bound axes system of the sectional contour is given in figure 4.5-a.
Velocities are positive if they are directed in the positive coordinate direction:

$$\frac{\partial \Phi}{\partial y} = v_y \qquad \frac{\partial \Phi}{\partial z} = v_z$$

The value of the stream function increases when - going in the positive direction - the flow goes in the negative y-direction:

$$\Psi_1 < \Psi_2 \rightarrow \qquad \frac{\partial \Phi}{\partial y} = +\frac{\partial \Psi}{\partial z}$$

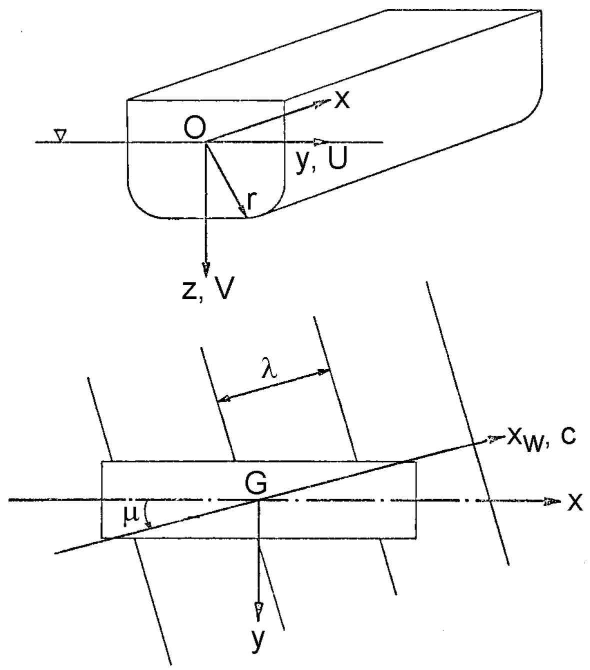

Figure 4.4: Axes System, as Used by Keil

$$\Psi_3 > \Psi_4 \rightarrow \qquad \begin{aligned} \frac{\partial \Phi}{\partial z} &= -\frac{\partial \Psi}{\partial y} \\ \frac{\partial \Phi}{\partial n} &= -\frac{\partial \Psi}{\partial s} \\ \frac{\partial \Phi}{\partial s} &= +\frac{\partial \Psi}{\partial n} \end{aligned}$$

For the imaginary parts, i and j have been used: i for geometrical variables (potential and stream function) and j for functions of time.

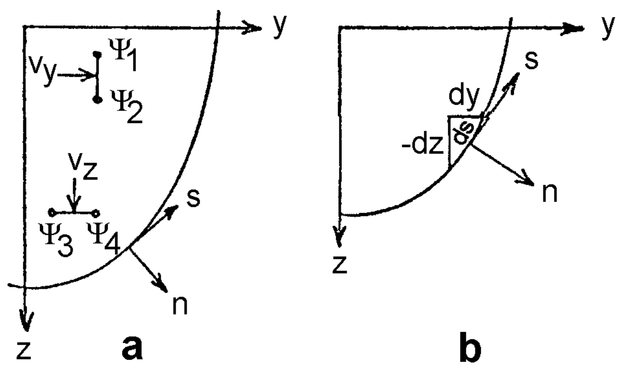

Figure 4.5: Definition of Sectional Contour

4.2.3 Vertical Motions

Boundary Conditions

The two-dimensional velocity potential of the fluid has to fulfil the following requirements:

1. The fluid is incompressible and the velocity potential must satisfy to the Continuity Condition and the Equation of Laplace:

$$\nabla^2\Phi = \frac{\partial^2\Phi}{\partial y^2} + \frac{\partial^2\Phi}{\partial z^2} = 0 \tag{Keil-1}$$

2. The linearized free surface condition follows from the condition that the pressure at the free surface is not time-depending:

$$\frac{\partial p}{\partial t} = \rho\left(g\frac{\partial\Phi}{\partial z} - \frac{\partial^2\Phi}{\partial t^2}\right) = 0 \qquad \text{for:} \quad |y| \geq \frac{B}{2} \quad \text{and} \quad z = 0 \tag{Keil-2/1}$$

   from which follows:

$$\frac{\omega^2}{g}\Phi + \frac{\partial\Phi}{\partial z} = 0 \qquad \text{for:} \quad |y| \geq \frac{B}{2} \quad \text{and} \quad z = 0$$

   or:

$$\nu\Phi + \frac{\partial\Phi}{\partial z} = 0 \qquad \text{for:} \quad |y| \geq \frac{B}{2} \quad \text{and} \quad z = 0 \tag{Keil-2/2}$$

3. The sea bottom is impervious, so the vertical fluid velocity at $z = h$ is zero:

$$\frac{\partial\Phi}{\partial z} = 0 \qquad \text{for:} \quad z = h \tag{Keil-3}$$

4. The harmonic oscillating cylinder produces a regular progressive wave system, travelling away from the cylinder, which fulfils the Sommerfeld Radiation Condition:

$$\lim_{y\to\infty}\left\{\sqrt{|y|}\left(\frac{\partial}{\partial y}\,\mathrm{Re}\,(\Phi) - \nu_0\,\mathrm{Im}\,(\Phi)\right)\right\} = 0 \tag{Keil-4}$$

   In here, $\nu_0 = 2\pi/\lambda$ is the wave number of the radiated wave.

5. The oscillating cylinder is impervious too; thus at the surface of the body is the fluid velocity equal to the body velocity, see figure 4.5-b. This yields that the boundary conditions on the surface of the body are given by:

$$\begin{aligned}\left[\frac{\partial\Phi}{\partial n}\right]_{body} &= [v_n]_{body} \\ &= \left[\frac{\partial\Phi}{\partial z}\cdot\frac{dy}{ds} - \frac{\partial\Phi}{\partial y}\cdot\frac{dz}{ds}\right]_{body} \\ &= -\left[\frac{d\Psi}{ds}\right]_{body}\end{aligned} \tag{Keil-5}$$

Two cases have to be distinguished:

a. The hydromechanical loads, which have to be obtained for the vertically oscillating cylinder in still water with a vertical velocity equal to:

$$V = \bar{V} \cdot e^{jwt}$$

The boundary condition on the surface of the body becomes:

$$-\left[\frac{d\Psi}{ds}\right]_{body} = \bar{V} \cdot \left[\frac{dy}{ds}\right]_{body} \cdot e^{jwt} \qquad \text{(Keil-5a/1)}$$

or:

$$\Psi_{body}\,(y,z,t) = -\bar{V} \cdot e^{jwt} \cdot y_{body} + C \qquad \text{(Keil-5a/2)}$$

b. The wave loads, which have to be obtained for the restrained cylinder in regular waves from the incoming undisturbed wave potential Φ_W and the diffraction potential Φ_S:

$$\begin{aligned}
\left[\frac{\partial \Phi_W}{\partial n} + \frac{\partial \Phi_S}{\partial n}\right]_{body} &= 0 \\
&= \left[\frac{\partial \Phi_W}{\partial z} \cdot \frac{dy}{ds} - \frac{\partial \Phi_W}{\partial y} \cdot \frac{dz}{ds} + \frac{\partial \Phi_S}{\partial z} \cdot \frac{dy}{ds} - \frac{\partial \Phi_S}{\partial y} \cdot \frac{dz}{ds}\right]_{body} \\
&= -\left[\frac{d\Psi_W}{ds} + \frac{d\Psi_S}{ds}\right]_{body} \qquad \text{(Keil-6/1)}
\end{aligned}$$

or:

$$\begin{aligned}
\left[d\Psi_S\,(y,z,t)\right]_{body} &= -\left[d\Psi_W\,(y,z,t)\right]_{body} \\
&= \left[\frac{\partial \Phi_W}{dz} dy - \frac{\partial \Phi_W}{dy} dz\right]_{body} \qquad \text{(Keil-6/2)}
\end{aligned}$$

The stream function of an incoming wave - which travels in the negative y-direction, so $\mu = 90^0$ - is given in Appendix 1 of this section by:

$$\Psi_W = j\frac{\bar{\zeta}\omega}{\nu} \cdot e^{j(wt+\nu_0 y)} \cdot \left(\sinh\left[\nu_0 z\right] - \tanh\left[\nu_0 h\right]\cosh\left[\nu_0 z\right]\right) \qquad \text{(Keil-A1)}$$

Because only vertical forces have to be determined, only the in y-symmetric part of the potential and stream functions have to be considered. From this follows the boundary condition on the surface of the body for beam waves, so wave direction $\mu = 90^0$:

$$\begin{aligned}
\left[\Psi_S\,(y,z,t)\right]_{body} &= -\left[\Psi_W\,(y,z,t)\right]_{body} \qquad \text{(Keil-6a)} \\
&= \frac{\bar{\zeta}\omega}{\nu} \cdot e^{jwt} \cdot \left[\left(\sinh\left[\nu_0 z\right] - \tanh\left[\nu_0 h\right]\cosh\left[\nu_0 z\right]\right)\sin(\nu_0 y)\right]_{body}
\end{aligned}$$

In case of another wave direction, this problem becomes three-dimensional and a stream function can not be written. However, boundary condition (Keil-6a) provides us a "quasi stream function" $\tilde{\Psi}_s$, i.e., this is the amount of fluid which has to come out of the body per unit of length so that totally no fluid of the incoming wave enters into the body. This function can be used as an approximation of the problem:

$$\left[\tilde{\Psi}_S\left(x_1, y_1, z_1, t\right)\right]_{body} = \left[\int_0^{y_1} \frac{\partial \Psi_W}{\partial z} dy - \int_{z_0}^{z_1} \frac{\partial \Psi_W}{\partial y} dz\right]_{body} = \tag{Keil-6b}$$

$$\begin{aligned} &= \frac{\bar{\zeta}\omega}{\nu} \cdot e^{jwt} \cdot \cos(\nu_0 x \cos\mu) \cdot \\ &\{\sin\mu \sin(\nu_0 y_1 \sin\mu) \left(\sinh\left[\nu_0 z_1\right] - \tanh\left[\nu_0 h\right] \cosh\left[\nu_0 z_1\right]\right) \\ &+\nu_0(1-\sin^2\mu) \int_0^{y_1} \cos(\nu_0 y \sin\mu) \left(\sinh\left[\nu_0 z\right] - \tanh\left[\nu_0 h\right]\cosh\left[\nu_0 z\right]\right) dy\} \end{aligned}$$

Potentials

3-D Radiation Potential Suppose a three-dimensional oscillating cylindrical body in previously still water. To find the potential of the resulting fluid motions, this body will be replaced by an oscillating pressure p at the free surface. The unknown amplitude $\bar{p}$ of this pressure has to follow from the boundary conditions.
This pressure is not supposed to act over the full breadth of the body; it is supposed to act - over the full length L of the body - only over a small distance $\Delta y/2$ to both sides of $y = 0$, so:

$$\begin{aligned} p(x, y, z &= z_0, t) = -j \cdot \bar{p}(x, y, z{=}z_0) e^{j\omega t} \\ \text{or} &: \\ \bar{p}(x, y, z &= z_0) = 0 \qquad \text{for } |y| \geqslant \Delta y/2 \text{ and } |x| \geqslant L/2 \end{aligned} \tag{Keil-7}$$

in which z_0 is the z-coordinate of the fluid surface.
The resulting force P in the z-direction becomes:

$$\begin{aligned} \bar{P} &= \int_{-L/2}^{+L/2} \int_{-\Delta y/2}^{+\Delta y/2} \bar{p}(x, y, z{=}z_0)\, dy dx \\ &= \int_{-L/2}^{+L/2} \bar{P}'(x) dx < \infty \end{aligned} \tag{Keil-7a}$$

Boundary condition (Keil-7) can be fulfilled by a pressure amplitude $\bar{p}(x, y, z{=}z_0)$ which is found by a superposition of an infinite number of harmonic pressures. From equation (Keil-7a) follows that the pressure amplitude $\bar{p}(x)$ can be integrated, so a Fourier series expansion follows from:

$$\bar{p}(x) = \frac{1}{\pi} \int_0^{\infty} \int_{-\infty}^{+\infty} \bar{p}(\xi) \cos\left[k_x(x-\xi)\right]\, d\xi\, dk_x$$

Because the pressure amplitude $\bar{p}$ depends on two variables, the Fourier series expansion has to be two-dimensional:

$$\bar{p}(x,y,z{=}z_0) = \frac{1}{\pi^2}\int_0^{\infty}\int_0^{\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\bar{p}(\xi,\eta)\cos\left[k_y(y-\eta)\right]\cos\left[k_x(x-\xi)\right]\,d\eta\,d\xi\,dk_y\,dk_x$$

in which k_x is the wave number in the x-direction and k_y is the wave number in the y-direction.
According to equation (Keil-7), the pressure amplitude $\bar{p}$ disappears for $|y| \geqslant \Delta y/2$ and $|x| \geqslant L/2$, so for this pressure expression remains:

$$\bar{p}(x,y,z{=}z_0) = \frac{1}{\pi^2}\int_0^{\infty}\int_0^{\infty}\int_{-L/2}^{+L/2}\int_{-\Delta y/2}^{+\Delta y/2}\bar{p}(\xi,\eta)\cos\left[k_y(y-\eta)\right]\cos\left[k_x(x-\xi)\right]\,d\eta\,d\xi\,dk_y\,dk_x$$

It is assumed that the value of Δy is small. This means that η remains small too, thus one can safely suppose that:

$$\cos\left[k_y(y-\eta)\right] \approx \cos(k_y y)$$

which results in:

$$\begin{aligned}\bar{p}(x,y,z{=}z_0) &= \frac{1}{\pi^2}\int_0^{\infty}\int_0^{\infty}\int_{-L/2}^{+L/2}\int_{-\Delta y/2}^{+\Delta y/2}\bar{p}(\xi,\eta)\,d\eta\cos\left[k_x(x-\xi)\right]\,d\xi\cos(k_y y)\,dk_y\,dk_x \\ &= \frac{1}{\pi^2}\int_0^{\infty}\int_0^{\infty}\cos(k_y y)\int_{-L/2}^{+L/2}\bar{P}'(\xi)\cos\left[k_x(x-\xi)\right]\,d\xi\,dk_y\,dk_x \qquad \text{(Keil-7b)}\end{aligned}$$

This pressure definition leads - as a start - to the following initial definition of the radiation potential:

$$\begin{aligned}\Phi_{0r}(x,y,z,t) &= e^{j\omega t}\int_0^{\infty}\int_0^{\infty}\int_0^{\infty}C(k_x,k_y)\cos\left[k_x(x-\xi)\right]\cos(k_y y)\,\cdot \\ &\quad\cdot\frac{\cosh\left[\sqrt{k_x^2+k_y^2}\cdot(z-h)\right]}{\sinh\left[\sqrt{k_x^2+k_y^2}\cdot h\right]}\,dk_x\,dk_y\,d\xi \qquad \text{(Keil-8)}\end{aligned}$$

in which the function $C(k_x,k_y)$ is still unknown.
This expression (Keil-8) for the radiation potential fulfils the Equation of Laplace:

$$\frac{\partial^2\Phi}{\partial x^2}+\frac{\partial^2\Phi}{\partial y^2}+\frac{\partial^2\Phi}{\partial z^2}=0$$

Now, the pressure at the free surface p_1 can be obtained from an integration of the - with the Bernoulli Equation obtained - derivative to the time of the pressure:

$$\begin{aligned}\frac{\partial p_1}{\partial t} &= e^{j\omega t}\rho\int_0^{\infty}\int_0^{\infty}\int_0^{\infty}C(k_x,k_y)\cos\left[k_x(x-\xi)\right]\cos(k_y y)\,\cdot \\ &\quad\cdot\frac{\omega^2-g\sqrt{k_x^2+k_y^2}\cdot\tanh\left[\sqrt{k_x^2+k_y^2}\cdot h\right]}{\tanh\left[\sqrt{k_x^2+k_y^2}\cdot h\right]}\,dk_x\,dk_y\,d\xi \qquad \text{(Keil-8a)}\end{aligned}$$

The harmonic oscillating pressure is given by:

$$p_1(x,y,z{=}z_0,t) = -j \cdot \bar{p}_1(x,y,z{=}0) \cdot e^{j\omega t}$$

and its amplitude becomes:

$$\bar{p}_1(x,y,z{=}z_0) = \frac{\rho}{\omega} \int_0^\infty \int_0^\infty \int_0^\infty C(k_x,k_y) \cos\left[k_x(x-\xi)\right] \cos(k_y y) \cdot \\ \cdot \frac{\omega^2 - g\sqrt{k_x^2+k_y^2} \cdot \tanh\left[\sqrt{k_x^2+k_y^2} \cdot h\right]}{\tanh\left[\sqrt{k_x^2+k_y^2} \cdot h\right]} \, dk_x \, dk_y \, d\xi \qquad \text{(Keil-8b)}$$

If this pressure amplitude $\bar{p}_1$ is supposed to be equal to the amplitude $\bar{p}$, then combining equations (Keil-7b) and (Keil-8b) provides the unknown function $C(k_x,k_y)$:

$$\begin{aligned} \bar{p}(x,y,z{=}z_0) &= \frac{1}{\pi^2} \int_0^\infty \int_0^\infty \cos(k_y y) \int_{-L/2}^{+L/2} \bar{P}'(\xi) \cos\left[k_x(x-\xi)\right] \, d\xi \, dk_y \, dk_x \\ &= \bar{p}_1(x,y,z{=}z_0) \\ &= \frac{\rho g}{\omega} \int_0^\infty \int_0^\infty \cos(k_y y) \int_0^\infty C(k_x,k_y) \cos\left[k_x(x-\xi)\right] \\ &\quad \cdot \frac{\nu - \sqrt{k_x^2+k_y^2} \cdot \tanh\left[\sqrt{k_x^2+k_y^2} \cdot h\right]}{\tanh\left[\sqrt{k_x^2+k_y^2} \cdot h\right]} \, dk_x \, dk_y \, d\xi \end{aligned}$$

Comparing the two integrands provides:

$$\int_0^\infty C(k_x,k_y) \cos\left[k_x(x-\xi)\right] \frac{\rho g}{\omega} \cdot \frac{\nu - \sqrt{k_x^2+k_y^2} \cdot \tanh\left[\sqrt{k_x^2+k_y^2} \cdot h\right]}{\tanh\left[\sqrt{k_x^2+k_y^2} \cdot h\right]} \, d\xi = \\ = \frac{1}{\pi^2} \int_{-L/2}^{+L/2} \bar{P}'(\xi) \cos\left[k_x(x-\xi)\right] \, d\xi$$

or:

$$\int_0^\infty C(k_x,k_y) \cos\left[k_x(x-\xi)\right] \, d\xi = \frac{\tanh\left[\sqrt{k_x^2+k_y^2} \cdot h\right]}{\nu - \sqrt{k_x^2+k_y^2} \cdot \tanh\left[\sqrt{k_x^2+k_y^2} \cdot h\right]} \cdot \\ \cdot \frac{\omega}{\rho g \pi^2} \int_{-L/2}^{+L/2} \bar{P}'(\xi) \, \cos\left[k_x(x-\xi)\right] \, d\xi \qquad \text{(Keil-9)}$$

When defining:

$$A_0(\xi) = \frac{\omega \bar{P}'(\xi)}{\rho g \pi^2}$$

and substituting equation (Keil-9) in equation (Keil-8) provides the radiation potential:

$$\Phi_{0r}(x,y,z,t) = e^{j\omega t} \int\limits_{-L/2}^{+L/2} A_0(\xi) \int\limits_{0}^{\infty} \cos\left[k_x(x-\xi)\right] \cdot \tag{Keil-10}$$

$$\cdot \int\limits_{0}^{\infty} \frac{\cosh\left[\sqrt{k_x^2+k_y^2}\cdot(z-h)\right]\cos(k_y y)}{\nu\cosh\left[\sqrt{k_x^2+k_y^2}\cdot h\right] - \sqrt{k_x^2+k_y^2}\cdot\sinh\left[\sqrt{k_x^2+k_y^2}\cdot h\right]}\, dk_x\, dk_y\, d\xi$$

This potential fulfills both the radiation conditions at infinity and the boundary conditions at the free surface.

2-D Radiation Potential In case of an oscillating two-dimensional body, no waves are travelling in the x-direction, so $k_x = 0$ and $k_y = \nu_0$. The distribution of $A(\xi)$ is constant over the full length of the body from $\xi = -\infty$ through $\xi = +\infty$ and the radiation potential - given in equation (Keil-10) - reduces to:

$$\Phi_{0r}(y,z,t) = e^{j\omega t} A_0 \int\limits_{0}^{\infty} \frac{\cosh\left[k(z-h)\right]}{\nu\cosh\left[kh\right] - k\sinh\left[kh\right]} \cos(ky)\, dk \tag{Keil-11}$$

To fulfil also the Sommerfeld Radiation Condition in (Keil-4), a term has to be added to equation (Keil-11). For this, use will be made here of the value of the potential given in equation (Keil-11) at a large distance from the body:

$$\begin{aligned} \Phi_{0r}(y,z,t) &= e^{j\omega t} A_0 \int\limits_{0}^{\infty} \frac{\cosh\left[k(z-h)\right]}{\nu\cosh\left[kh\right] - k\sinh\left[kh\right]} \cos(ky)\, dk \\ &= e^{j\omega t} A_0 \frac{1}{2} \left\{ \int\limits_{0}^{\infty} F(k) e^{+iky} dk + \int\limits_{0}^{\infty} F(k) e^{-iky} dk \right\} \end{aligned} \tag{K-11}$$

with:

$$F(k) = \frac{\cosh\left[k(z-h)\right]}{\nu\cosh\left[kh\right] - k\sinh\left[kh\right]} \tag{K-11a}$$

When substituting for k the term $u = k + il$, the first integral in equation (K-11) integrates for $y > 0$ over the closed line *I-II-III-IV* in the first quadrant of the complex domain and the second integral in equation (K-11) integrates for $y > 0$ over the closed line *I-V-VI-VII* in the fourth quadrant. So:

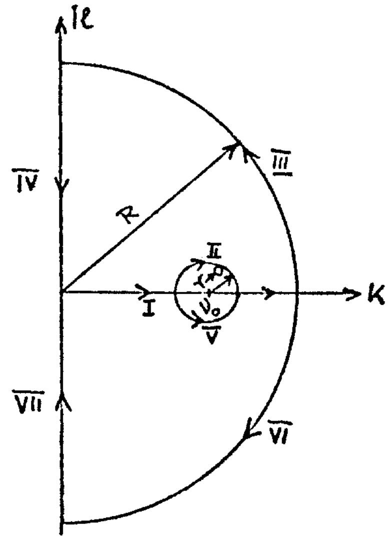

Figure 4.6: Treatment of Singularities

1. For line *I-II-III-IV:*

$$
\begin{aligned}
\oint F(u)e^{+iuy}du &= \int\limits_0^R ...dk + \int\limits_{II} ...du + \int\limits_{III} ...du + \int\limits_{IV} ...du \\
&= J_I + J_{II} + J_{III} + J_{IV} \\
&= 0 \\
\int\limits_0^\infty F(k)e^{+iky}dk &= -\lim_{R\to\infty} \left[J_{II} + J_{III} + J_{IV}\right]
\end{aligned}
$$

The location of the singular point follows from the denominator in the expression (K-11a) for $F(k)$:

$$\nu \cosh\left[\nu_0 h\right] - \nu_0 \sinh\left[\nu_0 h\right] = 0$$

Because $\lim_{R\to\infty}\left[J_{III}\right] = 0$ and J_{IV} disappears too for a large y; the singular point itself delivers a contribution only:

$$
\begin{aligned}
J_{II} &= -i\pi \cdot Residue(\nu_0) \\
&= -i\pi \frac{\cosh\left[\nu_0(z-h)\right]}{\nu h \sinh\left[\nu_0 h\right] - \sinh\left[\nu_0 h\right] - \nu_0 h \cosh\left[\nu_0 h\right]} e^{+i\nu_0 y} \\
&= +i\pi \frac{\cosh\left[\nu_0 h\right]\cosh\left[\nu_0(z-h)\right]}{\nu_0 h + \sinh\left[\nu_0 h\right]\cosh\left[\nu_0 h\right]} e^{+i\nu_0 y}
\end{aligned}
$$

and the searched integral becomes for $y \to \infty$:

$$\int\limits_0^\infty F(k)e^{+iky}dk = -i\pi \frac{\cosh\left[\nu_0 h\right]\cosh\left[\nu_0(z-h)\right]}{\nu_0 h + \sinh\left[\nu_0 h\right]\cosh\left[\nu_0 h\right]} e^{+i\nu_0 y}$$

2. For line *I-V-VI-VII*:

$$\begin{aligned}
\oint F(u)e^{-iuy}du &= \int_0^R ...dk + \int_V ...du + \int_{VI} ...du + \int_{VII} ...du \\
&= J_I + J_V + J_{VI} + J_{VII} \\
&= 0 \\
\int_0^\infty F(k)e^{-iky}dk &= -\lim_{R\to\infty}\left[J_V + J_{VI} + J_{VII}\right]
\end{aligned}$$

Because $\lim_{R\to\infty}[J_{VI}] = 0$ and J_{VII} disappears too for a large y; the singular point itself delivers a contribution only:

$$\begin{aligned}
J_V &= +i\pi \cdot \mathit{Residue}(\nu_0) \\
&= -i\pi \frac{\cosh[\nu_0 h]\cosh[\nu_0(z-h)]}{\nu_0 h + \sinh[\nu_0 h]\cosh[\nu_0 h]} e^{-i\nu_0 y}
\end{aligned}$$

and the searched integral becomes for $y \to \infty$:

$$\int_0^\infty F(k)e^{-iky}dk = +i\pi \frac{\cosh[\nu_0 h]\cosh[\nu_0(z-h)]}{\nu_0 h + \sinh[\nu_0 h]\cosh[\nu_0 h]} e^{-i\nu_0 y}$$

This provides for the potential in equation (Keil-11) for $y \to \infty$:

$$\Phi_{0r}(y,z,t) = e^{j\omega t} A_0 \pi \frac{\cosh[\nu_0 h]\cosh[\nu_0(z-h)]}{\nu_0 h + \sinh[\nu_0 h]\cosh[\nu_0 h]} \sin(\nu_0 y) \qquad \text{(Keil-11a)}$$

The Sommerfeld Radiation Condition in equation (Keil-4) will be fulfilled when:

$$e^{j\omega t} A_0 \pi \nu_0 \frac{\cosh[\nu_0 h]\cosh[\nu_0(z-h)]}{\nu_0 h + \sinh[\nu_0 h]\cosh[\nu_0 h]} \cos(\nu_0 y) - \nu_0 \operatorname{Im}\{\Phi_0(y,z,t)\} = 0$$

or:

$$\begin{aligned}
\operatorname{Im}\{\Phi_0(y,z,t)\} &= \Phi_{0j}(y,z,t) \\
&= e^{j\omega t} A_0 \pi \frac{\cosh[\nu_0 h]\cosh[\nu_0(z-h)]}{\nu_0 h + \sinh[\nu_0 h]\cosh[\nu_0 h]} \cos(\nu_0 y) \qquad \text{(Keil-11b)}
\end{aligned}$$

With equations (Keil-11a) and (Keil-11b), the radiation potential becomes:

$$\begin{aligned}
\Phi_{0r}(y,z,t) + j\Phi_{0j}(y,z,t) &= e^{j\omega t} A_0 \cdot \qquad \text{(Keil-12)} \\
&\cdot \left\{ \int_0^\infty \frac{\cosh[k(z-h)]}{\nu \cosh[kh] - k \sinh[kh]} \cos(ky)\, dk \right. \\
&\left. + j\pi \frac{\cosh[\nu_0 h]\cosh[\nu_0(z-h)]}{\nu_0 h + \sinh[\nu_0 h]\cosh[\nu_0 h]} \cos(\nu_0 y) \right\}
\end{aligned}$$

From this follows for $y \rightarrow \infty$:

$$\Phi_0(y \rightarrow \infty, z, t) = j \cdot e^{j\omega t} A_0 \pi \frac{\cosh[\nu_0 h] \cosh[\nu_0(z-h)]}{\nu_0 h + \sinh[\nu_0 h] \cosh[\nu_0 h]} e^{-j\nu_0 y}$$

This means that equation (Keil-12) describes a flow, consisting of waves with an amplitude:

$$\bar{\zeta} = A_0 \frac{\omega \pi}{g} \frac{\cosh^2[\nu_0 h]}{\nu_0 h + \sinh[\nu_0 h] \cosh[\nu_0 h]} \qquad \text{(Keil-12c)}$$

travelling away from both sides of the cylinder.
From the orthogonality condition:

$$\frac{\partial \Phi}{\partial y} = + \frac{\partial \Psi}{\partial z}$$

follows the stream function:

$$\begin{aligned} \Psi_{0r}(y,z,t) + j\Psi_{0j}(y,z,t) = & -e^{j\omega t} A_0 \cdot \qquad \text{(Keil-13)} \\ & \cdot \left\{ \int_0^\infty \frac{\sinh[k(z-h)]}{\nu \cosh[kh] - k \sinh[kh]} \sin(ky)\, dk \right. \\ & \left. + j\pi \frac{\cosh[\nu_0 h] \sinh[\nu_0(z-h)]}{\nu_0 h + \sinh[\nu_0 h] \cosh[\nu_0 h]} \sin(\nu_0 y) \right\} \end{aligned}$$

For an **infinite water depth**, equations (Keil-12) and (Keil-13) reduces to:

$$\Phi_{0r\infty}(y,z,t) + j\Phi_{0j\infty}(y,z,t) = e^{j\omega t} A_0 \left\{ \int_0^\infty \frac{e^{-kz}}{\nu - k} \cos(ky)\, dk + j\pi e^{-\nu z} \cos(\nu y) \right\} \qquad \text{(Keil-12a)}$$

$$\Psi_{0r\infty}(y,z,t) + j\Psi_{0j\infty}(y,z,t) = e^{j\omega t} A_0 \left\{ \int_0^\infty \frac{e^{-kz}}{\nu - k} \sin(ky)\, dk + j\pi e^{-\nu z} \sin(\nu y) \right\} \qquad \text{(Keil-13a)}$$

Now, the potential and stream functions can be written as:

$$\begin{aligned} \Phi_0 = & \Phi_{0r\infty} + \Phi_{0rad} + j\Phi_{0j} = e^{j\omega t} A_0 \cdot \qquad \text{(Keil-12-b)} \\ & \cdot \left\{ \int_0^\infty \frac{e^{-kz}}{\nu - k} \cos(ky)\, dk + \int_0^\infty \frac{e^{-kh}}{\nu - k} \cos(ky) \frac{\nu \sinh[kz] - k \cosh[kz]}{\nu \cosh[kh] - k \sinh[kh]}\, dk \right. \\ & \left. + j\pi \frac{\cosh[\nu_0 h] \cosh[\nu_0(z-h)]}{\nu_0 h + \sinh[\nu_0 h] \cosh[\nu_0 h]} \cos(\nu_0 y) \right\} \end{aligned}$$

$$\Psi_0 = \Psi_{0r\infty} + \Psi_{0rad} + j\Psi_{0j} = e^{j\omega t} A_0 \cdot \qquad \text{(Keil-13b)}$$

$$\cdot \left\{ \int_0^\infty \frac{e^{-kz}}{\nu - k} \sin(ky)\, dk - \int_0^\infty \frac{e^{-kh}}{\nu - k} \sin(ky) \frac{\nu \cosh[kz] - k \sinh[kz]}{\nu \cosh[kh] - k \sinh[kh]}\, dk \right.$$
$$\left. - j\pi \frac{\cosh[\nu_0 h] \sinh[\nu_0 (z-h)]}{\nu_0 h + \sinh[\nu_0 h] \cosh[\nu_0 h]} \sin(\nu_0 y) \right\}$$

In here, $\Phi_{0r\infty}$ is the potential at deep water and Φ_{0rad} is the additional potential due to the finite water depth. Φ_j can be written in the same way.

Alternative Derivation Assuming that the real part of the potential at an infinite water depth, $\Phi_{r\infty}$,is known, another derivation of the 2-D potential is given by [Porter, 1960]. The additional potential for a restricted water depth, Φ_{rad}, will be determined in such a way that it fulfills the free surface condition and - together with $\Phi_{r\infty}$ - also the boundary condition at the sea bed.
As an extension of the additional real potential will be chosen:

$$\Phi_{0rad}(y, z, t) = e^{j\omega t} A_0 \int_0^\infty \{C_1(k) \sinh[kz] + C_2(k) \cosh[k(z-h)]\} \cos(ky) dk \tag{Keil-14}$$

From the free surface condition (Keil-2) follows for $|y| \geqslant B/2$:

$$e^{j\omega t} A_0 \int_0^\infty \{\nu C_2(k) \cosh[kh] + kC_1(k) - kC_2(k) \sinh[kh]\} \cos(ky) dk = 0$$

The solution of this Fourier integral equation:

$$\int_0^\infty f(k) \cos(k\xi) dk = g(\xi)$$

is known:

$$f(k) = \frac{1}{\pi} \int_0^\infty g(\xi) \cos(k\xi)\, d\xi$$

Also will be obtained:

$$f(k) = kC_1(k) + C_2(k) \{\nu \cosh[kh] - k \sinh[kh]\} = 0$$

from which follows:

$$C_2(k) = \frac{-k}{\nu \cosh[kh] - k \sinh[kh]} C_1(k)$$

With this will be obtained:

$$\Phi_{0rad}(y, z, t) = e^{j\omega t} A_0 \int_0^\infty C_1(k) \left\{ \sinh[kz] - \frac{k \cosh[k(z-h)]}{\nu \cosh[kh] - k \sinh[kh]} \right\} \cos(ky)\, dk$$

The still unknown function $C_1(k)$ follows from the boundary condition at the sea bed:

$$\left[\frac{\partial \Phi_{0r\infty}}{\partial z} + \frac{\partial \Phi_{0rad}}{\partial z}\right]_{z=h} = 0 =$$

$$= e^{j\omega t} A_0 \left\{ -\int_0^\infty \frac{k e^{-kh}}{\nu - k} \cos(ky)\, dk + \int_0^\infty k C_1(k) \cosh[kh] \cos(ky)\, dk \right\}$$

So:

$$\begin{aligned} C_1(k) &= \frac{e^{-kh}}{(\nu - k)\cosh[kh]} \\ C_2(k) &= \frac{-k e^{-kh}}{(\nu - k)\{\nu \cosh[kh] - k \sinh[kh]\}\cosh[kh]} \end{aligned}$$

With this, the extension of the additional real potential, as given in equation (Keil-12b), becomes:

$$\Phi_{0rad}(y,z,t) = e^{j\omega t} A_0 \int_0^\infty \frac{e^{-kh}}{\nu - k} \cdot \frac{\nu \sinh[kz] - k \cosh[kz]}{\nu \cosh[kh] - k \sinh[kh]} \cos(ky)\, dk \tag{Keil-14a}$$

The imaginary part can be obtained as described before.

2-D Multi-Potential The free surface conditions can not be fulfilled with the potential (Keil-12b) - and the stream function (Keil-13b) respectively - only. Additional potentials Φ_n are required which fulfill the boundary conditions (Keil-1) through (Keil-4) and together with Φ_0 also fulfill the boundary conditions (Keil-5) or (Keil-6):

$$\begin{aligned} \Phi(y,z,t) &= A_0 \Phi_0'(y,z,t) + \sum_{n=1}^\infty A_n \Phi_n'(y,z,t) \qquad \text{(Keil-15)} \\ &= A_0 \left[\Phi_{0r\infty}'(y,z,t) + \Phi_{0rad}'(y,z,t) + j\Phi_{0j}'(y,z,t)\right] \\ &\quad + \sum_{n=1}^\infty A_n \left[\Phi_{nr\infty}'(y,z,t) + \Phi_{nrad}'(y,z,t) + j\Phi_{nj}'(y,z,t)\right] \end{aligned}$$

Use will be made here of multi-potentials given by [Grim, 1956] and [Grim, 1957], of which - using the Sommerfeld Radiation Condition - the real part of the additional potential Φ_{nrad} and the imaginary part potential Φ_{nj} will be determined. This results in:

$$\Phi_{nr\infty}(y,z,t) = +e^{j\omega t} A_n \int_0^\infty (k+\nu) k^{2(n-1)} e^{-kz} \cos(ky)\, dk$$

(Keil-16)

$$\Phi_{nrad}(y,z,t) = +e^{j\omega t}A_n \int_0^\infty (k+\nu)k^{2(n-1)}e^{-kh}\frac{\nu\sinh[kz]-k\cosh[kz]}{\nu\cosh[kh]-k\sinh[kh]}\cos(ky)\,dk$$

(Keil-16a)

$$\Phi_{nj}(y,z,t) = -e^{j\omega t}A_n\frac{\pi\nu_0^{2n}}{\cosh[\nu_0 h]}\cdot\frac{\cosh[\nu_0(z-h)]}{\nu_0 h+\sinh[\nu_0 h]\cosh[\nu_0 h]}\cos(\nu_0 y)$$

(Keil-16b)

The orthogonality condition provides the stream function:

$$\Psi_{nr\infty}(y,z,t) = +e^{j\omega t}A_n\int_0^\infty (k+\nu)k^{2(n-1)}e^{-kz}\sin(ky)\,dk$$

(Keil-17)

$$\Psi_{nrad}(y,z,t) = -e^{j\omega t}A_n \int_0^\infty (k+\nu)k^{2(n-1)}e^{-kh}\frac{\nu\cosh[kz]-k\sinh[kz]}{\nu\cosh[kh]-k\sinh[kh]}\sin(ky)\,dk$$

(Keil-17a)

$$\Psi_{nj}(y,z,t) = +e^{j\omega t}A_n\frac{\pi\nu_0^{2n}}{\cosh[\nu_0 h]}\cdot\frac{\sinh[\nu_0(z-h)]}{\nu_0 h+\sinh[\nu_0 h]\cosh[\nu_0 h]}\sin(\nu_0 y)$$

(Keil-17b)

The potentials Φ_{nj} and Φ_{nrad} disappear in deep water.

Total Potentials Only the complex constant A_n with $(0 \leqslant n \leqslant \infty)$ in the potential has to be determined:

$$\begin{aligned}
\Phi(y,z,t) &= \sum_{n=0}^{\infty}[A_{nr}+jA_{nj}]\left[\Phi'_{nr\infty}(y,z,t)+\Phi'_{nrad}(y,z,t)+j\Phi'_{nj}(y,z,t)\right]\\
&= \sum_{n=0}^{\infty}\left\{A_{nr}\left[\Phi'_{nr\infty}+\Phi'_{nrad}\right]-A_{nj}\Phi'_{nj}+j\left[A_{nj}(\Phi'_{nr\infty}+\Phi'_{nrad})+A_{nr}\Phi'_{nj}\right]\right\}\\
&= e^{j\omega t}\sum_{n=0}^{\infty}\left\{A_{nr}\left[\varphi_{nr\infty}+\varphi_{nrad}\right]-A_{nj}\varphi_{nj}+j\left[A_{nj}(\varphi_{nr\infty}+\varphi_{nrad})+A_{nr}\varphi_{nj}\right]\right\}
\end{aligned}$$

(Keil-18)

and

$$\begin{aligned}
\Psi(y,z,t) &= \sum_{n=0}^{\infty}[A_{nr}+jA_{nj}]\left[\Psi'_{nr\infty}(y,z,t)+\Psi'_{nrad}(y,z,t)+j\Psi'_{nj}(y,z,t)\right]\\
&= \sum_{n=0}^{\infty}\left\{A_{nr}\left[\Psi'_{nr\infty}+\Psi'_{nrad}\right]-A_{nj}\Psi'_{nj}+j\left[A_{nj}(\Psi'_{nr\infty}+\Psi'_{nrad})+A_{nr}\Psi'_{nj}\right]\right\}
\end{aligned}$$

$$= \quad e^{j\omega t}\sum_{n=0}^{\infty}\left\{A_{nr}\left[\psi_{nr\infty}+\psi_{nrad}\right]-A_{nj}\psi_{nj}+j\left[A_{nj}(\psi_{nr\infty}+\psi_{nrad})+A_{nr}\psi_{nj}\right]\right\} \tag{Keil-19}$$

Summarized, the complex total potential can now be written as:

$$\begin{aligned}\Phi(y,z,t)+i\Psi(y,z,t) \quad = \quad & \\ e^{j\omega t}\left[A_{0r}+jA_{0j}\right] \quad & \left\{\int_0^{\infty}\frac{\cos\left(k\left[y+i(z-h)\right]\right)}{\nu\cosh\left[kh\right]-k\sinh\left[kh\right]}\,dk\right. \\ & \left.+j\pi\frac{\cosh\left[\nu_0 h\right]\cos\left(\nu_0\left[y+i(z-h)\right]\right)}{\nu_0 h+\sinh\left[\nu_0 h\right]\cosh\left[\nu_0 h\right]}\right\} \\ +e^{j\omega t}\sum_{n=1}^{\infty}\left[A_{nr}+jA_{nj}\right] \quad & \left\{\int_0^{\infty}(\nu^2-k^2)k^{2(n-1)}\frac{\cos\left(k\left[y+i(z-h)\right]\right)}{\nu\cosh\left[kh\right]-k\sinh\left[kh\right]}\,dk\right. \\ & \left.-j\frac{\pi\nu_0^{2n}}{\cosh\left[\nu_0 h\right]}\cdot\frac{\cos\left\{\nu_0\left(y+i(z-h)\right)\right\}}{\nu_0 h+\sinh\left[\nu_0 h\right]\cosh\left[\nu_0 h\right]}\right\}\end{aligned} \tag{Keil-20}$$

The coefficients A_{nr} and A_{nj} with $(0 \leqslant n \leqslant \infty)$ have to be determined in such a way that the instantaneous boundary conditions on the body surface have been fulfilled. These coefficients are dimensional and it is very practical to determine them for the amplitude of the flow velocity $\bar{V}$; also if they then have the dimension $[L^{2n+1}]$:

$$A_r+jA_j=\bar{V}\frac{A_r+jA_j}{\bar{V}}=\bar{V}\left(A_r^{'}+jA_j^{'}\right)$$

Then, $A_n^{'}\varphi_n$ and $A_n^{'}\psi_n$ have the dimensions of a length $[L]$.

Expansion of Potential Parts

The expansion of the potential parts at an infinite water depth is given by Grim.
For $\nu r=\nu\sqrt{y^2+z^2}\rightarrow 0$:

$$\begin{aligned}\varphi_{0r\infty} \quad = \quad & e^{-\nu z}\left\{\left[\gamma+\ln(\nu r)+\sum_{m=1}^{\infty}\frac{\nu^m}{m\cdot m!}\operatorname{Re}\left\{(z+iy)^m\right\}\right]\cos(\nu y)\right. \\ & \left.+\left[\arctan\frac{y}{z}+\sum_{m=1}^{\infty}\frac{\nu^m}{m\cdot m!}\operatorname{Im}\left\{(z+iy)^m\right\}\right]\sin(\nu y)\right\}\end{aligned} \tag{Keil-21}$$

$$\begin{aligned}\psi_{0r\infty} \quad = \quad & e^{-\nu z}\left\{\left[\gamma+\ln(\nu r)+\sum_{m=1}^{\infty}\frac{\nu^m}{m\cdot m!}\operatorname{Re}\left\{(z+iy)^m\right\}\right]\sin(\nu y)\right. \\ & \left.-\left[\arctan\frac{y}{z}+\sum_{m=1}^{\infty}\frac{\nu^m}{m\cdot m!}\operatorname{Im}\left\{(z+iy)^m\right\}\right]\cos(\nu y)\right\}\end{aligned} \tag{Keil-21a}$$

with the Euler constant: $\gamma = 0.57722$.

For $\nu r = \nu\sqrt{y^2+z^2} \rightarrow \infty$:

$$\begin{aligned}
\varphi_{0r\infty} &= \sum_{m=1}^{M} \frac{(m-1)!}{\nu^m r^{2m}} \operatorname{Re}\{(z+iy)^m\} + \pi e^{-\nu z}\frac{y}{|y|}\sin(\nu y) \\
\psi_{0r\infty} &= \sum_{m=1}^{M} \frac{(m-1)!}{\nu^m r^{2m}} \operatorname{Im}\{(z+iy)^m\} - \pi e^{-\nu z}\frac{y}{|y|}\cos(\nu y)
\end{aligned}$$

Mind you that $\frac{(m-1)!}{\nu^m r^{2m}}\{(z+iy)^m\}$ is semi–convergent.

$$\varphi_{nr\infty} = (-1)^n(2n-1)!\operatorname{Re}\left\{(y+iz)^{-2n}\right\} + \frac{\nu}{2n-1}\operatorname{Im}\left\{(y+iz)^{-(2n-1)}\right\} \tag{Keil-22}$$

$$\psi_{nr\infty} = (-1)^n(2n-1)!\operatorname{Im}\left\{(y+iz)^{-2n}\right\} - \frac{\nu}{2n-1}\operatorname{Re}\left\{(y+iz)^{-(2n-1)}\right\} \tag{Keil-22a}$$

For the expansion of the remaining potential parts use has been made of the following relations as derived in Appendix 2:

$$\begin{aligned}
\cosh[kz]\cos(ky) &= \sum_{t=0}^{\infty}\frac{k^{2t}}{(2t)!}\operatorname{Re}\left\{(z+iy)^{2t}\right\} \\
\sinh[kz]\sin(ky) &= \sum_{t=0}^{\infty}\frac{k^{2t}}{(2t)!}\operatorname{Im}\left\{(z+iy)^{2t}\right\} \\
\sinh[kz]\cos(ky) &= \sum_{t=0}^{\infty}\frac{k^{2t+1}}{(2t+1)!}\operatorname{Re}\left\{(z+iy)^{2t+1}\right\} \\
\cosh[kz]\sin(ky) &= \sum_{t=0}^{\infty}\frac{k^{2t+1}}{(2t+1)!}\operatorname{Im}\left\{(z+iy)^{2t+1}\right\}
\end{aligned}$$

With these relations follows from equation (Keil-14a):

$$\begin{aligned}
\varphi_{0rad} &= \int_0^{\infty}\frac{e^{-kh}}{(\nu-k)\left(\nu\cosh[kh]-k\sinh[kh]\right)}\,dk\,\cdot \\
&\quad\cdot\left\{\nu\sum_{t=0}^{\infty}\frac{k^{2t+1}}{(2t+1)!}\operatorname{Re}\left\{(z+iy)^{2t+1}\right\} - k\sum_{t=0}^{\infty}\frac{k^{2t}}{(2t)!}\operatorname{Re}\left\{(z+iy)^{2t}\right\}\right\} \\
&= \sum_{t=0}^{\infty}\frac{1}{(2t+1)!}\int_0^{\infty}\frac{k^{2t+1}e^{-kh}}{(\nu-k)\left(\nu\cosh[kh]-k\sinh[kh]\right)}\,dk\,\cdot \\
&\quad\cdot\left\{\nu\operatorname{Re}\left\{(z+iy)^{2t+1}\right\} - (2t+1)\operatorname{Re}\left\{(z+iy)^{2t}\right\}\right\} \\
&= \sum_{t=0}^{\infty} -\frac{G(2t+1)}{(2t+1)!}\left\{\nu\operatorname{Re}\left\{(z+iy)^{2t+1}\right\} - (2t+1)\operatorname{Re}\left\{(z+iy)^{2t}\right\}\right\}
\end{aligned} \tag{Keil-23}$$

It is obvious that:

$$\Psi_{0rad} = \sum_{t=0}^{\infty} -\frac{G(2t+1)}{(2t+1)!} \left\{ (2t+1)\,\mathrm{Im}\left\{(z+iy)^{2t}\right\} - \nu\,\mathrm{Im}\left\{(z+iy)^{2t+1}\right\}\right\} \qquad \text{(Keil-23a)}$$

The function:

$$G(t) = \int_0^{\infty} \frac{k^t e^{-kh}}{(k-\nu)\left[\nu\cosh[kh] - k\sinh[kh]\right]}\,dk$$

will be treated in the next section.
Further, it follows from (Keil-16a):

$$\begin{aligned}
\varphi_{nrad} &= \int_0^{\infty} \frac{(k^2-\nu^2)\,k^{2(n-1)}e^{-kh}}{(k-\nu)\,(\nu\cosh[kh] - k\sinh[kh])}\,dk \,\cdot \\
&\quad \cdot \left\{ \nu \sum_{t=0}^{\infty} \frac{k^{2t+1}}{(2t+1)!}\,\mathrm{Re}\left\{(z+iy)^{2t+1}\right\} - k\sum_{t=0}^{\infty} \frac{k^{2t}}{(2t)!}\,\mathrm{Re}\left\{(z+iy)^{2t}\right\}\right\} \\
&= \sum_{t=0}^{\infty} \frac{1}{(2t+1)!} \left\{ \int_0^{\infty} \frac{k^{2t+2n+1}e^{-kh}}{(k-\nu)\,(\nu\cosh[kh] - k\sinh[kh])}\,dk \right. \\
&\quad \left. -\nu^2 \int_0^{\infty} \frac{k^{2t+2n-1}e^{-kh}}{(k-\nu)\,(\nu\cosh[kh] - k\sinh[kh])}\,dk \right\} \cdot \\
&\quad \cdot \left\{ \nu\,\mathrm{Re}\left\{(z+iy)^{2t+1}\right\} - (2t+1)\,\mathrm{Re}\left\{(z+iy)^{2t}\right\}\right\} \\
&= \sum_{t=0}^{\infty} \frac{G(2t+2n+1) - \nu^2\cdot G(2t+2n-1)}{(2t+1)!} \,\cdot \\
&\quad \cdot \left\{ \nu\,\mathrm{Re}\left\{(z+iy)^{2t+1}\right\} - (2t+1)\,\mathrm{Re}\left\{(z+iy)^{2t}\right\}\right\} \qquad \text{(Keil-24)}
\end{aligned}$$

It is clear that:

$$\begin{aligned}
\psi_{nrad} &= \sum_{t=0}^{\infty} \frac{G(2t+2n+1) - \nu^2\cdot G(2t+2n-1)}{(2t+1)!} \,\cdot \\
&\quad \cdot \left\{ (2t+1)\,\mathrm{Im}\left\{(z+iy)^{2t}\right\} - \nu\,\mathrm{Im}\left\{(z+iy)^{2t+1}\right\}\right\} \qquad \text{(Keil-24a)}
\end{aligned}$$

For the imaginary parts can be written:

$$\begin{aligned}
\varphi_{oj} &= +\pi \frac{\cosh^2(\nu_0 h)}{\nu_0 h + \sinh[\nu_0 h]\cosh[\nu_0 h]} \,\cdot \qquad \text{(Keil-25)} \\
&\quad \cdot \left\{ \sum_{t=0}^{\infty} \frac{\nu_0^{2t}}{(2t)!}\,\mathrm{Re}\left\{(z+iy)^{2t}\right\} - \tanh[\nu_0 h] \sum_{t=0}^{\infty} \frac{\nu_0^{2t+1}}{(2t+1)!}\,\mathrm{Re}\left\{(z+iy)^{2t+1}\right\}\right\}
\end{aligned}$$

$$\begin{aligned}
\psi_{oj} &= -\pi \frac{\cosh^2(\nu_0 h)}{\nu_0 h + \sinh[\nu_0 h]\cosh[\nu_0 h]} \cdot \qquad \text{(Keil-25a)}\\
&\quad \cdot \left\{ \sum_{t=0}^{\infty} \frac{\nu_0^{2t}}{(2t)!} \operatorname{Im}\left\{(z+iy)^{2t}\right\} - \tanh[\nu_0 h] \sum_{t=0}^{\infty} \frac{\nu_0^{2t+1}}{(2t+1)!} \operatorname{Im}\left\{(z+iy)^{2t+1}\right\} \right\}\\
\varphi_{nj} &= -\pi \frac{\nu_0^{2n}}{\nu_0 h + \sinh[\nu_0 h]\cosh[\nu_0 h]} \cdot \qquad \text{(Keil-26)}\\
&\quad \cdot \left\{ \sum_{t=0}^{\infty} \frac{\nu_0^{2t}}{(2t)!} \operatorname{Re}\left\{(z+iy)^{2t}\right\} - \tanh[\nu_0 h] \sum_{t=0}^{\infty} \frac{\nu_0^{2t+1}}{(2t+1)!} \operatorname{Re}\left\{(z+iy)^{2t+1}\right\} \right\}\\
&= -\nu_0^{2(n-1)}\left(\nu_0^2 - \nu^2\right) \cdot \varphi_{oj}\\
\psi_{nj} &= +\pi \frac{\nu_0^{2n}}{\nu_0 h + \sinh[\nu_0 h]\cosh[\nu_0 h]} \cdot \qquad \text{(Keil-26a)}\\
&\quad \cdot \left\{ \sum_{t=0}^{\infty} \frac{\nu_0^{2t}}{(2t)!} \operatorname{Im}\left\{(z+iy)^{2t}\right\} - \tanh[\nu_0 h] \sum_{t=0}^{\infty} \frac{\nu_0^{2t+1}}{(2t+1)!} \operatorname{Im}\left\{(z+iy)^{2t+1}\right\} \right\}\\
&= -\nu_0^{2(n-1)}\left(\nu_0^2 - \nu^2\right) \cdot \psi_{oj}
\end{aligned}$$

Function G(t)

The function:

$$G(t) = \int_0^{\infty} \frac{k^t e^{-kh}}{(k-\nu)(\nu\cosh[kh] - k\sinh[kh])}\, dk \qquad \text{with unit: } \left[L^{1-t}\right]$$

has two singular points: $k = \nu$ and $k = \nu_0$. Thus, it is not possible to solve this integral directly.
First, this integral will be normalized:

$$\begin{aligned}
G'(t) &= G(t) \cdot h^{t-1}\\
&= \int_0^{\infty} \frac{u^t e^{-u}}{(u-\nu h)(\nu h\cosh[u] - u\sinh[u])}\, du
\end{aligned}$$

A substitution of:

$$w = u + iv = \frac{\varsigma}{2}\sqrt{2} \cdot e^{i\sigma}$$

provides:

$$\begin{aligned}
J &= \oint \frac{w^t e^{-w}}{(w-\nu h)(\nu h\cosh[w] - w\sinh[w])}\, dw\\
&= \int_0^{\infty} ...du + \int_I ...dw + \int_{II} ...dw + \int_{III} ...dw + \int_{IV} ...dw
\end{aligned}$$

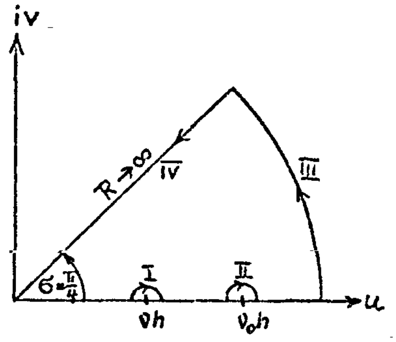

Figure 4.7: Singularities in G-Function

$$
\begin{aligned}
&= \int_0^\infty ...du + J_I + J_{II} + J_{III} + J_{IV} \\
&= 0 && \text{(Keil-27a)}
\end{aligned}
$$

From this follows:

$$\int_0^\infty \frac{u^t e^{-u}}{(u-\nu h)\,(\nu h \cosh[u] - u \sinh[u])}\,du = -\operatorname{Re}\{J_I + J_{II} + J_{III} + J_{IV}\}$$

J_I and J_{II} are imaginary because they are residues and $J_{III} = 0$ for $R \to \infty$.
So, it remains:

$$\int_0^\infty \frac{u^t e^{-u}}{(u-\nu h)\,(\nu h \cosh[u] - u \sinh[u])}\,du = -\operatorname{Re}\{J_{IV}\}$$

$$= -\operatorname{Re}\left\{\int_{IV} \frac{w^t e^{-w}}{(w-\nu h)\,(\nu h \cosh[w] - w \sinh[w])}\,dw\right\}$$

With the complex function:

$$(\tilde{w}-\nu h)\,(\nu h \cosh[\tilde{w}] - \tilde{w} \sinh[\tilde{w}]) \qquad \text{with: } \tilde{w} = \frac{\varsigma}{2}\sqrt{2}\cdot e^{i\sigma}$$

the nominator of this integral will be made real by removing.
So:

$$G'(t) =$$

$$-\operatorname{Re}\left\{\int_{IV} \frac{w^t e^{-w}\,(\tilde{w}-\nu h)\,(\nu h \cosh[\tilde{w}] - \tilde{w}\sinh[\tilde{w}])}{(w-\nu h)\,(\tilde{w}-\nu h)\,(\nu h \cosh[w] - w\sinh[w])\,(\nu h \cosh[\tilde{w}] - \tilde{w}\sinh[\tilde{w}])}\,dw\right\}$$

(Keil-28)

$$= -\cos\frac{t\pi}{4}\cdot$$

$$\int_0^\infty \left(\left(\frac{\varsigma}{\sqrt{2}}\right)^t \frac{\begin{array}{c} 2\,(\nu h)^2\left\{\left(1-\tan\frac{t\pi}{4}\right)\left(\cos\varsigma+e^{-\varsigma}\right)+\left(1+\tan\frac{t\pi}{4}\right)\sin\varsigma\right\} \\ -4\nu h\varsigma\left\{\cos\varsigma+\tan\frac{t\pi}{4}\sin\varsigma\right\} \\ +\varsigma^2\left\{\left(1+\tan\frac{t\pi}{4}\right)\left(\cos\varsigma-e^{-\varsigma}\right)-\left(1-\tan\frac{t\pi}{4}\right)\sin\varsigma\right\}\end{array}}{\begin{array}{c}\left\{\varsigma^2-2\nu h\left(\varsigma-\nu h\right)\right\}\cdot \\ \cdot\left\{\cosh\left[\varsigma\right]\left(2\nu^2h^2+\varsigma^2-2\nu h\varsigma\tanh\left[\varsigma\right]\right)\right. \\ \left.+\left(2\nu^2h^2-\varsigma^2\right)\cos\varsigma+2\nu h\varsigma\sin\varsigma\right\}\end{array}} \right) d\varsigma$$

Because t is always odd:

$$G'(t) = -\cos\frac{t\pi}{4}\int_0^\infty \frac{4\,(\nu h)^2\sin\varsigma+2\varsigma^2\left(\cos\varsigma-e^{-\varsigma}\right)-4\nu h\varsigma\left(\cos\varsigma+\sin\varsigma\right)}{denominator}\left(\frac{\varsigma}{\sqrt{2}}\right)^t d\varsigma$$

for $t = 1,\ 5,\ 9,\ ...$

$$= -\cos\frac{t\pi}{4}\int_0^\infty \frac{4\,(\nu h)^2\left(\cos\varsigma+e^{-\varsigma}\right)-2\varsigma^2\sin\varsigma-4\nu h\varsigma\left(\cos\varsigma-\sin\varsigma\right)}{denominator}\left(\frac{\varsigma}{\sqrt{2}}\right)^t d\varsigma$$

for $t = 3,\ 7,\ 11,\ ...$

For $t > 1$ the function $G'(t)$ becomes finite. However, $G'(1)$ does not converge for $\nu h \to 0$; the integral increases monotone with decreasing νh. This will be investigated first.

$$G'(1) = G(1) = \int_0^\infty \frac{ue^{-u}}{\left(u-\nu h\right)\left(\nu h\cosh\left[u\right]-u\sinh\left[u\right]\right)}\,du \qquad \text{(Keil-28a)}$$

This integral converges fast for small νh values. This will be approximated by:

$$\lim_{\nu h\to 0} G_1(1) = \int_0^\infty \frac{ue^{-u}}{\left(u-\nu h\right)\left(\nu h-u^2\right)}\,du \qquad \text{(Keil-29)}$$

This can be written as:

$$\lim_{\nu h\to 0} G_1(1) = \lim_{\nu h\to 0}\left\{\frac{1}{1-\nu h}\int_0^\infty\frac{e^{-u}}{u-\nu h}\,du + \frac{1}{2\sqrt{\nu h}\left(\sqrt{\nu h}-1\right)}\int_0^\infty\frac{e^{-u}}{u-\sqrt{\nu h}}\,du + \frac{1}{2\sqrt{\nu h}\left(\sqrt{\nu h}+1\right)}\int_0^\infty\frac{e^{-u}}{u+\sqrt{\nu h}}\,du\right\}$$

From:

$$\int_0^\infty \frac{e^{-u}}{u-a}\,du = -e^{-a}\cdot\left\{\gamma+\ln|a|+\sum_{m=1}^{\infty}\frac{a^m}{m\cdot m!}\right\}$$

follows:

$$
\begin{aligned}
\lim_{\nu h \to 0} G_1(1) &= \lim_{\nu h \to 0} \left\{ \frac{-e^{-\nu h}}{1-\nu h} \left[\gamma + \ln(\nu h) + \sum_{m=1}^{\infty} \frac{(\nu h)^m}{m \cdot m!} \right] \right. \\
&\quad + \frac{e^{-\sqrt{\nu h}}}{2\sqrt{\nu h}\left(1-\sqrt{\nu h}\right)} \left[\gamma + \frac{1}{2}\ln(\nu h) + \sum_{m=1}^{\infty} \frac{(\nu h)^{\frac{m}{2}}}{m \cdot m!} \right] \\
&\quad \left. - \frac{e^{\sqrt{\nu h}}}{2\sqrt{\nu h}\left(1+\sqrt{\nu h}\right)} \left[\gamma + \frac{1}{2}\ln(\nu h) + \sum_{m=1}^{\infty} (-1)^m \frac{(\nu h)^{\frac{m}{2}}}{m \cdot m!} \right] \right\} \\
&= \lim_{\nu h \to 0} \left\{ \frac{-e^{-\nu h}}{1-\nu h} \left[\gamma + \ln(\nu h) + \sum_{m=1}^{\infty} \frac{(\nu h)^m}{m \cdot m!} \right] \right. \\
&\quad + \frac{\left(1+\sqrt{\nu h}\right) e^{-\sqrt{\nu h}}}{2\sqrt{\nu h}(1-\nu h)} \left[\gamma + \frac{1}{2}\ln(\nu h) + \sqrt{\nu h} + \sum_{m=2}^{\infty} \frac{(\nu h)^{\frac{m}{2}}}{m \cdot m!} \right] \\
&\quad \left. - \frac{\left(1-\sqrt{\nu h}\right) e^{\sqrt{\nu h}}}{2\sqrt{\nu h}(1-\nu h)} \left[\gamma + \frac{1}{2}\ln(\nu h) - \sqrt{\nu h} + \sum_{m=2}^{\infty} (-1)^m \frac{(\nu h)^{\frac{m}{2}}}{m \cdot m!} \right] \right\} \\
&= \lim_{\nu h \to 0} \left\{ \frac{-e^{-\nu h}}{1-\nu h} \left[\gamma + \ln(\nu h) + \sum_{m=1}^{\infty} \frac{(\nu h)^m}{m \cdot m!} \right] \right. \\
&\quad + \frac{1}{1-\nu h} \left[\cosh\left[\sqrt{\nu h}\right] - \frac{\sinh\left[\sqrt{\nu h}\right]}{\sqrt{\nu h}} \right] \left[\gamma + \frac{1}{2}\ln(\nu h) \right] \\
&\quad + \frac{\cosh\left[\sqrt{\nu h}\right]}{1-\nu h} - \frac{\sinh\left[\sqrt{\nu h}\right]}{1-\nu h} \\
&\quad + \frac{1}{2(1-\nu h)} \left[e^{-\sqrt{\nu h}} \sum_{m=2}^{\infty} \frac{(\nu h)^{\frac{m-1}{2}}}{m \cdot m!} + e^{-\sqrt{\nu h}} \sum_{m=2}^{\infty} \frac{(\nu h)^{\frac{m}{2}}}{m \cdot m!} \right. \\
&\quad \left. \left. - e^{\sqrt{\nu h}} \sum_{m=2}^{\infty} (-1)^m \frac{(\nu h)^{\frac{m-1}{2}}}{m \cdot m!} + e^{\sqrt{\nu h}} \sum_{m=2}^{\infty} (-1)^m \frac{(\nu h)^{\frac{m}{2}}}{m \cdot m!} \right] \right\} \\
&= 1 - \gamma - \ln(\nu h) \qquad \text{(Keil-29a)}
\end{aligned}
$$

The imaginary part of integral (Keil-27a) has been treated in Appendix 3.

Hydrodynamic Loads

The hydrodynamic loads can be found from an integration of the pressures on the hull of the oscillating body in (previously) still water. With a known potential, these pressures can be found from the linear part of the instationary pressures as follows from the Bernoulli equation:

$$
p_{dyn} = p - p_{stat} = -\rho \frac{\partial \Phi}{\partial t}
$$

$$= -j\omega\rho\Phi$$

The potential is in-phase with the oscillation velocity. To obtain the phase of the pressures with respect to the oscillatory motion a phase shift of -90^0 is required, which means a multiplication with -j. Then the pressure is:

$$p_{dyn} = -\rho\omega\Phi$$

The hydrodynamic force on the body is equal to the integrated pressure on the body. This is - in the two-dimensional case - a force per unit length.
The vertical force becomes:

$$\begin{aligned} F_V = F_{Vr} + jF_{Vj} &= \int\limits_S p dy \\ &= \rho\omega\bar{V}\left[F'_{Vr} + jF'_{Vj}\right] \\ &= -\rho\omega\bar{V}e^{j\omega t}\sum_{n=0}^{\infty}\int\limits_S\left[A'_{nr}\varphi_{nr} - A'_{nj}\varphi_{nj} + j\left(A'_{nr}\varphi_{nj} + A'_{nj}\varphi_{nr}\right)\right]dy \end{aligned} \tag{Keil-30}$$

The real part of this force is equal to the hydrodynamic mass coefficient times the oscillatory acceleration, from which the hydrodynamic mass coefficient follows:

$$m" = \frac{F_{Vr}}{b} = \frac{\bar{F}_{Vr}}{\omega\bar{V}}$$

or non-dimensional:

$$C_V = \frac{m"}{\rho\frac{\pi}{8}B^2} = \frac{\bar{F}_{Vr}}{\rho\frac{\pi}{8}B^2\omega\bar{V}} \tag{Keil-30a}$$

The imaginary part of the force must be equal to the hydrodynamic damping coefficient times the oscillatory velocity, from which the damping coefficient follows:

$$N_V = \frac{\left|\bar{F}_{Vj}\right|}{\bar{V}}$$

Instead of this coefficient, generally the ratio between amplitude of the radiated wave $\bar{\zeta}$ and the oscillatory motion $\bar{z}$ will be used. The energy balance provides:

$$\begin{aligned} \bar{A}_V^2 = \frac{\bar{\zeta}^2}{\bar{z}^2} &= \frac{\omega^2}{\rho g}\cdot\frac{N_V}{2\cdot c_{group}} \\ &= \frac{\omega\nu_0}{\rho g}\cdot\frac{\sinh\left[2\nu_0 h\right]}{2\nu_0 h + \sinh\left[2\nu_0 h\right]}N_V \\ &= \frac{\nu^2}{\rho\omega\bar{V}}\cdot\frac{\cosh^2\left[\nu_0 h\right]}{\nu_0 h + \sinh\left[\nu_0 h\right]\cosh\left[\nu_0 h\right]}\bar{F}_{Vj} \end{aligned} \tag{Keil-30b}$$

In **deep water** becomes the hydrodynamic mass for $\nu \rightarrow 0$ infinite, because potential (Keil-21) becomes:

$$\lim_{\nu \to 0} \varphi_{0r\infty} = \gamma + \ln(\nu r) \tag{Keil-30c}$$

and the non-dimensional mass of a circle becomes:

$$\begin{aligned} \lim_{\nu \to 0} C_V &= \lim_{\nu \to 0} \frac{m'}{\rho \frac{\pi}{8} B^2} \\ &= \frac{8}{\pi^2} \left[-\gamma - \ln(\nu r) + \sum_{n=1}^{\infty} \frac{1}{n(4n^2-1)} \right] \end{aligned}$$

and the amplitude ratio - in this deep water case - becomes:

$$\lim_{\nu \to 0} \frac{d\bar{A}_V}{d\nu} = B$$

The hydrodynamic mass for $\nu \to 0$ in shallow water remains finite. Because the multi-potentials - just as in deep water - provide finite contributions, the radiation potential has to be discussed only, which is decisive (infinite mass) in deep water. the borderline change-over "deep to shallow" water provides for this radiation potential:

$$\begin{aligned} \lim_{\nu \to 0} \left[\varphi_{0r\infty} + \varphi_{0rad} \right] &= \gamma + \ln(\nu r) - \ln(\nu h) \\ &= \gamma + \ln\left(\frac{r}{h}\right) \end{aligned}$$

It is obvious that equation (Keil-30d) - just as equation (Keil-30c) - provides an infinite value.
When the contributions of the multi-potentials (which disappear here for the borderline case $\nu \to 0$) are ignored, it follows from equation (Keil–12c) for the amplitude ratio in shallow water:

$$\begin{aligned} \bar{A}_V = \frac{\bar{\zeta}}{\bar{z}} &= A_0 \frac{\omega \pi}{g \bar{z}} \cdot \frac{\cosh^2[\nu_0 h]}{\nu_0 h + \sinh[\nu_0 h] \cosh[\nu_0 h]} \\ &= A_0 \frac{\nu \pi}{\omega \bar{z}} \cdot \frac{\cosh^2[\nu_0 h]}{\nu_0 h + \sinh[\nu_0 h] \cosh[\nu_0 h]} \\ &= \frac{A_0}{\bar{V}} \pi \nu \frac{\cosh^2[\nu_0 h]}{\nu_0 h + \sinh[\nu_0 h] \cosh[\nu_0 h]} \\ &= A_0' \pi \frac{\nu_0 \sinh[\nu_0 h] \cosh[\nu_0 h]}{\nu_0 h + \sinh[\nu_0 h] \cosh[\nu_0 h]} \end{aligned}$$

Because:

$$\lim_{\nu \to 0} \frac{A_0' \pi}{B} = 1$$

it follows:

$$\lim_{\nu \to 0} \bar{A}_V = B \frac{\nu_0 \sinh[\nu_0 h] \cosh[\nu_0 h]}{\nu_0 h + \sinh[\nu_0 h] \cosh[\nu_0 h]}$$

and:

$$\begin{aligned}\lim_{\nu\to 0}\frac{d\bar{A}_V}{d\nu} &= \lim_{\nu_0\to 0}\frac{d\bar{A}_V}{d\nu_0}\\ &= \frac{B}{2}\lim_{\nu_0\to 0}\frac{1}{\nu_0 h+\sinh[\nu_0 h]\cosh[\nu_0 h]}\\ &= \infty\end{aligned}$$

Thus, $\bar{A}_V(\nu B/2)$ has at $\nu B/2 = 0$ a vertical tangent.
The fact that the hydrodynamic mass goes to infinity for zero frequency can be explained physically as follows. The smaller the frequency becomes, the longer becomes the radiated wave and the faster travels it away from the cylinder. In the borderline case $\nu = 0$ has the wave an infinite length and it travels away - just as the pressure (incompressible fluid) - with an infinite velocity. This means that all fluid particles are in phase with the motions of the body. This means that the hydrodynamic force is in phase with the motion of the body, which holds too that:

$$\varepsilon_{HT=\infty} = \arctan\frac{\bar{F}_{Vi}}{\bar{F}_{Vr}} = \arctan\frac{\bar{F}'_{Vj}}{\bar{F}'_{Vr}} = 0$$

This condition is fulfilled only when $\bar{F}'_{Vj} = 0$ or $\bar{F}'_{Vr} = m"/\rho = \infty$. However, $\bar{F}'_{Vj}$ is finite:

$$\bar{F}'_{Vj} = \frac{\bar{F}_{Vj}}{\rho\omega\bar{V}} = \frac{N_V}{\rho\omega} = \frac{g^2}{\omega^4}\bar{A}_V^2 = \frac{\bar{A}_V^2}{\nu^2}$$

Because $\lim_{\nu_0\to 0}\bar{A}_V = B_V$ follows $\lim_{\nu_0\to 0}\bar{F}'_{Vj} = B^2$. The term $\bar{F}'_{Vr} = m"/\rho$ has to be infinite.
The finite value of the hydrodynamic mass at shallow water is physically hard to interpret. A full explanation is not given here. However, it has been shown here that the result makes some sense. At shallow water can the wave (even in an incompressible fluid) not travel with an infinite velocity; its maximum velocity is $\sqrt{gh}$ In case of long waves at shallow water, the energy has the same velocity. From that can be concluded that at low decreasing frequencies the damping part in the hydrodynamic force will increase. This means that:

$$\varepsilon_{HT\neq\infty} = \arctan\frac{\bar{F}'_{Vi}}{\bar{F}'_{Vr}} \neq 0$$

So or $\bar{F}'_{Vr} = m"/\rho$ has to be finite.

Wave Loads

The wave forces on the restrained body in waves consists of forces F_1 in the undisturbed incoming waves (Froude-Krylov hypothesis), and the forces caused by the disturbance of the waves by the body, one part F_2 in phase with the accelerations of the water particles and another part F_3 in phase with the velocity of the water particles:

$$F_E = F_1 + F_2 + jF_3$$

These forces will be determined from the undisturbed wave potential Φ_W and the disturbance potential Φ_S. As mentioned before, for $\mu \neq 90^0$ only an approximation will be found.

$$\begin{aligned} F_E &= F_1 + F_2 + jF_3 = \\ &= -\rho\omega \int_S \left[\Phi_W + \Phi_S\right] \, dy \\ &= -\rho\omega e^{j\omega t} \int_S \left\{ \frac{\bar{\zeta}\omega}{\nu} e^{-j\nu_0 x \cos\mu} \left(\cosh\left[\nu_0 z\right] - \tanh\left[\nu_0 h\right] \sinh\left[\nu_0 z\right]\right) \cos\left(\nu_0 y \sin\mu\right) \right. \\ & \left. + \bar{V} \sum_{n=0}^{\infty} \left[A'_{nr}\varphi_{nr} - A'_{nj}\varphi_{nj} + j\left(A'_{nr}\varphi_{nj} + A'_{nj}\varphi_{nr}\right)\right] \right\} dy \end{aligned} \tag{Keil-31}$$

In here:

$$\begin{aligned} F_1 &= -\frac{\rho\omega^2\bar{\zeta}}{\nu} e^{j(\omega t - \nu_0 x \cos\mu)} \int_S \left\{ \left(\cosh\left[\nu_0 z\right] - \tanh\left[\nu_0 h\right] \sinh\left[\nu_0 z\right]\right) \cos\left(\nu_0 y \sin\mu\right) \right\} \, dy \\ F_2 &= -\rho\omega\bar{V} e^{j\omega t} \sum_{n=0}^{\infty} \left[A'_{nr}\varphi_{nr} - A'_{nj}\varphi_{nj} \right] \, dy \\ F_3 &= -\rho\omega\bar{V} e^{j\omega t} \sum_{n=0}^{\infty} \left[A'_{nr}\varphi_{nj} + A'_{nj}\varphi_{nr} \right] \, dy \end{aligned}$$

Using $\bar{V} = \bar{\zeta}\omega$, the non-dimensional amplitudes are:

$$\begin{aligned} \bar{E}_1 &= \frac{\bar{F}_1}{\rho g \bar{\zeta} B} = -\frac{1}{B} \int_S \left\{ \left(\cosh\left[\nu_0 z\right] - \tanh\left[\nu_0 h\right] \sinh\left[\nu_0 z\right]\right) \cos\left(\nu_0 y \sin\mu\right) \right\} \, dy \\ \bar{E}_2 &= \frac{\bar{F}_2}{\rho g \bar{\zeta} B} = -\frac{\nu}{B} \sum_{n=0}^{\infty} \left[A'_{nr}\varphi_{nr} - A'_{nj}\varphi_{nj} \right] \, dy \\ \bar{E}_3 &= \frac{\bar{F}_3}{\rho g \bar{\zeta} B} = -\frac{\nu}{B} \sum_{n=0}^{\infty} \left[A'_{nr}\varphi_{nj} + A'_{nj}\varphi_{nr} \right] \, dy \end{aligned}$$

In case of $\mu = 90^0$, so beam waves, the theory of [Haskind, 1957] can be used too to determine the amplitudes $\bar{E}_1$, $\bar{E}_2$ and $\bar{E}_3$. When $\Phi_W = \varphi_W e^{j\omega t}$ is the potential of the incoming wave and $\Phi_S = \varphi_S e^{j\omega t}$ is the potential of the disturbance by the body at a large distance from the body with a velocity amplitude $\bar{V} = 1$, then:

$$F_E = -\rho\omega e^{j\omega t} \int_0^h \left(\varphi_w \frac{\partial\varphi}{\partial y} - \varphi \frac{\partial\varphi_w}{\partial y} \right) \, dz \tag{Keil-32}$$

According to equation (Keil-A4) in Appendix 1 is:

$$\varphi_W = \frac{\bar{\zeta}\cdot\omega}{\nu} \left\{ \cosh\left[\nu_0 z\right] - \tanh\left[\nu_0 h\right] \sinh\left[\nu_0 z\right] \right\} \cos\left(\nu_0 y\right)$$

From the previous subsections follows the asymptotic expression for the disturbance potential in still water with $\bar{V} = 1$:

$$\begin{aligned}\varphi_{y\to\infty} &= \frac{\pi\cosh^2[\nu_0 h]}{\nu_0 h + \sinh[\nu_0 h]\cosh[\nu_0 h]}\left(\cosh[\nu_0 z] - \tanh[\nu_0 h]\sinh[\nu_0 z]\right)\sin(\nu_0 y)\cdot \\ &\cdot\left[A'_{0r} + jA'_{0j} - \left(\nu_0^2 - \nu^2\right)\sum_{n=1}^{\infty}\left(A'_{nr} + jA'_{nj}\right)\nu_0^{2(n-1)}\right]\end{aligned}$$

Substituting this in equation (Keil-32), provides:

$$\begin{aligned}F_E &= -\rho g\bar{\zeta}\nu_0 e^{j\omega t}\frac{2\pi\cosh^2[\nu_0 h]}{\nu_0 h + \sinh[\nu_0 h]\cosh[\nu_0 h]}\cdot \\ &\cdot\int_0^h\left(\cosh[\nu_0 z] - \tanh[\nu_0 h]\sinh[\nu_0 z]\right)^2 dz\cdot \\ &\cdot\left[A'_{0r} + jA'_{0j} - \left(\nu_0^2 - \nu^2\right)\sum_{n=1}^{\infty}\left(A'_{nr} + jA'_{nj}\right)\nu_0^{2(n-1)}\right] \\ &= -\rho g\bar{\zeta}\pi e^{j\omega t}\left[A'_{0r} + jA'_{0j} - \left(\nu_0^2 - \nu^2\right)\sum_{n=1}^{\infty}\left(A'_{nr} + jA'_{nj}\right)\nu_0^{2(n-1)}\right]\end{aligned}$$

Non-dimensional:

$$\begin{aligned}\bar{E}_1 + \bar{E}_2 &= \frac{\mathrm{Re}\left\{\bar{F}_E\right\}}{\rho g\bar{\zeta}B} = -\frac{\pi}{B}\left[A'_{0r} - \left(\nu_0^2 - \nu^2\right)\sum_{n=1}^{\infty}A'_{nr}\nu_0^{2(n-1)}\right] \\ \bar{E}_3 &= \frac{\mathrm{Im}\left\{\bar{F}_E\right\}}{\rho g\bar{\zeta}B} = -\frac{\pi}{B}\left[A'_{0j} - \left(\nu_0^2 - \nu^2\right)\sum_{n=1}^{\infty}A'_{nj}\nu_0^{2(n-1)}\right]\end{aligned} \qquad \text{(Keil-32a)}$$

Solution

The Lewis transformation of a cross section is given by:

$$y + iz = e^{+i\cdot\theta} + ae^{-i\cdot\theta} + be^{-i\cdot 3\theta} \qquad \text{(Keil-33)}$$

Then, the coordinates of the cross section are:

$$\begin{aligned}y &= (1 + a)\cos\theta + b\cos(3\theta) \\ z &= (1 - a)\sin\theta - b\sin(3\theta)\end{aligned} \qquad \text{(Keil-33a)}$$

Then:

$$z + iy = i(y - iz) = i\left(e^{-i\cdot\theta} + ae^{+i\cdot\theta} + be^{+i\cdot 3\theta}\right) \qquad \text{(Keil-33b)}$$

All calculations will be carried out in the Lewis domain. Scale factors are given in the table below.

	Ship	Lewis Form	$\frac{\text{Ship}}{\text{Lewis Form}}$
Breadth	BR	$2(1+a+b)$	$BR/\{2(1+a+b)\}$
Draft	TI	$1-a+b$	$TI/(1-a+b)$
Water depth	$HT \cdot TI$	$WT = HT \cdot (1-a+b)$	$TI/(1-a+b)$
Wave number	ν_0	$WF = \nu_0 \cdot TI/(1-a+b)$	$(1-a+b)/TI$
Acceleration	g	g	1
Forces	F_G	F	$\{TI/(1-a+b)\}^2$

The yet unknown complex coefficients A_n $(0 \leqslant n \leqslant \infty)$, the source strengths of the by the flow generated singularity, can be determined by substituting the stream function (Keil-19) and the coordinates of the cross section in the relevant boundary conditions (Keil-5a) through (Keil-6b).

$$\Psi\left[(y,z)_{body}, t\right] = \Psi_{body}\left[(y,z)_{body}, t\right] \qquad \text{(Keil-34)}$$

To determine the unknowns A_n, an equal number of equations has to be formulated. Because Lewis forms are used only here, a simple approach is possible.
All stream function parts and boundary conditions can be given as a Fourier series:

$$\psi_n = \sum_{m=0}^{\infty} \{c_{nm} \sin[2m\theta] + d_{nm} \cos[(2m+1)\theta]\}$$

or with

$$\cos[(2m+1)\theta] = 1 - \frac{2\theta}{\pi} + \frac{2}{\pi}\sum_{k=1}^{\infty} \frac{(2m+1)^2}{k\left[4k^2-(2m+1)^2\right]} \sin(2k\theta)$$

by:

$$\psi_n = a_{n0} + \sum_{m=1}^{\infty} a_{nm} \sin[2m\theta]$$

The solution of the by equating coefficients generated equations provide the unknowns A_n'.

4.2.4 Horizontal Motions

Boundary Conditions

The for the vertical motions made first four assumptions are valid for horizontal motions too. The potential must fulfil the motion-dependent boundary conditions which have been substituted in the equations (Keil-1) through (Keil-4). However, the fifth boundary condition needs here a new formulation.
Because two motions (a translation and a rotation) are considered here, follows in still water from:

$$\left[\frac{\partial \Phi}{\partial n}\right]_{body} = [v_n]_{body}$$

two boundary conditions:

1. For sway:

$$\begin{aligned} \left[\frac{\partial \Phi}{\partial n}\right]_{body} &= [v_n]_{body} \\ &= \left[\frac{\partial \Phi}{\partial z} \cdot \frac{dy}{ds} - \frac{\partial \Phi}{\partial y} \cdot \frac{dz}{ds}\right]_{body} \\ &= -\left[\frac{d\Psi}{ds}\right]_{body} \\ &= -\bar{U} e^{j\omega t} \left[\frac{dz}{ds}\right]_{body} \end{aligned}$$

   or:

$$d\Psi_{body} = \bar{U} e^{j\omega t} \left[dz\right]_{body}$$

   from which follows:

$$\Psi_{body}\left(y, z, t\right) = \bar{U} e^{j\omega t} \left[z\right]_{body} + C \qquad \text{(Keil-35a)}$$

2. For roll:

$$\begin{aligned} \left[\frac{\partial \Phi}{\partial n}\right]_{body} &= [v_n]_{body} \\ &= \left[\frac{\partial \Phi}{\partial z} \cdot \frac{dy}{ds} - \frac{\partial \Phi}{\partial y} \cdot \frac{dz}{ds}\right]_{body} \\ &= -\left[\frac{d\Psi}{ds}\right]_{body} \\ &= -\bar{\varphi} \omega e^{j\omega t} \left[\frac{dr}{ds} r\right]_{body} \end{aligned}$$

   or:

$$d\Psi_{body} = -\bar{\varphi} \omega e^{j\omega t} \left[r\, dr\right]_{body}$$

   from which follows:

$$\Psi_{body}\left(y, z, t\right) = -\frac{\bar{\varphi}\omega}{2} e^{j\omega t} \left[y^2 + z^2\right]_{body} + C \qquad \text{(Keil-35b)}$$

For the restrained body in waves, only the force in the horizontal direction and the moment about the longitudinal axis of the body will be calculated. One gets in beam waves only the in y point-symmetric part of the potential and the in y symmetric part of the stream function of the wave (see Appendix 1), respectively:

$$[\Psi_s(y,z,t)]_{body} = \frac{\bar{\zeta}\omega}{\nu} \cdot e^{jwt} \cdot [(\sinh[\nu_0 z] - \tanh[\nu_0 h]\cosh[\nu_0 z]) \cdot \cos(\nu_0 y)]_{body} \quad \text{(Keil-36a)}$$

and in oblique waves:

$$\begin{aligned} \left[\tilde{\Psi}_s(x,y_1,z_1,t)\right]_{body} = & \frac{\bar{\zeta}\omega}{\nu} \cdot e^{jwt} \cdot \sin(\nu_0 x \cos\mu) \cdot \qquad \text{(Keil-36b)} \\ & [\sin\mu \cos(\nu_0 y_1 \sin\mu) \cdot (\sinh[\nu_0 z_1] - \tanh[\nu_0 h]\cosh[\nu_0 z_1]) \\ & -\nu_0(1-\sin^2\mu)\int\limits_0^{y_1} \sin(\nu_0 y \sin\mu) \cdot (\sinh[\nu_0 z] - \tanh[\nu_0 h]\cosh[\nu_0 z])\,dy \Bigg]_{body} \end{aligned}$$

Potentials

2-D Radiation Potential In a similar way as equation (Keil-10) for heave, the three dimensional radiation potential for sway and roll can be derived as:

$$\begin{aligned} \Phi_{0r}(x,y,z,t) = e^{j\omega t} \int\limits_{-L/2}^{+L/2} A_0(\xi) \int\limits_0^{\infty} \cos[k_x(x-\xi)] \cdot \qquad \text{(Keil-37)} \\ \cdot \int\limits_0^{\infty} \frac{\sqrt{k_x^2+k_y^2} \cdot \cosh\left[\sqrt{k_x^2+k_y^2} \cdot (z-h)\right]}{\nu\cosh\left[\sqrt{k_x^2+k_y^2} \cdot h\right] - \sqrt{k_x^2+k_y^2} \cdot \sinh\left[\sqrt{k_x^2+k_y^2} \cdot h\right]} \sin(k_y y) dk_x\, dk_y\, d\xi \end{aligned}$$

This becomes for the two-dimensional case:

$$\Phi_{0r}(y,z,t) = e^{j\omega t} A_0 \int\limits_0^{\infty} \frac{k\cosh[k(z-h)]}{\nu\cosh[kh] - k\sinh[kh]} \sin(ky)\,dk \quad \text{(Keil-38)}$$

With the Sommerfeld radiation Condition (Keil-4) and Appendix 3 the total radiation potential becomes:

$$\begin{aligned} \Phi_{0r}(y,z,t) + j\Phi_{0j}(y,z,t) = & \; e^{j\omega t} A_0 \cdot \qquad \text{(Keil-39)} \\ & \cdot \left\{ \int\limits_0^{\infty} \frac{k\cosh[k(z-h)]}{\nu\cosh[kh] - k\sinh[kh]} \sin(ky)\,dk \right. \\ & \left. + j\pi\nu_0 \frac{\cosh[\nu_0 h]\cosh[\nu_0(z-h)]}{\nu_0 h + \sinh[\nu_0 h]\cosh[\nu_0 h]} \sin(\nu_0 y) \right\} \end{aligned}$$

and the stream function becomes:

$$\Psi_{0r}(y,z,t) + j\Psi_{0j}(y,z,t) = e^{j\omega t}A_0 \cdot \left\{ \int_0^\infty \frac{k\sinh[k(z-h)]}{\nu\cosh[kh] - k\sinh[kh]}\cos(ky)\,dk + j\pi\nu_0 \frac{\sinh[\nu_0(z-h)]\cosh[\nu_0 h]}{\nu_0 h + \sinh[\nu_0 h]\cosh[\nu_0 h]}\cos(\nu_0 y) \right\} \quad \text{(Keil-40)}$$

Potential and stream function are separated by:

$$\Phi_0 = \Phi_{0r\infty} + \Phi_{0rad} + j\Phi_{0j} = e^{j\omega t} \cdot \left\{ \int_0^\infty \frac{ke^{-kz}}{\nu - k}\sin(ky)\,dk + \int_0^\infty \frac{ke^{-kh}}{\nu-k} \cdot \frac{\nu\sinh[kz] - k\cosh[kz]}{\nu\cosh[kh] - k\sinh[kh]}\sin(ky)\,dk + j\pi\nu_0 \frac{\cosh[\nu_0(z-h)]\cosh[\nu_0 h]}{\nu_0 h + \sinh[\nu_0 h]\cosh[\nu_0 h]}\sin(\nu_0 y) \right\} \quad \text{(Keil-39-b)}$$

$$\Psi_0 = \Psi_{0r\infty} + \Psi_{0rad} + j\Psi_{0j} = e^{j\omega t} \cdot \left\{ \int_0^\infty \frac{ke^{-kh}}{\nu - k}\cos(ky)\,dk + \int_0^\infty \frac{ke^{-kh}}{\nu-k} \cdot \frac{\nu\cosh[kz] - k\sinh[kz]}{\nu\cosh[kh] - k\sinh[kh]}\cos(ky)\,dk + j\pi\nu_0 \frac{\sinh[\nu_0(z-h)]\cosh[\nu_0 h]}{\nu_0 h + \sinh[\nu_0 h]\cosh[\nu_0 h]}\cos(\nu_0 y) \right\} \quad \text{(Keil-40b)}$$

2-D Multi-Potential The two-dimensional multi-potential becomes:

$$\Phi_{nr\infty}(y,z,t) = -e^{j\omega t}A_n \int_0^\infty (k+\nu)k^{2n-1}e^{-kz}\sin(ky)\,dk \quad \text{(Keil-41)}$$

$$\Phi_{nrad}(y,z,t) = -e^{j\omega t}A_n \int_0^\infty (k+\nu)k^{2n-1}e^{-kh}\frac{\nu\sinh[kz] - k\cosh[kz]}{\nu\cosh[kh] - k\sinh[kh]}\sin(ky)\,dk \quad \text{(Keil-41a)}$$

$$\Phi_{nj}(y,z,t) = +e^{j\omega t}A_n \frac{\pi\nu_0^{2n+1}}{\cosh[\nu_0 h]} \cdot \frac{\cosh[\nu_0(z-h)]}{\nu_0 h + \sinh[\nu_0 h]\cosh[\nu_0 h]}\sin(\nu_0 y) \quad \text{(Keil-41b)}$$

The stream function related to it is:

$$\Psi_{nr\infty}(y,z,t) = +e^{j\omega t}A_n \int_0^\infty (k+\nu)k^{2n-1}e^{-kz}\cos(ky)\,dk$$

(Keil-42)

$$\Psi_{nrad}(y,z,t) = -e^{j\omega t}A_n \int_0^\infty (k+\nu)k^{2n-1}e^{-kh}\frac{\nu\cosh[kz]-k\sinh[kz]}{\nu\cosh[kh]-k\sinh[kh]}\cos(ky)\,dk$$

(Keil-42a)

$$\Psi_{nj}(y,z,t) = +e^{j\omega t}A_n\frac{\pi\nu_0^{2n+1}}{\cosh[\nu_0 h]}\cdot\frac{\sinh[\nu_0(z-h)]}{\nu_0 h+\sinh[\nu_0 h]\cosh[\nu_0 h]}\cos(\nu_0 y)$$

(Keil-42b)

Total Potentials With exception of the complex constant A_n with $(0 \leqslant n \leqslant \infty)$, the potential is known:

$$\begin{aligned}\Phi(y,z,t)+i\Psi(y,z,t) &= \qquad\qquad\qquad\qquad\qquad\qquad \text{(Keil-43)}\\ +e^{j\omega t}[A_{0r}+jA_{0j}] & \left\{\int_0^\infty \frac{k\sin(k[y+i(z-h)])}{\nu\cosh[kh]-k\sinh[kh]}\,dk \right.\\ & \left. +j\pi\nu_0\frac{\cosh[\nu_0 h]\sin(\nu_0[y+i(z-h)])}{\nu_0 h+\sinh[\nu_0 h]\cosh[\nu_0 h]}\right\}\\ -e^{j\omega t}\sum_{n=1}^{\infty}[A_{nr}+jA_{nj}] & \left\{\int_0^\infty (\nu^2-k^2)k^{2n-1}\frac{\sin(k[y+i(z-h)])}{\nu\cosh[kh]-k\sinh[kh]}\,dk\right.\\ & \left. -j\frac{\pi\nu_0^{2n+1}}{\cosh[\nu_0 h]}\cdot\frac{\sin\{\nu_0(y+i(z-h))\}}{\nu_0 h+\sinh[\nu_0 h]\cosh[\nu_0 h]}\right\}\end{aligned}$$

Writing in a similar way as for heave provides:

$$\Phi(y,z,t) = e^{j\omega t}\sum_{n=0}^{\infty}\left\{A_{nr}[\varphi_{nr\infty}+\varphi_{nrad}]-A_{nj}\varphi_{nj}+j[A_{nj}(\varphi_{nr\infty}+\varphi_{nrad})]+A_{nr}\varphi_{nj}\right\}$$

(Keil-43a)

and

$$\Psi(y,z,t) = e^{j\omega t}\sum_{n=0}^{\infty}\left\{A_{nr}[\psi_{nr\infty}+\psi_{nrad}]-A_{nj}\psi_{nj}+j[A_{nj}(\psi_{nr\infty}+\psi_{nrad})]+A_{nr}\psi_{nj}\right\}$$

(Keil-43b)

with:

$$\begin{aligned}A_r+jA_j &= \bar{U}\left[A_r^{'}+jA_j^{'}\right] \qquad \text{for sway}\\ & \quad\ \bar{\varphi}\omega\left[A_r^{'}+jA_j^{'}\right] \qquad \text{for roll}\end{aligned}$$

$A_n^{'}$ has the dimension $[L^{2n+2}]$. Then, $A_n^{'}\varphi_n$ and $A_n^{'}\psi_n$ have for sway the dimension $[L]$ and for roll the dimension $[L^2]$.
The determination of the coefficients $A_n^{'}$ follow from the conditions at the body contour.

Expansion of Potential Parts

For $\nu r = \nu\sqrt{y^2+z^2} \to 0$:

$$\begin{aligned}\varphi_{0r\infty} &= -\frac{y}{y^2+z^2} + \nu e^{-\nu z}\left\{\sin(\nu y)\left[\gamma + \ln(\nu r) + \sum_{m=1}^{\infty}\frac{\nu^m}{m\cdot m!}\operatorname{Re}\{(z+iy)^m\}\right]\right. \\ &\left. - \cos(\nu y)\left[\arctan\frac{y}{z} + \sum_{m=1}^{\infty}\frac{\nu^m}{m\cdot m!}\operatorname{Im}\{(z+iy)^m\}\right]\right\} \qquad \text{(Keil-44)} \\ \psi_{0r\infty} &= +\frac{z}{y^2+z^2} - \nu e^{-\nu z}\left\{\cos(\nu y)\left[\gamma + \ln(\nu r) + \sum_{m=1}^{\infty}\frac{\nu^m}{m\cdot m!}\operatorname{Re}\{(z+iy)^m\}\right]\right. \\ &\left. + \sin(\nu y)\left[\arctan\frac{y}{z} + \sum_{m=1}^{\infty}\frac{\nu^m}{m\cdot m!}\operatorname{Im}\{(z+iy)^m\}\right]\right\} \qquad \text{(Keil-44a)}\end{aligned}$$

with the Euler constant: $\gamma = 0.57722$.
For $\nu r = \nu\sqrt{y^2+z^2} \to \infty$:

$$\begin{aligned}\varphi_{0r\infty} &= -\frac{y}{y^2+z^2} + \nu\sum_{m=1}^{M}\frac{(m-1)!}{\nu^m r^{2m}}\operatorname{Im}\{(z+iy)^m\} - \pi\nu e^{-\nu z}\frac{y}{|y|}\cos(\nu y) \\ \psi_{0r\infty} &= +\frac{z}{y^2+z^2} - \nu\sum_{m=1}^{M}\frac{(m-1)!}{\nu^m r^{2m}}\operatorname{Re}\{(z+iy)^m\} - \pi\nu e^{-\nu z}\frac{y}{|y|}\sin(\nu y)\end{aligned}$$

Mind you that $\frac{(m-1)!}{\nu^m r^{2m}}\{(z+iy)^m\}$ is semi–convergent.

$$\varphi_{nr\infty} = (-1)^{n+1}(2n)!\left\{\operatorname{Re}\left\{(y+iz)^{-(2n+1)}\right\} + \frac{\nu}{2n}\operatorname{Im}\left\{(y+iz)^{-2n}\right\}\right\} \qquad \text{(Keil-45)}$$

$$\psi_{nr\infty} = (-1)^{n+1}(2n)!\left\{\operatorname{Im}\left\{(y+iz)^{-(2n+1)}\right\} - \frac{\nu}{2n}\operatorname{Re}\left\{(y+iz)^{-2n}\right\}\right\} \qquad \text{(Keil-45a)}$$

$$\varphi_{0rad} = \sum_{t=0}^{\infty}\left\{\frac{G(2t+3)}{(2t+1)!}\operatorname{Im}\left\{(z+iy)^{2t+1}\right\} - \nu\frac{G(2t+1)}{(2t)!}\operatorname{Im}\left\{(z+iy)^{2t}\right\}\right\} \qquad \text{(Keil-46)}$$

$$\Psi_{0rad} = \sum_{t=0}^{\infty}\left\{\frac{G(2t+3)}{(2t+1)!}\operatorname{Re}\left\{(z+iy)^{2t+1}\right\} - \nu\frac{G(2t+1)}{(2t)!}\operatorname{Re}\left\{(z+iy)^{2t}\right\}\right\} \qquad \text{(Keil-46a)}$$

$$\varphi_{nrad} = \sum_{t=0}^{\infty}\left\{\frac{G(2t+2n+3) - \nu^2\cdot G(2t+2n+1)}{(2t+1)!}\operatorname{Im}\left\{(z+iy)^{2t+1}\right\}\right.$$

$$-\nu\frac{G(2t+2n+1)-\nu^2\cdot G(2t+2n-1)}{(2t)!}\,\mathrm{Im}\left\{(z+iy)^{2t}\right\}\Bigg\}$$

(Keil-47)

$$\psi_{nrad} = \sum_{t=0}^{\infty}\left\{\frac{G(2t+2n+3)-\nu^2\cdot G(2t+2n+1)}{(2t+1)!}\,\mathrm{Re}\left\{(z+iy)^{2t+1}\right\}\right.$$
$$\left.-\nu\frac{G(2t+2n+1)-\nu^2\cdot G(2t+2n-1)}{(2t)!}\,\mathrm{Re}\left\{(z+iy)^{2t}\right\}\right\} \quad \text{(Keil-47a)}$$

$$\varphi_{oj} = +\pi\nu_0\frac{\cosh^2(\nu_0 h)}{\nu_0 h+\sinh[\nu_0 h]\cosh[\nu_0 h]}\cdot \quad \text{(Keil-48)}$$
$$\cdot\left\{\sum_{t=0}^{\infty}\frac{\nu_0^{2t+1}}{(2t+1)!}\,\mathrm{Im}\left\{(z+iy)^{2t+1}\right\}-\tanh[\nu_0 h]\sum_{t=0}^{\infty}\frac{\nu_0^{2t}}{(2t)!}\,\mathrm{Im}\left\{(z+iy)^{2t}\right\}\right\}$$

$$\psi_{oj} = +\pi\nu_0\frac{\cosh^2(\nu_0 h)}{\nu_0 h+\sinh[\nu_0 h]\cosh[\nu_0 h]} \quad \text{(Keil-48a)}$$
$$\cdot\left\{\sum_{t=0}^{\infty}\frac{\nu_0^{2t+1}}{(2t+1)!}\,\mathrm{Re}\left\{(z+iy)^{2t+1}\right\}-\tanh[\nu_0 h]\sum_{t=0}^{\infty}\frac{\nu_0^{2t}}{(2t)!}\,\mathrm{Re}\left\{(z+iy)^{2t}\right\}\right\}$$

$$\varphi_{nj} = +\nu_0^{2(n-1)}\left(\nu_0^2-\nu^2\right)\cdot\varphi_{oj} \quad \text{(Keil-49)}$$
$$\psi_{nj} = +\nu_0^{2(n-1)}\left(\nu_0^2-\nu^2\right)\cdot\psi_{oj} \quad \text{(Keil-49a)}$$

Zero-Frequency Potential

[Grim, 1956] and [Grim, 1957] give for the horizontal motions at zero frequency the complex potential:

$$\varphi+i\psi = \sum_{n=0}^{\infty}A_n\left\{(y+iz)^{-(2n+1)}+\right. \quad \text{(Keil-50)}$$
$$\left.\sum_{m=1}^{\infty}\left\{(y+iz+i2mh)^{-(2n+1)}+(y+iz-i2mh)^{-(2n+1)}\right\}\right\}$$

For Lewis forms this becomes:

$$\varphi+i\psi = \sum_{n=0}^{\infty}A_n\left\{e^{-i(2n+1)\theta}\sum_{p=0}^{\infty}(-1)^p\binom{2n+p}{p}\left(ae^{-i2\theta}+be^{-i4\theta}\right)^p\right.$$
$$+i2\sum_{m=1}^{\infty}\left\{(i2mH)^{-(2n+1)}\cdot\right.$$
$$\left.\left.\cdot\sum_{p=0}^{\infty}(-1)^p\binom{2p+2n+1}{2p+1}\left(\frac{e^{+i\theta}+ae^{-i\theta}+be^{-i3\theta}}{2mH}\right)^{2p+1}\right\}\right\}$$

$$= \sum_{n=0}^{\infty} A_n \left\{ \sum_{p=0}^{\infty} \left\{ (-1)^p \sum_{l=0}^{p} \binom{2n+p}{p} \binom{p}{l} a^{p-l} b^l e^{-i(2n+2p+2l+1)\theta} \right\} \right.$$
$$+2 \sum_{m=1}^{\infty} \sum_{p=0}^{\infty} (-1)^{p+n} \left[(2mH)^{-2(p+n+1)} \sum_{l=0}^{2p+1} \sum_{k=0}^{l} \right. \qquad \text{(Keil-50a)}$$
$$\left. \left. \binom{2n+2p+1}{2p+1} \binom{2p+1}{l} \binom{l}{\nu_0} a^{l-k} b^k e^{i(2p-2l-2k+1)\theta} \right] \right\}$$

These sums converge as long as:

$$\frac{e^{+i\theta} + ae^{-i\theta} + be^{-i3\theta}}{2m \cdot H} = \frac{e^{+i\theta} + ae^{-i\theta} + be^{-i3\theta}}{2m \cdot HT \cdot (1-a+b)} < 1 \qquad \text{(Keil-51)}$$

Because $m \geqslant 1$, it follows:

$$2HT \geqslant \frac{1 + ae^{-i2\theta} + be^{-i4\theta}}{1-a+b}$$

The potential converges too when:

$$2HT \quad \geqslant \quad \frac{1+a+b}{1-a+b} = \frac{B}{2T} \qquad \text{(Keil-51a)}$$

$$\text{breadth} \quad \leqslant \quad 4 \text{ x water depth} \qquad \text{(Keil-51b)}$$

Hydrodynamic Loads

The hydrodynamic force at sway oscillations in still water becomes:

$$F_Q \;=\; F_{Qr} + jF_{Qj} = -\rho\omega\bar{U}e^{j\omega t} \cdot \qquad \text{(Keil-52)}$$
$$\cdot \sum_{n=0}^{\infty} \int_S \left[A'_{nr}\varphi_{nr} - A'_{nj}\varphi_{nj} + j\left(A'_{nr}\varphi_{nj} + A'_{nj}\varphi_{nr} \right) \right] dz$$

and at roll oscillations:

$$F_R \;=\; F_{Rr} + jF_{Rj} = -\rho\omega^2\bar{\varphi}e^{j\omega t} \cdot \qquad \text{(Keil-52a)}$$
$$\cdot \sum_{n=0}^{\infty} \int_S \left[A'_{nr}\varphi_{nr} - A'_{nj}\varphi_{nj} + j\left(A'_{nr}\varphi_{nj} + A'_{nj}\varphi_{nr} \right) \right] dz$$

The hydrodynamic moment at sway oscillations in still water becomes:

$$M_Q \;=\; M_{Qr} + jM_{Qj} = -\rho\omega\bar{U}e^{j\omega t} \cdot \qquad \text{(Keil-53)}$$
$$\cdot \sum_{n=0}^{\infty} \int_S \left[A'_{nr}\varphi_{nr} - A'_{nj}\varphi_{nj} + j\left(A'_{nr}\varphi_{nj} + A'_{nj}\varphi_{nr} \right) \right] [ydy + zdz]$$

and at roll oscillations:

$$\begin{aligned} M_R &= M_{Rr} + jM_{Rj} = -\rho\omega^2\bar{\varphi}e^{j\omega t}\cdot \\ &\cdot\sum_{n=0}^{\infty}\int_S \left[A'_{nr}\varphi_{nr} - A'_{nj}\varphi_{nj} + j\left(A'_{nr}\varphi_{nj} + A'_{nj}\varphi_{nr}\right)\right][ydy + zdz] \end{aligned} \tag{Keil-53a}$$

Of course, the coefficients A'_{nr} and A'_{nj} of sway and roll will differ.
Fictive moment levers are defined by:

$$\begin{aligned} \bar{H}_{Qr} &= \frac{\bar{M}_{Qr}}{\bar{U}\omega m"} & \bar{H}_{Qj} &= \frac{\bar{M}_{Qj}}{\bar{U}N_Q} \\ \bar{H}_{Rr} &= \frac{I"\omega^2\bar{\varphi}}{\bar{F}_{Rr}} & \bar{H}_{Rj} &= \frac{N_R\omega\bar{\varphi}B}{2\bar{F}_{Rj}} \end{aligned}$$

Non-dimensional values for the sway motions are:

$$\begin{aligned} C_H &= \frac{m"}{\rho\frac{\pi}{2}T^2} = \frac{\bar{F}_{Qr}}{\rho\omega\bar{U}\frac{\pi}{2}T^2} \\ \bar{A}_Q^2 &= \frac{\bar{\zeta}^2}{\bar{y}^2} = \frac{\nu^2}{\rho\omega\bar{U}}\cdot\frac{\cosh^2[\nu_0 h]}{\nu_0 h + \sinh[\nu_0 h]\cosh[\nu_0 h]}\bar{F}_{Qj} \\ \frac{\bar{H}_{Qr}}{T} &= \frac{\bar{M}_{Qr}}{T\cdot\bar{F}_{Qr}} \\ \frac{\bar{H}_{Qj}}{T} &= \frac{\bar{M}_{Qj}}{T\cdot\bar{F}_{Qj}} \end{aligned} \tag{Keil-54}$$

and for roll motions:

$$\begin{aligned} C_R &= \frac{I"}{\rho\frac{\pi}{8}T^4} = \frac{\bar{M}_{Rr}}{\rho\omega^2\bar{\varphi}\frac{\pi}{8}T^4} \\ \bar{A}_R^2 &= \frac{\bar{\zeta}^2}{\bar{\varphi}^2\frac{B^2}{4}} = \frac{4\nu^2}{\rho\omega^2\bar{\varphi}B^2}\cdot\frac{\cosh^2[\nu_0 h]}{\nu_0 h + \sinh[\nu_0 h]\cosh[\nu_0 h]}\bar{M}_{Rj} \\ \frac{\bar{H}_{Rr}}{T} &= \frac{\bar{M}_{Rr}}{T\cdot\bar{F}_{Rr}} \\ \frac{\bar{H}_{Rj}}{T} &= \frac{\bar{M}_{Rj}}{T\cdot\bar{F}_{Rj}} \end{aligned} \tag{Keil-54a}$$

Wave Loads

The wave loads are separated in contributions of the undisturbed wave and diffraction:

$$F_E = F_1 + F_2 + jF_3 =$$

$$
\begin{aligned}
&= -\rho\omega \int_S \left[\Phi_W + \Phi_S\right] \, dz \\
&= -\rho\omega e^{j\omega t} \int_S \Bigg\{ -\frac{\bar{\zeta}\omega}{\nu} e^{-j\nu_0 x \cos\mu} \left(\cosh\left[\nu_0 z\right] - \tanh\left[\nu_0 h\right] \sinh\left[\nu_0 z\right]\right) \sin\left(\nu_0 y \sin\mu\right) \\
&\quad + \bar{U} \sum_{n=0}^{\infty} \left[A'_{nr}\varphi_{nr} - A'_{nj}\varphi_{nj} + j\left(A'_{nr}\varphi_{nj} + A'_{nj}\varphi_{nr}\right)\right] \Bigg\} \, dz \qquad \text{(Keil-55)}
\end{aligned}
$$

$$
\begin{aligned}
M_E &= M_1 + M_2 + jM_3 = \\
&= -\rho\omega e^{j\omega t} \int_S \Bigg\{ -\frac{\bar{\zeta}\omega}{\nu} e^{-j\nu_0 x \cos\mu} \left(\cosh\left[\nu_0 z\right] - \tanh\left[\nu_0 h\right] \sinh\left[\nu_0 z\right]\right) \sin\left(\nu_0 y \sin\mu\right) \\
&\quad + \bar{U} \sum_{n=0}^{\infty} \left[A'_{nr}\varphi_{nr} - A'_{nj}\varphi_{nj} + j\left(A'_{nr}\varphi_{nj} + A'_{nj}\varphi_{nr}\right)\right] \Bigg\} \left[ydy + zdz\right] \qquad \text{(Keil-56)}
\end{aligned}
$$

The separate parts are:

$$
\begin{aligned}
F_1 &= +\frac{\rho\omega^2\bar{\zeta}}{\nu} e^{j(\omega t - \nu_0 x \cos\mu)} \int_S \left\{ \left(\cosh\left[\nu_0 z\right] - \tanh\left[\nu_0 h\right] \sinh\left[\nu_0 z\right]\right) \sin\left(\nu_0 y \sin\mu\right) \right\} \, dz \\
F_2 &= -\rho\omega\bar{U} e^{j\omega t} \sum_{n=0}^{\infty} \left[A'_{nr}\varphi_{nr} - A'_{nj}\varphi_{nj} \right] \, dz \\
F_2 &= -\rho\omega\bar{U} e^{j\omega t} \sum_{n=0}^{\infty} \left[A'_{nr}\varphi_{nj} + A'_{nj}\varphi_{nr} \right] \, dz \\
M_1 &= +\frac{\rho\omega^2\bar{\zeta}}{\nu} e^{j(\omega t - \nu_0 x \cos\mu)} \int_S \left\{ \left(\cosh\left[\nu_0 z\right] - \tanh\left[\nu_0 h\right] \sinh\left[\nu_0 z\right]\right) \sin\left(\nu_0 y \sin\mu\right) \right\} \cdot \\
&\quad \cdot \left[ydy + zdz\right] \\
M_2 &= -\rho\omega\bar{U} e^{j\omega t} \sum_{n=0}^{\infty} \left[A'_{nr}\varphi_{nr} - A'_{nj}\varphi_{nj} \right] \left[ydy + zdz\right] \\
M_2 &= -\rho\omega\bar{U} e^{j\omega t} \sum_{n=0}^{\infty} \left[A'_{nr}\varphi_{nj} + A'_{nj}\varphi_{nr} \right] \left[ydy + zdz\right]
\end{aligned}
$$

Dimensionless:

$$
\begin{aligned}
\bar{E}_1 &= \frac{\bar{F}_1}{\rho g \bar{\zeta} \nu_0 A_x} = +\frac{\tanh\left[\nu_0 h\right]}{\nu A_x} \int_S \left\{ \left(\cosh\left[\nu_0 z\right] - \tanh\left[\nu_0 h\right] \sinh\left[\nu_0 z\right]\right) \sin\left(\nu_0 y \sin\mu\right) \right\} \, dz \\
\bar{E}_2 &= \frac{\bar{F}_2}{\rho g \bar{\zeta} \nu_0 A_x} = -\frac{\tanh\left[\nu_0 h\right]}{A_x} \sum_{n=0}^{\infty} \left[A'_{nr}\varphi_{nr} - A'_{nj}\varphi_{nj} \right] \, dz \\
\bar{E}_3 &= \frac{\bar{F}_3}{\rho g \bar{\zeta} \nu_0 A_x} = -\frac{\tanh\left[\nu_0 h\right]}{A_x} \sum_{n=0}^{\infty} \left[A'_{nr}\varphi_{nj} + A'_{nj}\varphi_{nr} \right] \, dz \qquad \text{(Keil-55a)}
\end{aligned}
$$

$$\frac{\bar{H}_{Wr}}{T} = \frac{\bar{M}_1 + \bar{M}_2}{T \cdot \left(\bar{F}_1 + \bar{F}_2\right)} \qquad \frac{\bar{H}_{Wj}}{T} = \frac{\bar{M}_3}{T \cdot \bar{F}_3} \tag{Keil-56a}$$

The Haskind-Newman relations are valid too here:

$$\begin{aligned} \bar{E}_1 + \bar{E}_2 &= \frac{\text{Re}\left\{\bar{F}_E\right\}}{\rho g \bar{\zeta} \nu_0 A_x} = +\frac{\pi}{A_x}\left[A'_{0r} + \left(\nu_0^2 - \nu^2\right)\sum_{n=1}^{\infty} A'_{nr}\nu_0^{2(n-1)}\right] \\ \bar{E}_3 &= \frac{\text{Im}\left\{\bar{F}_E\right\}}{\rho g \bar{\zeta} \nu_0 A_x} = +\frac{\pi}{A_x}\left[A'_{0j} + \left(\nu_0^2 - \nu^2\right)\sum_{n=1}^{\infty} A'_{nj}\nu_0^{2(n-1)}\right] \end{aligned} \tag{Keil-55b}$$

Solution

To determine the unknowns A_n, an equal number of equations has to be formulated. Because Lewis forms are used only here, a simple approach is possible.
All stream function parts and boundary conditions can be given as a Fourier series:

$$\psi_n = \sum_{m=0}^{\infty} \left\{c_{nm} \sin\left[(2m+1)\,\theta\right] + d_{nm} \cos\left[2m\theta\right]\right\}$$

or with

$$\cos\left[2m\theta\right] = 1 + \frac{16}{\pi}\sum_{k=0}^{\infty} \frac{m^2}{(2k+2m+1)\,(2k-2m+1)\,(2k+1)} \sin\left[(2k+1)\,\theta\right]$$

in:

$$\psi_n = \sum_{m=0}^{\infty} a_{nm} \sin\left[(2m+1)\,\theta\right]$$

The solution of the by equating coefficients generated equations provide the unknowns A'_n.

4.2.5 Appendices

Appendix 1: Undisturbed Wave Potential

The general expression of the complex potential of a shallow wave, travelling in the negative y-direction, is:

$$\begin{aligned}
\Phi_W + i\Psi_W &= \bar{\zeta} \cdot c \cdot \frac{\cos\{\nu_0 (y + i(z-h)) + \omega t\}}{\sinh(\nu_0 h)} \qquad \text{with: } c = \frac{\omega}{\nu_0} \qquad \text{(Keil-A1)} \\
&= \frac{\bar{\zeta}\omega}{\nu_0 \sinh[\nu_0 h]} \{\cosh[\nu_0 (z-h)] \cos(\nu_0 y + \omega t) \\
&\qquad -i \sinh[\nu_0 (z-h)] \sin(\nu_0 y + \omega t)\} \\
&= \frac{\bar{\zeta}\omega}{\nu \cosh[\nu_0 h]} \{\cosh[\nu_0 (z-h)] \cos(\nu_0 y + \omega t) \\
&\qquad -i \sinh[\nu_0 (z-h)] \sin(\nu_0 y + \omega t)\}
\end{aligned}$$

$$\begin{aligned}
\Phi_W &= \frac{\bar{\zeta}\omega}{\nu \cosh[\nu_0 h]} \cosh[\nu_0 (z-h)] \cos(\nu_0 y + \omega t) \\
&= \frac{\bar{\zeta}\omega}{\nu \cosh[\nu_0 h]} e^{j\omega t} \cdot \cosh[\nu_0 (z-h)]\, e^{j\nu_0 y} \\
&= \frac{\bar{\zeta}\omega}{\nu} e^{j\omega t} \{\cosh[\nu_0 z] - \tanh[\nu_0 h] \sinh[\nu_0 z]\}\, e^{j\nu_0 y}
\end{aligned}$$

$$\begin{aligned}
\Psi_W &= -\frac{\bar{\zeta}\omega}{\nu \cosh[\nu_0 h]} \sinh[\nu_0 (z-h)] \sin(\nu_0 y + \omega t) \\
&= j \frac{\bar{\zeta}\omega}{\nu \cosh[\nu_0 h]} e^{j\omega t} \cdot \sinh[\nu_0 (z-h)]\, e^{j\nu_0 y} \\
&= j \frac{\bar{\zeta}\omega}{\nu} e^{j\omega t} \{\sinh[\nu_0 z] - \tanh[\nu_0 h] \cosh[\nu_0 z]\}\, e^{j\nu_0 y}
\end{aligned}$$

For the vertical motions is the in y symmetrical part of the potential significant. For the horizontal motions is the in y point symmetrical part - multiplied with j, so a phase shift of 90^0 - important.

$$\begin{aligned}
\Phi_{WV} &= +\frac{\bar{\zeta}\omega}{\nu} e^{j\omega t} \{\cosh[\nu_0 z] - \tanh[\nu_0 h] \sinh[\nu_0 z]\} \cos(\nu_0 y) && \text{(Keil-A2)} \\
\Psi_{WV} &= -\frac{\bar{\zeta}\omega}{\nu} e^{j\omega t} \{\sinh[\nu_0 z] - \tanh[\nu_0 h] \cosh[\nu_0 z]\} \sin(\nu_0 y) && \text{(Keil-A2a)} \\
\Phi_{WH} &= -\frac{\bar{\zeta}\omega}{\nu} e^{j\omega t} \{\cosh[\nu_0 z] - \tanh[\nu_0 h] \sinh[\nu_0 z]\} \sin(\nu_0 y) && \text{(Keil-A3)} \\
\Psi_{WH} &= -\frac{\bar{\zeta}\omega}{\nu} e^{j\omega t} \{\sinh[\nu_0 z] - \tanh[\nu_0 h] \cosh[\nu_0 z]\} \cos(\nu_0 y) && \text{(Keil-A3a)}
\end{aligned}$$

When the wave travels in the x_w-direction, the potential becomes:

$$\Phi_W = \frac{\bar{\zeta}\omega}{\nu} e^{j(\omega t - \nu_0 x_w)} \left\{ \cosh\left[\nu_0 z\right] - \tanh\left[\nu_0 h\right] \sinh\left[\nu_0 z\right] \right\}$$

With:

$$\begin{aligned} x_w &= x\cos\mu - y\sin\mu \\ y_w &= x\sin\mu + y\cos\mu \end{aligned}$$

the potential becomes:

$$\Phi_W = \frac{\bar{\zeta}\omega}{\nu} e^{j\omega t} e^{j\nu_0 (y\sin\mu - x\cos\mu)} \left\{ \cosh\left[\nu_0 z\right] - \tanh\left[\nu_0 h\right] \sinh\left[\nu_0 z\right] \right\}$$

This results for the vertical motions in:

$$\Phi_{WV} = +\frac{\bar{\zeta}\omega}{\nu} e^{j(\omega t - \nu_0 x\cos\mu)} \left\{ \cosh\left[\nu_0 z\right] - \tanh\left[\nu_0 h\right] \sinh\left[\nu_0 z\right] \right\} \cos\left(\nu_0 y \sin\mu\right) \qquad \text{(Keil-A4)}$$

and for the horizontal motions in:

$$\Phi_{WH} = -\frac{\bar{\zeta}\omega}{\nu} e^{j(\omega t - \nu_0 x\cos\mu)} \left\{ \cosh\left[\nu_0 z\right] - \tanh\left[\nu_0 h\right] \sinh\left[\nu_0 z\right] \right\} \sin\left(\nu_0 y \sin\mu\right) \qquad \text{(Keil-A5)}$$

Appendix 2: Series Expansions of Hyperbolic Functions

With:

$$\begin{aligned} z \pm iy &= \sqrt{y^2 + z^2} \cdot e^{\pm i \arctan \frac{y}{z}} \\ &= r \cdot e^{\pm i\beta} \end{aligned}$$

the following series expansions can be found.

$$\begin{aligned} \cosh\left[kz\right]\cos\left(ky\right) &= \cosh\left[kz\right] \cdot \cosh\left[iky\right] \\ &= \frac{1}{2}\left\{ \cosh\left[k\left(z+iy\right)\right] + \cosh\left[k\left(z-iy\right)\right] \right\} \\ &= \frac{1}{2}\left\{ \cosh\left[kre^{+i\beta}\right] + \cosh\left[kre^{-i\beta}\right] \right\} \\ &= \frac{1}{2}\left\{ \sum_{t=0}^{\infty} \frac{(kr)^{2t}}{(2t)!} e^{+i2t\beta} + \sum_{t=0}^{\infty} \frac{(kr)^{2t}}{(2t)!} e^{-i2t\beta} \right\} \\ &= \sum_{t=0}^{\infty} \frac{(kr)^{2t}}{(2t)!} \cos\left(2t\beta\right) \\ &= \sum_{t=0}^{\infty} \frac{k^{2t}}{(2t)!} \mathrm{Re}\left\{ (z+iy)^{2t} \right\} \end{aligned}$$

$$
\begin{aligned}
\sinh[kz]\cos(ky) &= \sinh[kz]\cdot\cosh[iky] \\
&= \frac{1}{2}\left\{\sinh[k(z+iy)]+\sinh[k(z-iy)]\right\} \\
&= \frac{1}{2}\left\{\sinh\left[kre^{+i\beta}\right]+\sinh\left[kre^{-i\beta}\right]\right\} \\
&= \frac{1}{2}\left\{\sum_{t=0}^{\infty}\frac{(kr)^{2t+1}}{(2t+1)!}e^{+i(2t+1)\beta}+\sum_{t=0}^{\infty}\frac{(kr)^{2t+1}}{(2t+1)!}e^{-i(2t+1)\beta}\right\} \\
&= \sum_{t=0}^{\infty}\frac{(kr)^{2t+1}}{(2t+1)!}\cos((2t+1)\beta) \\
&= \sum_{t=0}^{\infty}\frac{k^{2t+1}}{(2t+1)!}\mathrm{Re}\left\{(z+iy)^{2t+1}\right\}
\end{aligned}
$$

$$
\begin{aligned}
\sinh[kz]\sin(ky) &= -i\sinh[kz]\cdot\sinh[iky] \\
&= -\frac{i}{2}\left\{\cosh[k(z+iy)]-\cosh[k(z-iy)]\right\} \\
&= -\frac{i}{2}\left\{\cosh\left[kre^{+i\beta}\right]-\cosh\left[kre^{-i\beta}\right]\right\} \\
&= -\frac{i}{2}\left\{\sum_{t=0}^{\infty}\frac{(kr)^{2t}}{(2t)!}e^{+i2t\beta}-\sum_{t=0}^{\infty}\frac{(kr)^{2t}}{(2t)!}e^{-i2t\beta}\right\} \\
&= \sum_{t=0}^{\infty}\frac{(kr)^{2t}}{(2t)!}\sin(2t\beta) \\
&= \sum_{t=0}^{\infty}\frac{k^{2t}}{(2t)!}\mathrm{Im}\left\{(z+iy)^{2t}\right\}
\end{aligned}
$$

$$
\begin{aligned}
\cosh[kz]\sin(ky) &= -i\cosh[kz]\cdot\sinh[iky] \\
&= -\frac{i}{2}\left\{\sinh[k(z+iy)]-\sinh[k(z-iy)]\right\} \\
&= -\frac{i}{2}\left\{\sinh\left[kre^{+i\beta}\right]-\sinh\left[kre^{-i\beta}\right]\right\} \\
&= -\frac{i}{2}\left\{\sum_{t=0}^{\infty}\frac{(kr)^{2t+1}}{(2t+1)!}e^{+i(2t+1)\beta}-\sum_{t=0}^{\infty}\frac{(kr)^{2t+1}}{(2t+1)!}e^{-i(2t+1)\beta}\right\} \\
&= \sum_{t=0}^{\infty}\frac{(kr)^{2t+1}}{(2t+1)!}\sin((2t+1)\beta) \\
&= \sum_{t=0}^{\infty}\frac{k^{2t+1}}{(2t+1)!}\mathrm{Im}\left\{(z+iy)^{2t+1}\right\}
\end{aligned}
$$

Appendix 3: Treatment of Singular Points

The determination of Φ_{0j} and his terms - which can be added to Φ_{0r} in equation (Keil-11) with which the by $\Phi_{0r}+j\Phi_{0j}$ described flow of the waves (travelling from both sides of

the body away) is given - is also possible in another way. This approach is based on work carried out by Rayleigh and is given in the literature by [Lamb, 1932] for an infinite water depth.
In this approach, an viscous force $\rho\mu w$ will be included in the Euler equations, μ is the dynamic viscosity and w is the velocity. Because the fluid is assumed to be non-viscous, in a later stage this dynamic viscosity μ will be set to zero.
From the Euler equation follows with this viscosity force the with time changing pressure change:

$$\frac{\partial p}{\partial t} = \rho \lim_{\mu \to 0} \left[g\frac{\partial \Phi}{\partial z} - \mu \frac{\partial \Phi}{\partial t} - \frac{\partial^2 \Phi}{\partial t^2} \right]$$

From this follows the approach as given in a subsection before for the two-dimensional radiation potential:

$$\begin{aligned}
\Phi_{0r}(y,z,t) \quad & +j\Phi_{0j}(y,z,t) = \\
&= e^{j\omega t} A_0 \lim_{\mu \to 0} \int_0^\infty \frac{\cosh\left[k\left(z-h\right)\right]}{\left(\nu - j\frac{\omega\mu}{g}\right)\cosh\left[kh\right] - k\sinh\left[kh\right]} \cos\left(ky\right)\, dk \\
&= e^{j\omega t} A_0 \lim_{\mu \to 0} \left\{ \int_0^\infty \frac{e^{-kz}}{\nu - j\frac{\omega\mu}{g} - k} \cos(ky)\, dk \right. \\
&\quad \left. + \int_0^\infty \frac{e^{-kh}}{\nu - j\frac{\omega\mu}{g} - k} \cdot \frac{\left(\nu - j\frac{\omega\mu}{g}\right)\sinh\left[kz\right] - k\cosh\left[kz\right]}{\left(\nu - j\frac{\omega\mu}{g}\right)\cosh\left[kh\right] - k\sinh\left[kh\right]} \cos(ky)\, dk \right\} \\
&= e^{j\omega t} A_0 \left\{ \left(\varphi_{0r\infty} + j\varphi_{0j\infty}\right) + \left(\varphi_{0rad} + j\varphi_{0jad}\right) \right\}
\end{aligned}$$

The first integral leads to equation (Keil-12a) and the second integral can be expanded as follows:

$$\begin{aligned}
\Phi_{0rad} \quad & +j\Phi_{0jad} = \\
&= \lim_{\mu \to 0} \int_0^\infty \frac{e^{-kh}}{\nu - j\frac{\omega\mu}{g} - k} \cdot \frac{\left(\nu - j\frac{\omega\mu}{g}\right)\sinh\left[kz\right] - k\cosh\left[kz\right]}{\left(\nu - j\frac{\omega\mu}{g}\right)\cosh\left[kh\right] - k\sinh\left[kh\right]} \cos(ky)\, dk \\
&= \lim_{\mu \to 0} \sum_{t=0}^{\infty} \left\{ \left[\left(\nu - j\frac{\omega\mu}{g}\right) \frac{\mathrm{Re}\left\{(z+iy)^{2t+1}\right\}}{(2t+1)!} - \frac{\mathrm{Re}\left\{(z+iy)^{2t}\right\}}{(2t)!} \right] \cdot \right. \\
&\quad \left. \cdot \int_0^\infty \frac{e^{-kh} k^{2t+1}}{\left(\nu - j\frac{\omega\mu}{g} - k\right)\left[\left(\nu - j\frac{\omega\mu}{g}\right)\cosh\left[kh\right] - k\sinh\left[kh\right]\right]}\, dk \right\} \\
&= -\lim_{\mu \to 0} \sum_{t=0}^{\infty} \left\{ \left[\left(\nu - j\frac{\omega\mu}{g}\right) \frac{\mathrm{Re}\left\{(z+iy)^{2t+1}\right\}}{(2t+1)!} - \frac{\mathrm{Re}\left\{(z+iy)^{2t}\right\}}{(2t)!} \right] \cdot \right. \\
&\quad \left. \cdot \left[G\left(2t+1\right) + jH\left(2t+1\right)\right] \right\}
\end{aligned}$$

The function:

$$G(t) + iH(t) = \lim_{\mu \to 0} \int_0^\infty \frac{e^{-kh}}{k - \left(\nu - i\frac{\omega\mu}{g}\right)} \cdot \frac{k^t}{\left[\left(\nu - i\frac{\omega\mu}{g}\right)\cosh[kh] - k\sinh[kh]\right]} dk$$

will be normalized as done before:

$$\begin{aligned} G'(t) + iH'(t) &= h^{t-1}\left[G(t) + iH(t)\right] \\ &= \lim_{\mu \to 0} \int_0^\infty \frac{u^t e^{-u}}{u - \left(\nu h - i\frac{\omega\mu h}{g}\right)\left[\left(\nu h - i\frac{\omega\mu h}{g}\right)\cosh[hu] - u\sinh[hu]\right]} du \end{aligned}$$

This is a complex integral and must be solved in the complex domain with $w = u + iv$. The integrand has singularity for:

$$w_1 = \nu h - i\frac{\omega\mu h}{g} \qquad \text{and} \qquad w_2$$

where w_2 is the solution of the equation:

$$\left(\nu h - i\frac{\omega\mu h}{g}\right)\cosh[w] - w\sinh[w] = 0$$

see figure 4.8-a.

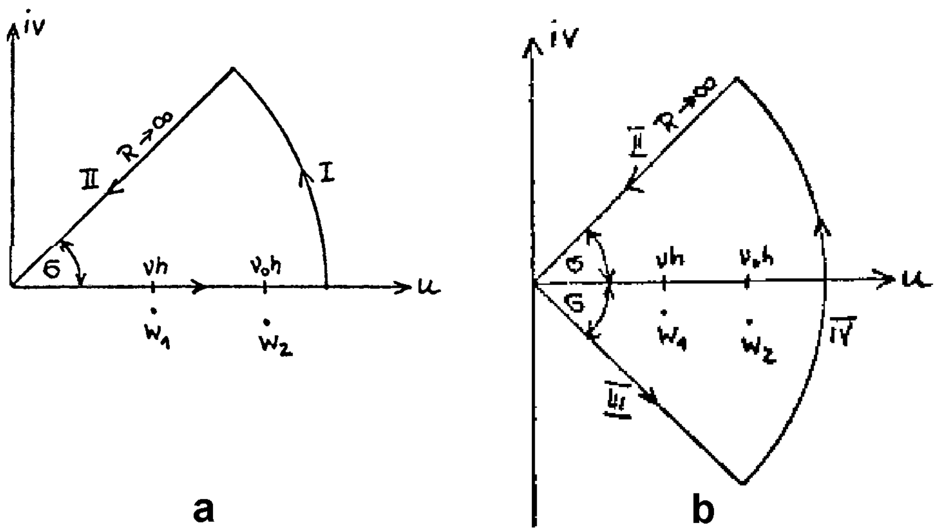

Figure 4.8: Treatment of Singularities

$$\begin{aligned} \lim_{\mu \to 0} \oint \frac{e^{-w}w^t}{(w - w_1)(w_1\cosh w - w\sinh w)} dw &= \lim_{\mu \to 0}\left\{\int_0^R ...du + \int_I ...dw + \int_{II} ...dw\right\} \\ &= \lim_{\mu \to 0}\left\{\int_0^R ...du + J_I + J_{II}\right\} \\ &= 0 \end{aligned}$$

Thus:

$$\begin{aligned} G'(t) + iH'(t) &= \lim_{\mu\to 0} \int_0^\infty \frac{u^t e^{-u}}{u - \left(\nu h - i\frac{\omega\mu h}{g}\right)\left[\left(\nu h - i\frac{\omega\mu h}{g}\right)\cosh[hu] - u\sinh[hu]\right]}\, du \\ &= \lim_{\mu\to 0}\lim_{R\to\infty} \{J_I + J_{II}\} \end{aligned}$$

Because:

$$\lim_{R\to\infty} J_I = 0$$

follows:

$$G'(t) + iH'(t) = -\lim_{\mu\to 0} \int_{II} \frac{w^t e^{-w}}{(w - w_1)(w_1\cosh[w] - w\sinh[w])}\, dw$$

The real part of this integral:

$$G'(t) = -\,\mathrm{Re}\left\{\lim_{\mu\to 0} \int_{II} \frac{w^t e^{-w}}{(w - w_1)(w_1\cosh[w] - w\sinh[w])}\, dw\right\}$$

will be calculated as done before. This integral has no singularity and the boundary $\mu \longrightarrow 0$ can be passed before integration; see figure 4.8-b.
The imaginary part of this integral:

$$H'(t) = -\,\mathrm{Im}\left\{\lim_{\mu\to 0} \int_{II} \frac{w^t e^{-w}}{(w - w_1)(w_1\cosh[w] - w\sinh[w])}\, dw\right\}$$

can be calculated numerically in an analog way. It is also possible to solve this integral independently by using another integral path:

$$\begin{aligned} \lim_{\mu\to 0} \oint \frac{w^t e^{-w}}{(w - w_1)(w_1\cosh[w] - w\sinh[w])}\, dw &= \lim_{\mu\to 0} \{J_{II} + J_{III} + J_{IV}\} \\ &= 2\pi i \lim_{\mu\to 0} \{Residue\,(w_1) + Residue\,(w_2)\} \end{aligned}$$

J_{IV} disappears for $R \rightarrow \infty$.
It can be found that:

$$\begin{aligned} \mathrm{Re}\{J_{II}\} &= -\,\mathrm{Re}\{J_{III}\} \\ \mathrm{Im}\{J_{II}\} &= +\,\mathrm{Im}\{J_{III}\} \end{aligned}$$

Then it follows:

$$
\begin{aligned}
H^{'}(t) &= -\operatorname{Im}\left\{\lim_{\mu\to 0} J_{II}\right\} \\
&= -\pi \lim_{\mu\to 0}\left\{Residue\left(w_1\right) + Residue\left(w_2\right)\right\} \\
&= \pi\left(\nu_0 h\right)^{t-1}\left\{\frac{\cosh^2\left[\nu_0 h\right]}{\nu_0 h + \sinh\left[\nu_0 h\right]\cosh\left[\nu_0 h\right]} - \tanh^{t-1}\left[\nu_0 h\right]\right\}
\end{aligned}
$$

and the imaginary additional potential becomes:

$$
\begin{aligned}
\Phi_{0jad} &= \sum_{t=0}^{\infty} H\left(2t+1\right)\left\{\frac{\operatorname{Re}\left\{\left(z+iy\right)^{2t}\right\}}{(2t)!} - \nu\frac{\operatorname{Re}\left\{\left(z+iy\right)^{2t+1}\right\}}{(2t+1)!}\right\} \\
&= \sum_{t=0}^{\infty} \pi\left\{\frac{\nu_0^{2t}\cosh^2\left[\nu_0 h\right]}{\nu_0 h + \sinh\left[\nu_0 h\right]\cosh\left[\nu_0 h\right]} - \nu^{2t}\right\}\cdot \\
&\qquad \cdot\left\{\frac{\operatorname{Re}\left\{\left(z+iy\right)^{2t}\right\}}{(2t)!} - \nu\frac{\operatorname{Re}\left\{\left(z+iy\right)^{2t+1}\right\}}{(2t+1)!}\right\} \\
&= \frac{\pi\cosh^2\left[\nu_0 h\right]}{\nu_0 h + \sinh\left[\nu_0 h\right]\cosh\left[\nu_0 h\right]}\cdot \\
&\quad \cdot\left\{\sum_{t=0}^{\infty}\frac{\nu_0^{2t}}{(2t)!}\operatorname{Re}\left\{\left(z+iy\right)^{2t}\right\} - \tanh\left[\nu_0 h\right]\sum_{t=0}^{\infty}\frac{\nu_0^{2t+1}}{(2t+1)!}\operatorname{Re}\left\{\left(z+iy\right)^{2t+1}\right\}\right\} \\
&\quad -\pi e^{-\nu z}\cos\left(\nu y\right)
\end{aligned}
$$

The same will be found as a difference between (Keil-25) and (Keil-12a).

4.3 Theory of Frank

As a consequence of conformal mapping to the unit circle, the cross sections need to have a certain breadth at the water surface; fully submersed cross sections, such as at the bulbous bow, cannot be mapped. Mapping problems can also appear for cross sections with too high or too low an area coefficient. These cases require another approach: the pulsating source method of Frank, also called Frank's Close-Fit Method. The report of [Frank, 1967] has used here to explain this method.

Hydrodynamic research of horizontal cylinders oscillating in or below the free surface of a deep fluid has increased in importance in the last decades and has been studied by a number of investigators. The history of this subject began with [Ursell, 1949], who formulated and solved the boundary-value problem for the semi-immersed heaving circular cylinder within the framework of linearized free-surface theory. He represented the velocity potential as the sum of an infinite set of multi-poles, each satisfying the linear free-surface condition and each being multiplied by a coefficient determined by requiring the series to satisfy the kinematic boundary condition at a number of points on the cylinder.
[Grim, 1953] used a variation of the Ursell method to solve the problem for two-parameter Lewis form cylinders by conformal mapping onto a circle. [Tasai, 1959] and [Porter, 1960], using the Ursell approach obtained the added mass and damping for oscillating contours mappable onto a circle by the more general Theodorsen transformation. [Ogilvie, 1963] calculated the hydrodynamic forces on completely submerged heaving circular cylinders.
Despite the success of the multipole expansion-mapping methods, [Frank, 1967] discusses the problem from a different view. The velocity potential is represented by a distribution of sources over the submerged cross section. The density of the sources is an unknown function (of position along the contour) to be determined from integral equations found by applying the kinematic boundary condition on the submerged part of the cylinder. The hydrodynamic pressures are obtained from the velocity potential by means of the linearized Bernoulli equation. Integration of these pressures over the immersed portion of the cylinder yields the hydrodynamic forces or moments.
A simpler approximation to the solution of the two-dimensional hydrodynamic problem was used in the strip theory of ship motions introduced by Korvin-Kroukovsky. The solution of two-dimensional water-wave problems for ship sections by multipole expansion and mapping techniques have been applied to this strip theory by several authors to predict the motions of surface vessels.
The work of [Frank, 1967] has largely been motivated by the desirability of devising a computer program, based on strip theory and independent of mapping techniques, to predict the response of surface ships moving with steady forward speed in oblique as well as head or following seas for all six degrees of freedom.

4.3.1 Notation

The notation of Frank is as follows:

$A^{(m)}$	=	oscillation amplitude in the m-th mode
B	=	beam of cross section C_0
C_0	=	submerged part of cross sectional contour in rest position
g	=	acceleration of gravity
$I_{ij}^{(m)}$	=	influence coefficient in-phase with displacement on the i-th midpoint due to the j-th segment in the m-th mode of oscillation
$J_{ij}^{(m)}$	=	influence coefficient in-phase with velocity on the i-th midpoint due to the j-th segment in the m-th mode of oscillation
$M_{(\omega)}^{(m)}$	=	added mass force or moment for the m-th mode of oscillation at frequency ω
N	=	number of line segments defining submerged portion of half section in rest position
$N_{(\omega)}^{(m)}$	=	damping force or moment for the m-th mode of oscillation at frequency ω
$n_i^{(m)}$	=	direction cosine of the normal velocity at i-th midpoint for the m-th mode of oscillation
PV	=	Cauchy pricipal value of integral
$p_a^{(m)}$	=	hydrodynamic pressure in-phase with displacement for the m-th mode of oscillation
$p_v^{(m)}$	=	hydrodynamic pressure in-phase with velocity for the m-th mode of oscillation
$Q_j^{(m)}$	=	source strength in-phase with displacement along the j-th segment for the m-th mode of oscillation
$Q_{j+N}^{(m)}$	=	source strength in-phase with velocity along the j-th segment for the m-th mode of oscillation
s	=	length variable along C_0
s_j	=	j-th line segment
T	=	draft of cross section
t	=	time
$v_i^{(m)}$	=	normal velocity component at the i-th midpoint for the m-th mode of oscillation
x_1	=	abscissa of the i-th midpoint
y_i	=	ordinate of the i-th midpoint
y_0	=	ordinate of the center of roll
$z = x + iy$	=	complex field point in region of fluid domain
$z_i = x_i + iy_i$	=	complex midpoint of i-th segment
α_i	=	angle between i-th segment and positive x-axis
ζ	=	complex variable along C_0
ζ_j	=	j-th complex input point along C_0
η_j	=	ordinate of j-th input point

$\nu = \omega^2/g$	=	wave number
ν_k	=	k-th irregular wave number for ajoint interior problem
ξ_j	=	abscissa of the j-th input point
ρ	=	density of fluid
$\Phi^{(m)}$	=	velocity potential for th m-th mode of oscillation
ω	=	radian frequency of oscillation
ω_k	=	k-th irregular frequency for adjoint interior problem (k-th eigen-frequency)

4.3.2 Formulation of the problem

Consider a cylinder, whose cross section is a simply connected region, which is fully or partially immersed horizontally in a previously undisturbed fluid of infinite depth. The body is forced into simple harmonic motion and it is assumed that steady state conditions have been attained.
The two-dimensional nature of the problem implies three degrees of freedom of motion. Therefore, consider the following three types of oscillatory motions: vertical or heave, horizontal or sway and rotational about a horizontal axis or roll.
To use linearized free-surface theory, the following assumptions are made:

1. The fluid is incompressible and inviscid.
2. The effects of surface tension are negligible.
3. The fluid is irrotational.
4. The motion amplitudes and velocities are small enough that all but the linear terms of the free-surface condition, the kinematic boundary condition on the cylinder and the Bernoulli equation may be neglected.

Given the above conditions and assumptions, the problem reduces to the following boundary-value problem of potential theory. The cylinder is forced into simple harmonic motion $A^{(m)} \cdot \cos(\omega t)$ with a prescribed radian frequency of oscillation ω, where the superscript may take on the values 2, 3 and 4, denoting swaying, heaving and rolling motions, respectively. It is required to find a velocity potential:

$$\Phi^{(m)}(x, y; t) = \mathrm{Re}\left\{\phi^{(m)}(x, y) \cdot e^{-i\omega t}\right\} \qquad \text{(Frank-1)}$$

satisfying the following conditions:

1. The Laplace equation:

$$\nabla^2 \Phi^{(m)} = \frac{\partial^2 \Phi^{(m)}}{\partial x^2} + \frac{\partial^2 \Phi^{(m)}}{\partial y^2} = 0 \qquad \text{(Frank-2)}$$

   in the fluid domain, i.e., for $y < 0$ outside the cylinder;

2. The free surface condition:

$$\frac{\partial^2 \Phi^{(m)}}{\partial t^2} + g \frac{\partial \Phi^{(m)}}{\partial y} = 0 \qquad \text{(Frank-3)}$$

   on the free surface $y = 0$ outside the cylinder, while g is the acceleration of gravity.

3. The sea bed boundary condition for deep water:

$$\lim_{y\to-\infty}\left|\nabla\Phi^{(m)}\right| = \lim_{y\to-\infty}\left|\frac{\partial\Phi^{(m)}}{\partial y}\right| = 0 \tag{Frank-4}$$

4. The condition of the normal velocity component of the fluid at the surface of the oscillating cylinder being equal to the normal component of the forced velocity of the cylinder. i.e., if v_n is the component of the forced velocity of the cylinder in the direction of the outgoing unit normal vector $\vec{n}$, then

$$\vec{n}\cdot\vec{\nabla}\Phi^{(m)} = v_n \tag{Frank-5}$$

   this kinematic boundary condition on the oscillating body surface the kinematic boundary condition being satisfied at the mean (rest) position of the cylindrical surface.

5. The radiation condition that the disturbed surface of the fluid takes the form of regular progressive outgoing gravity waves at large distances from the cylinder.

According to Wehausen and Laitone, the complex potential at z of a pulsating point source of unit strength at the point ζ in the lower half plane is:

$$\begin{aligned} G^*(z,\zeta;t) = \frac{1}{2\pi}\left\{\ln(z-\zeta) - \ln(z-\widehat{\zeta}) + 2PV\int_0^\infty \frac{e^{-ik(z-\widehat{\zeta})}}{\nu-k}dk\right\} & \qquad \cos\omega t \\ -\left\{e^{-i\nu(z-\widehat{\zeta})}\right\} & \qquad \sin\omega t \end{aligned} \tag{Frank-6}$$

so that the real point-source potential is:

$$H(x,y,\xi,\eta;t) = \mathrm{Re}\left\{G^*(z,\zeta;t)\right\} \tag{Frank-7}$$

where:

$$z = x+iy \qquad \zeta = \xi+i\eta \qquad \widehat{\zeta} = \xi - i\eta \qquad \nu = \frac{\omega^2}{g}$$

Letting:

$$\begin{aligned} G(z,\zeta) &= \frac{1}{2\pi}\,\mathrm{Re}\left\{\ln(z-\zeta) - \ln(z-\widehat{\zeta}) + 2PV\int_0^\infty \frac{e^{-ik(z-\widehat{\zeta})}}{\nu-k}dk\right\} \\ &\quad -i\,\mathrm{Re}\left\{e^{-i\nu(z-\widehat{\zeta})}\right\} \end{aligned} \tag{Frank-8}$$

then:

$$H(x,y,\xi,\eta;t) = \mathrm{Re}\left\{G(z,\zeta)\cdot e^{-i\omega t}\right\} \tag{Frank-9}$$

Equation (Frank-9) satisfies the radiation condition and also the equations (Frank-1) through (Frank-4).
Another expression satisfying all these conditions is:

$$H(x, y, \xi, \eta; t - \frac{\pi}{2\omega}) = \text{Re}\left\{iG(z, \zeta) \cdot e^{-i\omega t}\right\} \tag{Frank-10}$$

Since the problem is linear, a superposition of equations (Frank-9) and (Frank-10) results in the velocity potential:

$$\Phi^{(m)}(x, y; t) = \text{Re}\left\{\int\limits_{C_0} Q(s) \cdot G(z, \zeta) \cdot e^{-i\omega t} \cdot ds\right\} \tag{Frank-11}$$

where C_0 is the submerged contour of the cylindrical cross section at its mean (rest) position and $Q(s)$ represents the complex source density as a function of the position along C_0.
Application of the kinematic boundary condition on the oscillating cylinder at z yields:

$$\begin{array}{rcl} \text{Re}\left\{(\vec{n} \cdot \vec{\nabla}) \int\limits_{C_0} Q(s) \cdot G(z, \zeta) \cdot ds\right\} & = & 0 \\ \text{Im}\left\{(\vec{n} \cdot \vec{\nabla}) \int\limits_{C_0} Q(s) \cdot G(z, \zeta) \cdot ds\right\} & = & A^{(m)} \cdot \omega \cdot n^{(m)} \end{array} \tag{Frank-12}$$

where $A^{(m)}$ denotes the amplitude of oscillation and $n^{(m)}$ the direction cosine of the normal velocity at z on the cylinder. Both $A^{(m)}$ and $n^{(m)}$ depend on the mode of motion of the cylinder, as will be shown in the following section.
The fact that $Q(s)$ is complex implies that equations (Frank-12) represent a set of coupled integral equations for the real functions $\text{Re}\{Q(s)\}$ and $\text{Im}\{Q(s)\}$. The solution of these integral equations and the evaluation of the kernel and potential integrals are described in the following section and in Appendices B and C, respectively.

4.3.3 Solution of the Problem

Since ship sections are symmetrical, this investigation is confined to bodies with right and left symmetry.
Take the x-axis to be coincident with the undisturbed free surface of a conventional two-dimensional Cartesian coordinate system. Let the cross sectional contour C_0 of the submerged portion of the cylinder be in the lower half plane, the y-axis being the axis of symmetry of C_0; see figure 4.9.
Select $N + 1$ points (ξ_i, η_i) of C_0 to lie in the fourth quadrant so that (ξ_1, η_1) is located on the negative y-axis. For partially immersed cylinders, (ξ_{N+1}, η_{N+1}) is on the positive x-axis. For fully submerged bodies, $\xi_{N+1} = \xi_1$ and $\eta_{N+1} < 0$.
Connecting these $N + 1$ points by successive straight lines, N straight line segments are obtained which, together with their reflected images in the third quadrant, yield an approximation to the given contour as shown in figure 4.9.

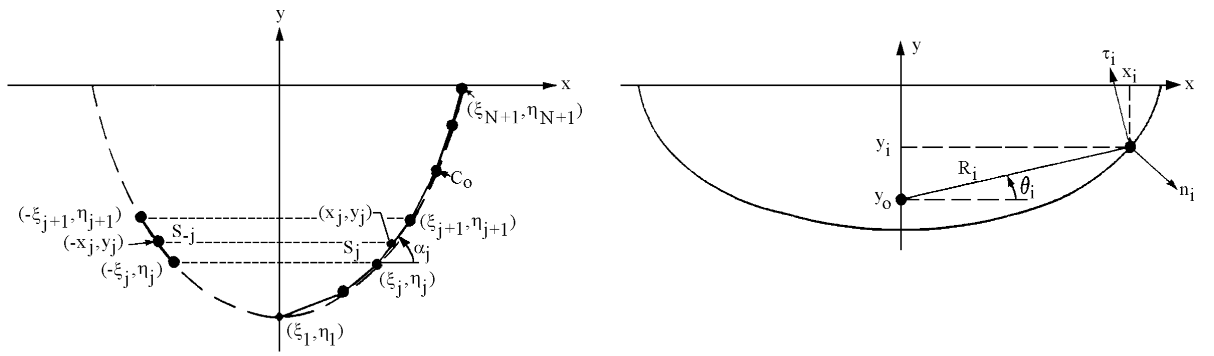

Figure 4.9: Axes System and Notation, as Used by Frank

The coordinates, length and angle associated with the j-th segment are identified by the subscript j, whereas the corresponding quantities for the reflected image in the third quadrant are denoted by the subscript $-j$, so that by symmetry $\xi_{-j} = -\xi_j$ and $\eta_{-j} = -\eta_j$ for $1 \leqslant j \leqslant N+1$.
Potentials and pressures are to be evaluated at the midpoint of each segment. The coordinates of the midpoint of the i-th segment are:

$$x_i = \frac{\xi_i + \xi_{i+1}}{2} \qquad \text{and} \qquad y_i = \frac{\eta_i + \eta_{i+1}}{2} \tag{Frank-13}$$

for $1 \leqslant i \leqslant N$.
The length of the i-th segment is:

$$|s_i| = \sqrt{\left(\xi_{i+1} - \xi_i\right)^2 + \left(\eta_{i+1} - \eta_i\right)^2} \tag{Frank-14}$$

while the angle made by the i-th segment with the positive x-axis is given by:

$$\alpha_i = \arctan\left\{\frac{\eta_{i+1} - \eta_i}{\xi_{i+1} - \xi_i}\right\} \tag{Frank-15}$$

The outgoing unit vector normal to the cross section at the i-th midpoint (x_i, y_i) is:

$$\vec{n}_i = \vec{i}\sin\alpha_i - \vec{j}\cos\alpha_i \tag{Frank-16}$$

where $\vec{i}$ and $\vec{j}$ are unit vectors in the directions of increasing x and y, respectively.
The cylinder is forced into simple harmonic motion with radian frequency ω, according to the displacement equation:

$$S^{(m)} = A^{(m)} \cdot \cos\omega t \tag{Frank-17}$$

for $m = 2$, 3 or 4, corresponding to sway, heave or roll, respectively. The rolling motions are about an axis through a point $(0, y_0)$ in the symmetry plane of the cylinder.
In the translational modes, any point on the cylinder moves with the velocity:

$$\text{sway:} \qquad \vec{v}^{(2)} = -\vec{i}\,A^{(2)}\omega\sin\omega t \tag{Frank-18}$$
$$\text{heave:} \qquad \vec{v}^{(3)} = -\vec{j}\,A^{(3)}\omega\sin\omega t \tag{Frank-19}$$

The rolling motion is illustrated in figure 4.9. Considering a point (x_i, y_i) on C_0, an inspection of this figure yields:

$$R_i = \sqrt{{x_i}^2 + (y_i - y_0)^2} \qquad \text{and} \qquad \begin{aligned} \theta_i &= \arctan\left\{\frac{y_i - y_0}{x_i}\right\} \\ &= \arcsin\left\{\frac{y_i - y_0}{R_i}\right\} \\ &= \arccos\left\{\frac{x_i}{R_i}\right\} \end{aligned}$$

Therefore, by elementary two-dimensional kinematics, the unit vector in the direction of increasing θ is:

$$\begin{aligned} \vec{\tau}_i &= -\vec{i}\sin\theta_i + \vec{j}\cos\theta_i \\ &= -\frac{y_i - y_0}{R_i}\vec{i} + \frac{x_i}{R_i}\vec{j} \end{aligned}$$

so that:

$$\begin{aligned} \text{roll:} \qquad \vec{v}^{(4)} &= R_i S^{(4)} \vec{\tau}_i \\ &= \omega A^{(4)} \left\{ (y_i - y_0)\,\vec{i} - x_i \vec{j} \right\} \sin\omega t \end{aligned} \qquad \text{(Frank-20)}$$

The normal components of the velocity ${v_i}^{(m)} = \vec{n}_i \cdot \vec{v}^{(m)}$ at the midpoint of the i-th segment (x_i, y_i) are:

$$\begin{aligned} \text{sway:} \quad {v_i}^{(2)} &= -\omega A^{(2)} \sin\alpha_i \sin\omega t \\ \text{heave:} \quad {v_i}^{(3)} &= +\omega A^{(3)} \cos\alpha_i \sin\omega t \\ \text{roll:} \quad {v_i}^{(4)} &= +\omega A^{(4)} \left\{ (y_i - y_0)\sin\alpha_i + x_i \cos\alpha_i \right\} \sin\omega t \end{aligned} \qquad \text{(Frank-21)}$$

Defining:

$${n_i}^{(m)} = \frac{{v_i}^{(m)}}{A^{(m)}\omega\sin\omega t}$$

then, consistent with the previously mentioned notation, the direction cosines for the three modes of motion are:

$$\begin{aligned} \text{sway:} \quad {n_i}^{(2)} &= -\sin\alpha_i \\ \text{heave:} \quad {n_i}^{(3)} &= +\cos\alpha_i \\ \text{roll:} \quad {n_i}^{(4)} &= +(y_i - y_0)\sin\alpha_i + x_i\cos\alpha_i \end{aligned} \qquad \text{(Frank-22)}$$

Equations (Frank-22) illustrate that heaving is symmetrical, i.e., ${n_{-i}}^{(3)} = {n_i}^{(3)}$. Swaying and rolling, however, are anti-symmetrical modes, i.e., ${n_{-i}}^{(2)} = -{n_i}^{(2)}$ and ${n_{-i}}^{(4)} = -{n_i}^{(4)}$. Equations (Frank-12) are applied at the midpoints of each of the N segments and it is assumed that over an individual segment the complex source strength $Q(s)$ remains constant, although it varies from segment to segment. With these stipulations, the set of coupled integral equations (Frank-12) becomes a set of $2N$ linear algebraic equations in the unknowns:

$$\begin{aligned} \text{Re}\left\{Q^{(m)} \cdot (s_j)\right\} &= {Q_j}^{(m)} \\ \text{Im}\left\{Q^{(m)} \cdot (s_j)\right\} &= {Q_{N+j}}^{(m)} \end{aligned}$$

Thus, for i = 1, 2, ..., N:

$$+\sum_{j=1}^{N}\left\{Q_j{}^{(m)}\cdot I_{ij}{}^{(m)}\right\}+\sum_{j=1}^{N}\left\{Q_{N+j}{}^{(m)}\cdot J_{ij}{}^{(m)}\right\} = 0$$

$$-\sum_{j=1}^{N}\left\{Q_j{}^{(m)}\cdot J_{ij}{}^{(m)}\right\}+\sum_{j=1}^{N}\left\{Q_{N+j}{}^{(m)}\cdot I_{ij}{}^{(m)}\right\} = \omega\cdot A^{(m)}\cdot n_i{}^{(m)} \qquad \text{(Frank-23)}$$

where the superscript $^{(m)}$ denotes the mode of motion.
The "influence coefficients" $I_{ij}{}^{(m)}$ and $J_{ij}{}^{(m)}$ and the potential $\Phi^{(m)}(x_i, y_i; t)$ are evaluated in the appendix. The resulting velocity potential consists of a term in-phase with the displacement and a term in-phase with the velocity.
The hydrodynamic pressure at (x_i, y_i) along the cylinder is obtained from the velocity potential by means of the linearized Bernoulli equation:

$$p^{(m)}(x_i, y_i, \omega; t) = -\rho\cdot\frac{\partial\Phi^{(m)}}{\partial t}(x_i, y_i, \omega; t) \qquad \text{(Frank-24)}$$

as:

$$p^{(m)}(x_i, y_i, \omega; t) = p_a{}^{(m)}(x_i, y_i; \omega)\cdot\cos\omega t + p_v{}^{(m)}(x_i, y_i; \omega)\cdot\sin\omega t \qquad \text{(Frank-25)}$$

where $p_a{}^{(m)}$ and $p_v{}^{(m)}$ are the hydrodynamic pressures in-phase with the displacement and in-phase with the velocity, respectively and ρ denotes the density of the fluid.
As indicated by the notation of equations (Frank-24) and (Frank-25), the pressure as well as the potential is a function of the oscillation frequency ω.
The hydrodynamic force or moment (when $m = 4$) per unit length along the cylinder, necessary to sustain the oscillations, is the integral of $p^{(m)}\cdot n^{(m)}$ over the submerged contour of the cross section C_0. It is assumed that the pressure at the i-th midpoint is the mean pressure for the i-th segment, so that the integration reduces to a summation, whence:

$$M^{(m)}(\omega) = 2\sum_{i=1}^{N} p_a{}^{(m)}(x_i, y_i; \omega)\cdot n_i{}^{(m)}\cdot|s_i| \qquad \text{(Frank-26)}$$

$$N^{(m)}(\omega) = 2\sum_{i=1}^{N} p_v{}^{(m)}(x_i, y_i; \omega)\cdot n_i{}^{(m)}\cdot|s_i| \qquad \text{(Frank-27)}$$

for the added mass and damping forces or moments, respectively.
The velocity potentials for very small and very large frequencies are derived and discussed in the next section.

4.3.4 Low and High Frequencies

For very small frequencies, i.e., as $\omega \to 0$, the free-surface condition equation (Frank-3) of the section formulating the problem degenerates into the wall-boundary condition:

$$\frac{\partial\Phi}{\partial y} = 0 \qquad \text{(Frank-49)}$$

on the surface of the fluid outside the cylinder, whereas for extremely large frequencies, i.e., when $\omega \rightarrow \infty$, the free-surface condition becomes the "impulsive" surface condition:

$$\Phi = 0 \qquad \text{(Frank-50)}$$

on $y = 0$ outside the cylinder.
Equations (Frank-2), (Frank-4) and (Frank-5) remain valid for both asymptotic cases. The radiation condition is replaced by a condition of boundedness at infinity.
Therefore, there is a Neumann problem for the case $\omega \rightarrow 0$ and a mixed problem when $\omega \rightarrow \infty$. The appropriate complex potentials for a source of unit strength at a point ζ in the lower half plane are:

$$G_0\left(z,\zeta\right) = \frac{1}{2\pi}\left\{\ln(z-\zeta) + \ln(z-\widehat{\zeta})\right\} + K_0 \qquad \text{(Frank-51)}$$

and:

$$G_\infty\left(z,\zeta\right) = \frac{1}{2\pi}\left\{\ln(z-\zeta) - \ln(z-\widehat{\zeta})\right\} + K_\infty \qquad \text{(Frank-52)}$$

for the Neuman and mixed problems, respectively, where K_0 and K_∞ are constants not yet specified.
Let:

$$\phi_a\left(x,y;\xi,\eta\right) = \mathrm{Re}\left\{G_a\left(z,\zeta\right)\right\}$$

so that the velocity potentials for the m-th mode of motion are:

$$\Phi_a^{(m)}\left(x,y\right) = \int\limits_{C_0} Q_a^{(m)}\left(s\right)\cdot\phi_a\left(x,y;\xi,\eta\right)\,ds \qquad \text{(Frank-53)}$$

for $a = 0$, and $a = \infty$, where $Q_a^{(m)}$ is the expression for the source strength as a function of position along the submerged contour of the cross section C_0.
An analysis similar to the one in the section on formulating the problem leads to the integral equation:

$$(\vec{n}\cdot\vec{\nabla})\int\limits_{C_0} Q_a^{(m)}(s)\cdot\phi_a\left(x,y;\xi,\eta\right)\,ds = A^{(m)}n^{(m)} \qquad \text{(Frank-54)}$$

which - after application at the N segmental midpoint - yields a set of N linear algebraic equations in the N unknown source strengths Q_j.
It remains to be shown whether these two problems are, in the language of potential theory, well posed, i.e., whether the solutions to these problems lead to unique forces or moments. The mixed problem raises no difficulty, since as $z \rightarrow \infty$, $G_\infty\left(z,\zeta\right) \rightarrow 0$. In fact $K_\infty = 0$, which can be inferred from the pulsating source-potential equation (Frank-8) by letting $\nu \rightarrow \infty$.
Considering the Neumann problem, note that the constant K_0 in the Green's function equation (Frank-51) yields by integration an additive constant K to the potential. However, for a completely submerged cylinder the cross sectional contour C_0 is a simply closed curve, so that the contribution of K in integrating the product of the pressure with the direction cosine of the body-surface velocity vanishes. For partially submerged bodies C_0 is no longer closed. But since $n_{-i}^{(m)} = -n_i^{(m)}$ for m being even,

$$\int_{C_0} K\, n^{(m)}\, ds = 0$$

so that the swaying force and rolling moment are unique.
The heaving force on a partially submerged cylinder is not unique for, in this case, $n_{-i}^{(3)} = n_i^{(3)}$, so that:

$$\int_{C_0} K\, n^{(3)}\, ds \neq 0$$

The constant K_0 may be obtained by letting $\nu \to 0$ in the pulsating source-potential equation (Frank-8).

4.3.5 Irregular Frequencies

[John, 1950] proved the existence and uniqueness of the solutions to the three- and two-dimensional potential problems pertaining to oscillations of rigid bodies in a free surface. The solutions were subject to the provisions that no point of the immersed surface of the body would be outside a cylinder drawn vertically downward from the intersection of the body with the free surface and that the free surface would be intersected orthogonally by the body in its mean or rest position.
[John, 1950] also showed that for a set of discrete "irregular" frequencies the Green's function-integral equation method failed to give a solution. He demonstrated that the irregular frequencies occurred when the following adjoint interior-potential problem had eigen-frequencies..
Let $\psi(x, y)$ be such that:

1. $\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = 0$ inside the cylinder in the region bounded by the immersed surface of the body and the extension of the free surface inside the cylinder;

2. $\frac{\partial\psi}{\partial y} - \nu_k\psi = 0$ on the extension of the free surface inside the cylinder, ν_k being the wave number corresponding to the irregular frequency ω_k, $k = 1, 2, 3, ...$;

3. $\psi = 0$ on the surface of the cylinder below the free surface.

For a rectangular cylinder with beam B and draft T, the irregular wave numbers may be easily obtained by separation of variables in the Laplace equation. Separating variables gives the eigen-frequencies:

$$\psi_k = B_k \sin\left(\frac{k\pi x}{B}\right) \sinh\left[\frac{k\pi y}{B}\right] \qquad \text{for: } k = 1, 2, 3, \text{ ...... etc.}$$

where B_k are Fourier coefficients to be determined from an appropriate boundary condition. Applying the free surface condition (Frank-2) on $y = T$ for $0 < x < B$, the eigen-wave numbers (or irregular wave numbers):

$$\nu_k = \frac{k\pi}{B} \coth\left[\frac{k\pi T}{B}\right] \qquad \text{(Frank-28)}$$

are obtained for k = 1, 2, 3, ..., etc. In particular, the lowest such irregular wave number is given by:

$$k_1 = \frac{\pi}{B} \coth\left[\frac{\pi T}{B}\right] \tag{Frank-29}$$

Keeping T fixed in equation (Frank-29) but letting B vary and setting $b = \pi/B$, then from the Taylor expansion:

$$b \coth[bT] = b\left[\frac{1}{bT} + \frac{bT}{3} - \frac{(bT)^3}{45} + \;......\right]$$

it is seen that as $b \to 0$, which is equivalent to $B \to \infty$, $\nu_1 \to 1/T$.
Therefore, for rectangular cylinders of draft T:

$$\nu_1 = \frac{1}{T} \tag{Frank-30}$$

a relation that John proved for general shapes complying with the restrictions previously outlined. For a beam-to-draft ratio of $B/T = 2.5$: $\nu_1 = 1.48$, while for $B/T = 2$: $\nu_1 = 1.71$.
At an irregular frequency the matrix of influence coefficients of equations (Frank-23) becomes singular as the number of defining points per cross section increases without limit, i,e., as $N \to \infty$. In practice, with finite N, the determinant of this matrix becomes very small, not only at the irregular frequency but also at an interval about this frequency. This interval can be reduced by increasing the number of defining points N for the cross section.
Most surface vessels have nearly constant draft over the length of the ship and the maximum beam occurs at or near amidships, where the cross section is usually almost rectangular, so that for most surface ships the first irregular frequency ω_1 is less for the midsection than for any other cross section.
For a ship with a 7:1 length-to-beam and a 5:2 beam-to-draft, the first irregular wave encounter frequency - in non-dimensional form with L denoting the ship length - occurs at:

$$\omega_1\sqrt{\frac{L}{g}} \approx 5.09$$

which is beyond the range of practical interest for ship-motion analysis.
Therefore, for slender surface vessels, the phenomenon of the first irregular frequency of wave encounter is not too important.

An effective method to reduce the effects of irregular frequencies is, among others, to "close" the body by means of discretization of the free surface inside the body (putting a "lid" on the free surface inside the body); see the added mass and damping of a hemisphere in figure 4.10. The solid line in this figure results from including the "lid".
Increasing the number of panels does not remove the irregular frequency but tends to restrict the effects to a narrower band around it; see for instance [Huijsmans, 1996]. It should be mentioned that irregular frequencies only occur for free surface piercing bodies; fully submerged bodies do not display these characteristics.

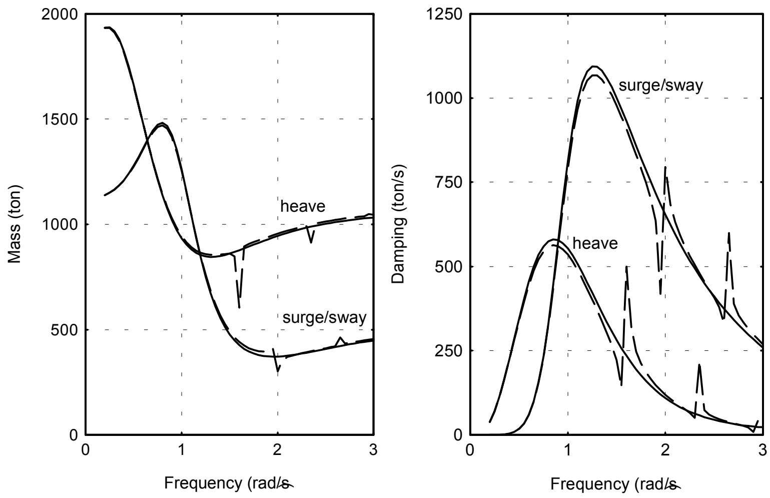

Figure 4.10: Effect of "Lid-Method" on Irregular Frequencies

4.3.6 Appendices

Appendix A: Principle Value Integrals

The real and imaginary parts of the principle value integral:

$$PV \int_0^{\infty} \frac{e^{-ik(z-\widehat{\zeta})}}{\nu - k} \, dk$$

are used in evaluating some of the kernel and potential integrals.
The residue of the integrand at $k = \nu$ is $e^{-i\nu(z-\zeta)}$, so that:

$$PV \int_0^{\infty} \frac{e^{-ik(z-\widehat{\zeta})}}{\nu - k} \, dk = \int_{\frown 0}^{\infty} \frac{e^{-ik(z-\widehat{\zeta})}}{\nu - k} \, dk - i\pi e^{-i\nu(z-\widehat{\zeta})} \qquad \text{(Frank-31)}$$

where the path of integration is the positive real axis indented into the upper half plane about $k = \nu$.
Note that $\nu = \omega^2/g > 0$, $\mathrm{Im}\{z\} < 0$ and $\mathrm{Im}\{\zeta\} \leqslant 0$.
The transformation $\omega = i(k-\nu)\left(z - \widehat{\zeta}\right)$ converts the contour integral on the right hand side of equation (Frank-31) to:

$$\begin{aligned}
\int_{\frown 0}^{\infty} \frac{e^{-ik(z-\widehat{\zeta})}}{\nu - k} \, dk &= -e^{-i\nu(z-\widehat{\zeta})} \int_{-i\nu(z-\widehat{\zeta})}^{\infty} \frac{e^{-w}}{w} \, dw \\
&= -e^{-i\nu(z-\widehat{\zeta})} E_1\left[-i\nu(z-\widehat{\zeta})\right] + \begin{pmatrix} 2\pi i \\ 0 \end{pmatrix} \text{for } \begin{matrix} x - \xi > 0 \\ x - \xi < 0 \end{matrix} \\
&= -e^{-i\nu(z-\zeta)} \left\{\gamma + \ln\left[-i\nu(z-\widehat{\zeta})\right]\right. \qquad \text{(Frank-32)}
\end{aligned}$$

$$+\sum_{n=1}^{\infty}\frac{(-1)^n\left[-i\nu(z-\widehat{\zeta})\right]^n}{n\cdot n!}\Bigg\}+\begin{pmatrix}2\pi i & \text{for} & x-\xi>0\\ 0 & & x-\xi<0\end{pmatrix}$$

where $\gamma = 0.5772...$ is the well known Euler-Mascheroni constant and the value of E_1 is defined by Abramowitz and Stegun.
Setting:

$$r=\left|-i\nu(z-\widehat{\zeta})\right| \qquad \text{and} \qquad \theta=\arctan\left[\frac{\text{Im}\left\{-i\nu(z-\widehat{\zeta})\right\}}{\text{Re}\left\{-i\nu(z-\widehat{\zeta})\right\}}\right]+\pi$$

the following expression is obtained for equation (Frank-31):

$$\begin{aligned}PV\int_0^{\infty}\frac{e^{-ik(z-\widehat{\zeta})}}{\nu-k}\,dk &= e^{\nu(y+\eta)}\left[\cos\nu\,(x-\xi)-i\sin\nu\,(x-\xi)\right]\cdot\\ &\cdot\left\{\left[\gamma+\ln r+\sum_{n=1}^{\infty}\frac{r^n\cos{(n\theta)}}{n\cdot n!}\right]\right.\\ &\left.+i\left[\begin{pmatrix}\theta & \text{for} & x-\xi>0\\ \theta-2\pi & & x-\xi<0\end{pmatrix}+\sum_{n=1}^{\infty}\frac{r^n\sin{(n\theta)}}{n\cdot n!}\right]\right\}\end{aligned} \quad \text{(Frank-33)}$$

Separating equation (Frank-33) into its real and imaginary parts yields:

$$\begin{aligned}PV\int_0^{\infty}\frac{e^{k(y+\eta)}\cos k\,(x-\xi)}{\nu-k}\,dk &= e^{\nu(y+\eta)}\left[C\,(r,\theta)\cos\nu\,(x-\xi)\right.\\ &\left.+S\,(r,\theta)\sin\nu\,(x-\xi)\right]\\ PV\int_0^{\infty}\frac{e^{k(y+\eta)}\sin k\,(x-\xi)}{\nu-k}\,dk &= e^{\nu(y+\eta)}\left[C\,(r,\theta)\sin\nu\,(x-\xi)\right.\\ &\left.-S\,(r,\theta)\cos\nu\,(x-\xi)\right]\end{aligned} \quad \text{(Frank-34)}$$

provided that:

$$\begin{aligned}C\,(r,\theta) &= \gamma+\ln r+\sum_{n=1}^{\infty}\frac{r^n\cos{(n\theta)}}{n\cdot n!}\\ S\,(r,\theta) &= \theta+\sum_{n=1}^{\infty}\frac{r^n\sin{(n\theta)}}{n\cdot n!}+\begin{pmatrix}\theta & \text{for} & x-\xi>0\\ \theta-2\pi & & x-\xi<0\end{pmatrix}\end{aligned}$$

Appendix B: Kernel Integrals

The influence coefficients of equations (Frank-23) are:

$$\begin{aligned} I_{ij}^{(m)} &= \mathrm{Re}\left\{ (\vec{n}_i \cdot \vec{\nabla}) \left[\int\limits_{s_j} \left(\frac{1}{2\pi} \left(\ln(z-\zeta) - \ln(z-\widehat{\zeta}) \right) + \frac{1}{\pi} PV \int\limits_0^\infty \frac{e^{-ik(z-\widehat{\zeta})}}{\nu - k} dk \right) ds \right.\right. \\ & \left.\left. - (-1)^m \int\limits_{s_{-j}} \left(\frac{1}{2\pi} \left(\ln(z+\widehat{\zeta}) - \ln(z+\zeta) \right) + \frac{1}{\pi} PV \int\limits_0^\infty \frac{e^{-ik(z+\zeta)}}{\nu - k} dk \right) ds \right]_{z=z_i} \right\} \end{aligned} \qquad \text{(Frank-35)}$$

and:

$$J_{ij}^{(m)} = \mathrm{Re}\left\{ (\vec{n}_i \cdot \vec{\nabla}) \left[\int\limits_{s_j} e^{-i\nu(z-\widehat{\zeta})} ds - (-1)^m \int\limits_{s_{-j}} e^{-i\nu(z+\zeta)} ds \right]_{z=z_i} \right\} \qquad \text{(Frank-36)}$$

Note that in the complex plane with z_i on s_i:

$$\mathrm{Re}\left\{ \left[(\vec{n}_i \cdot \vec{\nabla}) F(z) \right]_{z=z_1} \right\} = \mathrm{Re}\left\{ \left[-ie^{i\alpha_i} \frac{dF(z)}{dz} \right]_{z=z_1} \right\}$$

Considering the term containing $\ln(z-\zeta)$, it is evident that the kernel integral is singular when $i=j$, so that the indicated differentiation cannot be performed under the integral sign. However, in that case one may proceed as follows:
since:

$$\begin{aligned} \zeta &= \xi + i\eta \\ d\zeta &= d\xi + i d\eta \\ &= ds\cos\alpha_j + i ds \sin\alpha_j \\ &= e^{i\alpha_j} ds \end{aligned}$$

for ζ along the j-th segment.
Therefore, $ds = e^{-i\alpha_j} d\zeta$ and:

$$\begin{aligned} \mathrm{Re}\left\{ (\vec{n}_j \cdot \vec{\nabla}) \left[\int\limits_{s_j} \ln(z-\zeta)\, ds \right]_{z=z_j} \right\} &= \mathrm{Re}\left\{ -ie^{i\alpha_j} \frac{d}{dz} \left[\int\limits_{\zeta_j}^{\zeta_{j+1}} e^{-i\alpha_j} d\zeta \ln(z-\zeta) \right]_{z=z_j} \right\} \\ &= \mathrm{Re}\left\{ -i \frac{d}{dz} \left[\int\limits_{\zeta_j}^{\zeta_{j+1}} d\zeta \ln(z-\zeta)\, d_\zeta \right]_{z=z_j} \right\} \end{aligned}$$

Setting $\zeta' = z - \zeta$, the last integral becomes:

$$\begin{aligned}
\mathrm{Re}\left\{-i\frac{d}{dz}\left[\int\limits_{z-\zeta_{j+1}}^{z-\zeta_j} \ln\zeta' d\zeta'\right]_{z=z_j}\right\} &= \arg\left(z_j-\zeta_j\right) - \arg\left(z_j - \zeta_{j+1}\right) \\
&= \pi \qquad \text{(Frank-37)}
\end{aligned}$$

If $i \neq j$, differentiation under the integral sign may be performed, so that:

$$\begin{aligned}
(L_1) &= \mathrm{Re}\left\{(\vec{n}_i\cdot\vec{\nabla})\left[\int\limits_{s_j}\ln(z-\zeta)\,ds\right]_{z=z_i}\right\} \\
&= \sin(\alpha_i-\alpha_j)\ln\sqrt{\frac{\left(x_i-\xi_j\right)^2+\left(y_i-\eta_j\right)^2}{\left(x_i-\xi_{j+1}\right)^2+\left(y_i-\eta_{j+1}\right)^2}} \qquad \text{(Frank-38)} \\
&\quad +\cos(\alpha_i-\alpha_j)\left[\arctan\frac{y_i-\eta_j}{x_i-\xi_j}-\arctan\frac{y_i-\eta_{j+1}}{x_i-\xi_{j+1}}\right]
\end{aligned}$$

For the integral containing the $\ln\left(z-\widehat{\zeta}\right)$ term, $ds = e^{i\alpha_j}d\widehat{\zeta}$, so that:

$$\begin{aligned}
(L_2) &= \mathrm{Re}\left\{(\vec{n}_i\cdot\vec{\nabla})\left[\int\limits_{s_j}\ln\left(z-\widehat{\zeta}\right)ds\right]_{z=z_i}\right\} \\
&= \sin(\alpha_i+\alpha_j)\ln\sqrt{\frac{\left(x_i-\xi_j\right)^2+\left(y_i+\eta_j\right)^2}{\left(x_i-\xi_{j+1}\right)^2+\left(y_i+\eta_{j+1}\right)^2}} \qquad \text{(Frank-39)} \\
&\quad +\cos(\alpha_i+\alpha_j)\left[\arctan\frac{y_i+\eta_j}{x_i-\xi_j}-\arctan\frac{y_i+\eta_{j+1}}{x_i-\xi_{j+1}}\right]
\end{aligned}$$

The kernel integral containing the principal value integrals is:

$$\begin{aligned}
(L5) &= \mathrm{Re}\left\{(\vec{n}_i\cdot\vec{\nabla})\left[\int\limits_{s_j}ds\cdot PV\int\limits_0^\infty\frac{e^{-ik(z_i-\widehat{\zeta})}}{\nu-k}dk\right]_{z=z_i}\right\} \\
&= \mathrm{Re}\left\{-ie^{i(\alpha_i+\alpha_j)}\int\limits_{\widehat{\zeta}_{j+1}}^{\widehat{\zeta}_j}d\widehat{\zeta}\frac{d}{d\widehat{\zeta}}PV\int\limits_0^\infty\frac{e^{-ik(z_i-\widehat{\zeta})}}{\nu-k}dk\right\} \\
&= \quad +\sin(\alpha_i+\alpha_j)\left[PV\int\limits_0^\infty\frac{e^{k(y_i+\eta_j)}\cos k\left(x_i-\xi_j\right)}{\nu-k}dk\right.
\end{aligned}$$

$$-PV\int_0^\infty \frac{e^{k(y_i+\eta_{j+1})}\cos k\left(x_i-\xi_{j+1}\right)}{\nu-k}dk\Bigg]$$

$$-\cos\left(\alpha_i+\alpha_j\right)\Bigg[PV\int_0^\infty \frac{e^{k(y_i+\eta_j)}\sin k\left(x_i-\xi_j\right)}{\nu-k}dk \qquad \text{(Frank-40)}$$

$$-PV\int_0^\infty \frac{e^{k(y_i+\eta_{j+1})}\sin k\left(x_i-\xi_{j+1}\right)}{\nu-k}dk\Bigg]$$

The first integral on the right hand side of equation (Frank-36) becomes:

$$\begin{aligned}(L7) &= \mathrm{Re}\left\{(\vec{n}_i\cdot\vec{\nabla})\left[\int_{s_j} e^{-i\nu(z-\widehat{\zeta})}ds\right]_{z=z_i}\right\} \qquad \text{(Frank-41)}\\ &= -\sin\left(\alpha_i+\alpha_j\right)\left[e^{\nu(y_i+\eta_j)}\cos\nu\left(x_i-\xi_j\right)-e^{\nu(y_i+\eta_{j+1})}\cos\nu\left(x_i-\xi_{j+1}\right)\right]\\ &\quad +\cos\left(\alpha_i+\alpha_j\right)\left[e^{\nu(y_i+\eta_j)}\sin\nu\left(x_i-\xi_j\right)-e^{\nu(y_i+\eta_{j+1})}\sin\nu\left(x_i-\xi_{j+1}\right)\right]\end{aligned}$$

The kernel integrals over the image segments are obtained from equations (Frank-38) through (Frank-41) by replacing ξ_j, ξ_{j+1} and α_j with $\xi_{-j}=-\xi_j$, $\xi_{-(j+1)}=-\xi_{j+1}$ and $\alpha_{-j}=-\alpha_j$, respectively.

Appendix C: Potential Integrals

The velocity potential of the m-th mode of oscillation at the i-th midpoint (x_i,y_i) is:

$$\begin{aligned}\Phi^{(m)}(x_i,y_i;t) &= \frac{1}{2\pi}\sum_{j=1}^{N}Q_j\,\mathrm{Re}\left\{\int_{s_j}\left[\ln(z_i-\zeta)-\ln(z_i-\widehat{\zeta})+2PV\int_0^\infty\frac{e^{-ik(z_i-\widehat{\zeta})}}{\nu-k}dk\right]ds\right.\\ &\quad \left.-(-1)^m\int_{s_{-j}}\left[\ln(z_i+\widehat{\zeta})-\ln(z_i+\zeta)+2PV\int_0^\infty\frac{e^{-ik(z_i+\zeta)}}{\nu-k}dk\right]ds\right\}\\ &\quad \mp\sum_{j=1}^{N}Q_{N+j}\,\mathrm{Re}\left\{\int_{s_j}e^{-i\nu(z_i-\widehat{\zeta})}ds-(-1)^m\int_{s_{-j}}e^{-i\nu(z_i+\zeta)}ds\right\}\cdot\begin{matrix}\cos\omega t\\ \sin\omega t\end{matrix}\end{aligned}$$

(Frank-42)

The integration of the $\ln(z_i-\zeta)$ term is straight forward, yielding:

$$\mathrm{Re}\left\{\int_{s_j}\ln(z_i-\zeta)ds\right\} = +\cos\alpha_j\left[\left(x_i-\xi_j\right)\ln\sqrt{\left(x_i-\xi_j\right)^2+\left(y_i-\eta_j\right)^2}+\xi_j-\xi_{j-1}\right.$$

$$
\begin{aligned}
&- \left(x_i - \xi_{j+1}\right) \ln \sqrt{\left(x_i - \xi_{j+1}\right)^2 + \left(y_i - \eta_{j+1}\right)^2} \\
&- \left(y_i - \eta_j\right) \arctan \frac{y_i - \eta_j}{x_i - \xi_j} + \left(y_i - \eta_{j+1}\right) \arctan \frac{y_i - \eta_{j+1}}{x_i - \xi_{j+1}} \Bigg] \\
&+ \sin \alpha_j \Bigg[\left(y_i - \eta_j\right) \ln \sqrt{\left(x_i - \xi_j\right)^2 + \left(y_i - \eta_j\right)^2} + \eta_j - \eta_{j-1} \\
&- \left(y_i - \eta_{j+1}\right) \ln \sqrt{\left(x_i - \xi_{j+1}\right)^2 + \left(y_i - \eta_{j+1}\right)^2} \qquad \text{(Frank-43)} \\
&+ \left(x_i - \xi_j\right) \arctan \frac{y_i - \eta_j}{x_i - \xi_j} - \left(x_i - \xi_{j+1}\right) \arctan \frac{y_i - \eta_{j+1}}{x_i - \xi_{j+1}} \Bigg]
\end{aligned}
$$

In the integration of the $\ln(z - \widehat{\zeta})$ term, note that η_j and η_{j+1} are replaced by $-\eta_j$ and $-\eta_{j+1}$, respectively.
To evaluate the potential integral containing the principal value integral, proceed in the following manner. For an arbitrary z in the fluid domain:

$$
\begin{aligned}
\int_{\widehat{\zeta}_j}^{\widehat{\zeta}_{j+1}} ds \cdot PV \int_0^\infty \frac{e^{-ik(z-\widehat{\zeta})}}{\nu - k} dk &= e^{i\alpha_j} \cdot PV \int_0^\infty \frac{dk}{\nu - k} \int_{\widehat{\zeta}_j}^{\widehat{\zeta}_{j+1}} e^{-ik(z-\widehat{\zeta})} d\widehat{\zeta} \\
&= e^{i\alpha_j} \cdot PV \int_0^\infty \frac{e^{-ikz}}{\nu - k} dk \int_{\widehat{\zeta}_j}^{\widehat{\zeta}_{j+1}} e^{ik\widehat{\zeta}} d\widehat{\zeta} \\
&= -e^{i\alpha_j} \cdot PV \int_0^\infty \frac{e^{-ikz}}{\nu - k} \frac{e^{ik\widehat{\zeta}_{j+1}} - e^{ik\widehat{\zeta}_j}}{k} dk
\end{aligned}
$$

where the change of integration is permissible since only one integral requires a principle value interpretation.
After dividing by ν and multiplying by $\nu - k + k$ under the integral sign, the last expression becomes:

$$
-\frac{ie^{i\alpha_j}}{\nu} \left[\int_0^\infty e^{-ikz} \frac{e^{ik\widehat{\zeta}_{j+1}} - e^{ik\widehat{\zeta}_j}}{k} dk + PV \int_0^\infty \frac{e^{-ik(z-\widehat{\zeta}_{j+1})}}{\nu - k} dk - PV \int_0^\infty \frac{e^{-ik(z-\widehat{\zeta}_j)}}{\nu - k} dk \right] \qquad \text{(Frank-44)}
$$

Regarding the first integral in equation (Frank-44) as a function of z:

$$
F(z) = \int_0^\infty e^{-ikz} \frac{e^{ik\widehat{\zeta}_{j+1}} - e^{ik\widehat{\zeta}_j}}{k} dk \qquad \text{(Frank-45)}
$$

Differentiating equation (Frank-45) with respect to z gives:

$$
F'(z) \;=\; -i \left\{ \int_0^\infty e^{-ik(z-\widehat{\zeta}_{j+1})} dk - \int_0^\infty e^{-ik(z-\widehat{\zeta}_j)} dk \right\}
$$

$$= \frac{1}{z-\widehat{\zeta}_j} - \frac{1}{z-\widehat{\zeta}_{j+1}}$$

So:

$$F(z) = \ln\left(z-\widehat{\zeta}_j\right) - \ln\left(z-\widehat{\zeta}_{j+1}\right) + \nu \tag{Frank-46}$$

where ν is a constant of integration to be determined presently. Since $F(z)$ is defined and analytic for all z in the lower half plane and since by equation (Frank-45), $\lim_{z\to -i\infty} F(z) = 0$, it follows from equation (Frank-46) that $\nu = 0$.
Therefore:

$$\begin{aligned}
(K5) &= \mathrm{Re}\left\{\int\limits_{s_j} ds \cdot PV\int\limits_0^\infty \frac{e^{-ik(z_i-\widehat{\zeta})}}{\nu-k}dk\right\} \\
&= \mathrm{Re}\left\{-\frac{ie^{i\alpha_j}}{\nu}\left[\ln(z_i-\widehat{\zeta}_j) - \ln(z_i-\widehat{\zeta}_{j+1})\right.\right. \\
&\quad \left.\left. +PV\int\limits_0^\infty \frac{e^{-ik(z_i-\widehat{\zeta}_{j+1})}}{\nu-k}dk - PV\int\limits_0^\infty \frac{e^{-ik(z_i-\widehat{\zeta}_j)}}{\nu-k}dk\right]\right\} \\
&= \frac{1}{\nu}\left\{\sin\alpha_j\left[\ln\sqrt{\frac{\left(x_i-\xi_j\right)^2+\left(y_i+\eta_j\right)^2}{\left(x_i-\xi_{j+1}\right)^2+\left(y_i+\eta_{j+1}\right)^2}}\right.\right. \qquad \text{(Frank-47)} \\
&\quad \left. +PV\int\limits_0^\infty \frac{e^{k(y_i+\eta_{j+1})}\cos k\left(x_i-\xi_{j+1}\right)}{\nu-k}dk - PV\int\limits_0^\infty \frac{e^{k(y_i+\eta_j)}\cos k\left(x_i-\xi_j\right)}{\nu-k}dk\right] \\
&\quad +\cos\alpha_j\left[\arctan\frac{y_i+\eta_j}{x_i-\xi_j} - \arctan\frac{y_i+\eta_{j+1}}{x_i-\xi_{j+1}}\right. \\
&\quad \left.\left. +PV\int\limits_0^\infty \frac{e^{k(y_i+\eta_j)}\sin k\left(x_i-\xi_j\right)}{\nu-k}dk - PV\int\limits_0^\infty \frac{e^{k(y_i+\eta_{j+1})}\sin k\left(x_i-\xi_{j+1}\right)}{\nu-k}dk\right]\right\}
\end{aligned}$$

The integration of the potential component in-phase with the velocity over s_j gives:

$$\begin{aligned}
(K7) &= \mathrm{Re}\left\{\int\limits_{s_j} e^{-i\nu(z_i-\widehat{\zeta})}ds\right\} \qquad \text{(Frank-48)} \\
&= \frac{1}{\nu}\left\{e^{\nu(y_i+\eta_j)}\sin\left[\nu\left(x_i-\xi_j\right)-\alpha_j\right] - e^{\nu(y_i+\eta_{j+1})}\sin\left[\nu\left(x_i-\xi_{j+1}\right)-\alpha_j\right]\right\}
\end{aligned}$$

4.4 Surge Coefficients

An equivalent longitudinal section, being constant over the ship's breadth B, is defined by:

sectional breadth B_x	= ship length L
sectional draught d_x	= midship draught d
sectional area coefficient C_{M_x}	= block coefficient C_B

By using a Lewis transformation of this equivalent longitudinal section to the unit circle, the two-dimensional potential mass M_{11}^* and damping N_{11}^* can be calculated in an analog manner as has been described for the two-dimensional potential mass and damping of sway, M_{22}' and N_{22}'.
With these two-dimensional values, the total potential mass and damping of surge are defined by:

$$\begin{aligned} M_{11} &= B \cdot M_{11}^* \\ N_{11} &= B \cdot N_{11}^* \end{aligned}$$

in which B is the breadth of the ship.
These frequency-dependent hydrodynamic coefficients do not include three-dimensional effects. Only the hydrodynamic mass coefficient, of which a large three-dimensional effect is expected, will be adapted here empirically. According to [Tasai, 1961] the zero-frequency potential mass for sway can be expressed in Lewis-coefficients:

$$M_{22}'(\omega = 0) = \rho \frac{\pi}{2} \left(\frac{d_x}{1 - a_1 + a_3} \right)^2 \left((1 - a_1)^2 + 3a_3^2 \right)$$

When using this formula for surge, the total potential mass of surge is defined by:

$$M_{11}(\omega = 0) = B \cdot M_{11}^*(\omega = 0)$$

A frequency-independent total hydrodynamic mass coefficient is estimated empirically by [Sargent and Kaplan, 1974] as a proportion of the total mass of the ship $\rho\nabla$:

$$M_{11}(S\&K) = \alpha \cdot \rho\nabla$$

The factor α is depending on the breadth-length ratio B/L of the ship:

$$\alpha = \frac{a}{2 - a}$$

in which:

$$a = \frac{1 - b^2}{b^3} \left(\ln \left\{ \frac{1 + b}{1 - b} \right\} - 2b \right) \qquad \text{with:} \qquad b = \sqrt{1 - \left(\frac{B}{L} \right)^2}$$

With this hydrodynamic mass value, a correction factor β for three-dimensional effects has been determined:

$$\beta = \frac{M_{11}(S\&K)}{M_{11}(\omega = 0)}$$

The three-dimensional effects for the potential damping of surge are ignored.

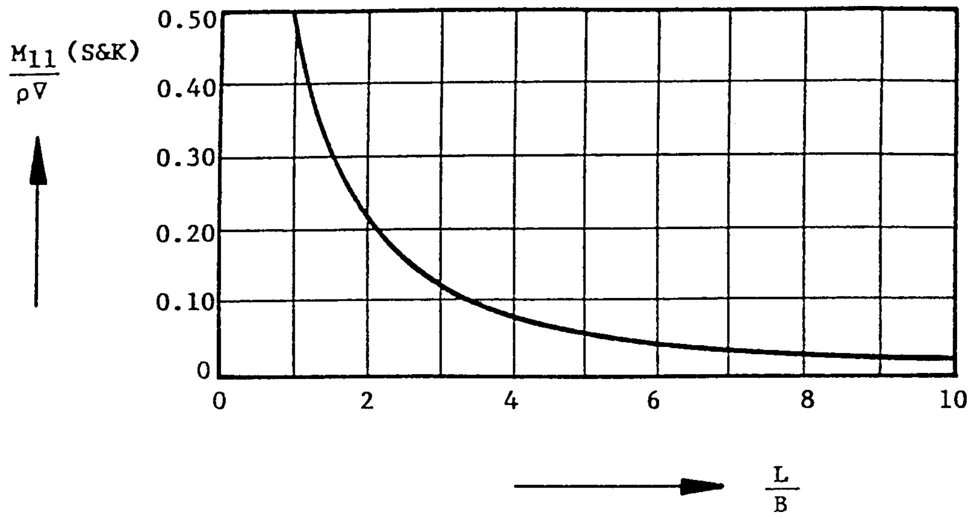

Figure 4.11: Surge Hydrodynamic Mass

So, the potential mass and damping of surge are defined by:

$$\begin{aligned} M_{11} &= B \cdot M_{11}^{*} \cdot \beta \\ N_{11} &= B \cdot N_{11}^{*} \end{aligned}$$

To obtain a uniform approach during all ship motions calculations, the cross sectional two-dimensional values of the hydrodynamic mass and damping have to be obtained. Based on the results of numerical 3-D studies with a Wigley hull form, a proportionality of both the two-dimensional hydrodynamic mass and damping with the absolute values of the derivatives of the cross sectional areas A_x in the x_b-direction is assumed:

$$M_{11}' = \frac{\left|\frac{dA_x}{dx_b}\right|}{\int\limits_L \left|\frac{dA_x}{dx_b}\right| dx_b} \cdot M_{11} \qquad\qquad N_{11}' = \frac{\left|\frac{dA_x}{dx_b}\right|}{\int\limits_L \left|\frac{dA_x}{dx_b}\right| dx_b} \cdot N_{11}$$

4.5 Comparative Results

Figure 4.12 compares the calculated coefficients for an amidships cross section of a container vessel with the three previous methods:

- Ursell-Tasai's method with 2-parameter Lewis conformal mapping.
- Ursell-Tasai's method with 10-parameter close-fit conformal mapping.
- Frank's pulsating source method.

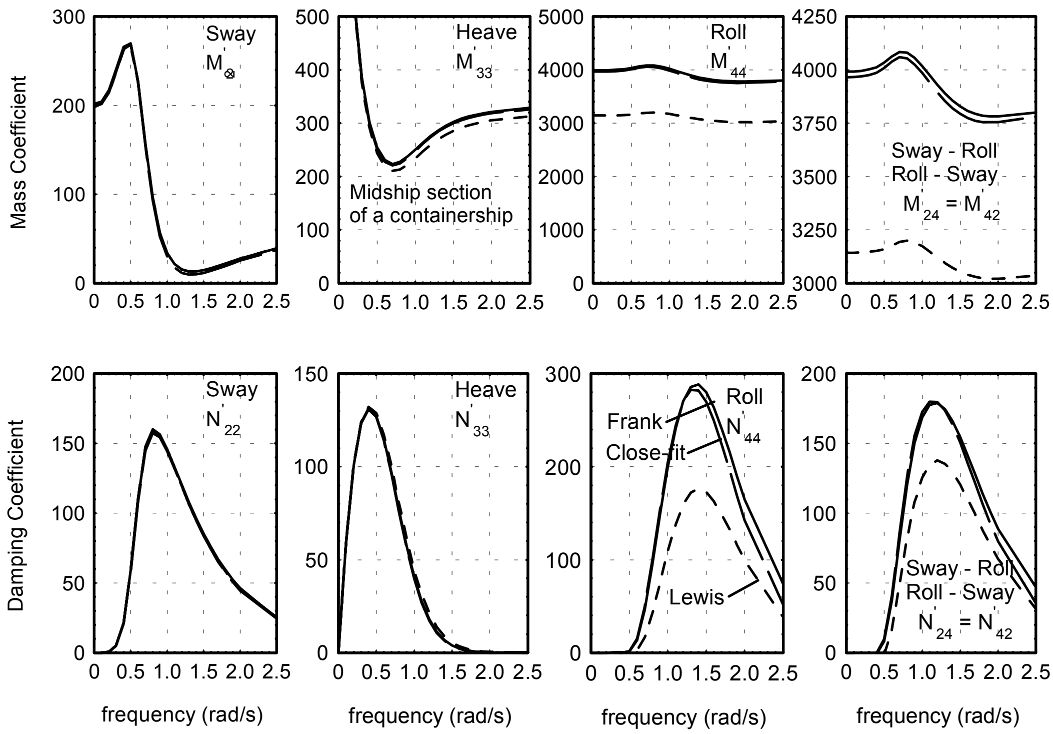

Figure 4.12: Comparison of Calculated Coefficients

With the exception of the roll motions, the three results are very close. The roll motion deviation, predicted with the Lewis conformal mapping method, is caused by the description of the "bilge" by the simple Lewis transformation.

A disadvantage of Frank's method can be the large computing time, when compared with Ursell-Tasai's method. Generally, it is advised to use Ursell-Tasai's method with 10 parameter close-fit conformal mapping. For submerged sections, bulbous sections and sections with an area coefficient, σ_s, less than 0.5, Frank's pulsating source method should be used.

Chapter 5

Viscous Damping

The strip theory is based on the potential flow theory. This holds that viscous effects are neglected, which can deliver serious problems when predicting roll motions at resonance frequencies. In practice, viscous roll damping effects can be accounted for by empirical formulas. For surge and roll, additional damping coefficients have to be introduced. Because of these additional contributions to the damping are from a viscous origin mainly, it is not possible to calculate the total damping in a pure theoretical way.

5.1 Surge Damping

The total damping for surge $B_{11t} = B_{11} + B_{11v}$ consists of a potential part, B_{11}, and an additional viscous part, B_{11v}. At forward ship speed V, the total damping coefficient, B_{11t}, can be determined simply from the resistance-speed curve of the ship in still water, $R_{sw}(V)$:

$$B_{11t} = B_{11} + B_{11v} = \frac{d\{R_{sw}(V)\}}{dV}$$

5.1.1 Total Surge Damping

For a rough estimation of the still water resistance use can be made of a modified empiric formula of [Troost, 1955], in principle valid at the ship's service speed for hull forms with a block coefficient C_B between 0.60 and 0.80:

$$R_{sw} = C_t \cdot \rho \nabla^{\frac{2}{3}} V^2$$

with:

$$C_t \approx 0.0036 + \frac{0.0152}{\log_{10}\{L\} + 0.60} \qquad \text{(with } L \text{ in meter)}$$

in which:

$$\begin{array}{lcl} \nabla & = & \text{volume of displacement of the ship in m}^3 \\ L & = & \text{length of the ship in m} \\ V & = & \text{forward ship speed in m/s.} \end{array}$$

This total resistance coefficient C_t is given in figure 5.1 as a function of the ship length.

[0] J.M.J. Journée, *"Theoretical Manual of SEAWAY, Release 4.19"*, Report 1216a, February 2001, Ship Hydromechanics Laboratory, Delft University of Technology, Mekelweg 2, 2628 CD Delft, The Netherlands. For updates see web site: http://dutw189.wbmt.tudelft.nl/~johan or http://www.shipmotions.nl.

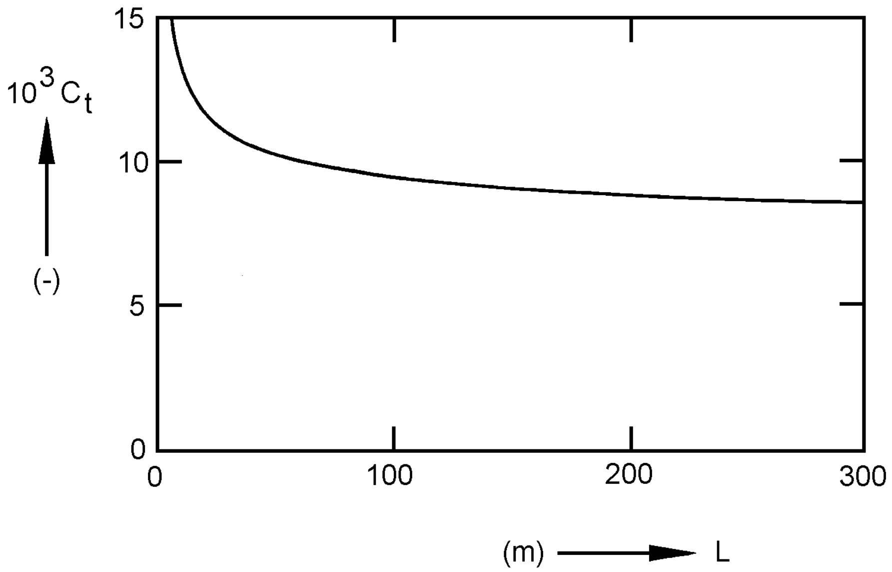

Figure 5.1: Total Still Water Resistance Coefficient of Troost

Then the total surge damping coefficient at forward ship speed V becomes:

$$B_{11t} = 2C_t \cdot \rho \nabla^{\frac{2}{3}} V$$

5.1.2 Viscous Surge Damping

This total damping coefficient includes a viscous part, which can be derived from the frictional part of the ship's resistance, defined by the ITTC-line:

$$R_f(V) = \frac{1}{2}\rho V^2 S \cdot \frac{0.075}{\left(\ln\{Rn\} - 2\right)^2} \qquad \text{with:} \quad Rn = \frac{VL}{\nu}$$

in which:

$$\begin{array}{lcl} \nu & = & \text{kinematic viscosity of seawater} \\ S & = & \text{wetted surface of the hull of the ship} \\ Rn & = & \text{Reynolds number} \end{array}$$

From this empiric formula follows the pure viscous part of the additional damping coefficient at forward ship speed V:

$$B_{11v} = \frac{d\left\{R_f(V)\right\}}{dV}$$

which can be obtained numerically.

5.2 Roll Damping

In case of pure free rolling in still water (free decay test), the uncoupled linear equation of the roll motion about the centre of gravity G is given by:

$$(I_{xx} + A_{44}) \cdot \ddot{\phi} + (B_{44} + B_{44v}) \cdot \dot{\phi} + C_{44} \cdot \phi = 0$$

with:

$$\begin{array}{rcll}
A_{44} & = & a_{44} + \overline{OG} \cdot a_{42} + \overline{OG} \cdot a_{24} + \overline{OG}^2 \cdot a_{22} & \text{(potential mass coefficient)} \\
B_{44} & = & b_{44} + \overline{OG} \cdot b_{42} + \overline{OG} \cdot b_{24} + \overline{OG}^2 \cdot b_{22} & \text{(potential damping coefficiernt)} \\
B_{44_V} & = & b_{44_V} & \text{(viscous damping coefficient)} \\
C_{44} & = & \rho g \nabla \cdot \overline{GM} & \text{(restoring term coefficient)}
\end{array}$$

while for zero forward speed:

$$a_{42} = a_{24} \qquad \text{and} \qquad b_{42} = b_{24}$$

These equations can be rewritten as:

$$\ddot{\phi} + 2\nu \cdot \dot{\phi} + \omega_0^2 \cdot \phi = 0$$

in which:

$$\begin{array}{rcll}
2\nu & = & \dfrac{B_{44} + B_{44v}}{I_{xx} + A_{44}} & \text{(quotient of damping and moment of inertia)} \\
\omega_0^2 & = & \dfrac{C_{44}}{I_{xx} + A_{44}} & \text{(natural roll frequency squared)}
\end{array}$$

The non-dimensional roll damping coefficient, κ, is given by:

$$\kappa = \frac{\nu}{\omega_0} = \frac{B_{44} + B_{44v}}{2\sqrt{(I_{xx} + A_{44}) \cdot C_{44}}}$$

This damping coefficient is written as a fraction between the actual damping coefficient, $B_{44} + B_{44v}$, and the **c**ritical damping coefficient, $B_{44cr} = 2\sqrt{(I_{xx} + A_{44}) \cdot C_{44}}$; so for critical damping: $\kappa_{cr} = 1$.

Herewith, the equation of motion can be re-written as:

$$\ddot{\phi} + 2\kappa\omega_0 \cdot \dot{\phi} + \omega_0^2 \cdot \phi = 0$$

Suppose the vessel is deflected to an initial heel angle, ϕ_a, in still water and then released. The solution of the equation of motion of this decay becomes:

$$\phi = \phi_a e^{-\nu t} \left(\cos \omega_\phi t + \frac{\nu}{\omega_\phi} \sin \omega_\phi t \right)$$

Then the logarithmic decrement of the motion is:

$$\nu T_\phi = \kappa \omega_0 T_\phi = \ln \left\{ \frac{\phi(t)}{\phi(t + T_\phi)} \right\}$$

Because $\omega_\phi^2 = \omega_0^2 - \nu^2$ for the natural frequency oscillation and the damping is small so that $\nu^2 \ll \omega_0^2$, one can neglect ν^2 here and use $\omega_\phi \approx \omega_0$; this leads to:

$$\omega_0 T_\phi \approx \omega_\phi T_\phi = 2\pi$$

The non-dimensional total roll damping is given now by:

$$\kappa = \frac{1}{2\pi} \ln \left\{ \frac{\phi(t)}{\phi(t + T_\phi)} \right\} = (B_{44} + B_{44v}) \cdot \frac{\omega_0}{2C_{44}}$$

The non-potential part of the total roll damping coefficient follows from the average value of κ by:

$$B_{44v} = \kappa \cdot \frac{2C_{44}}{\omega_0} - B_{44}$$

5.2.1 Experimental Determination

The κ-values can easily been found when results of free rolling experiments with a model in still water are available, see figure 5.2.

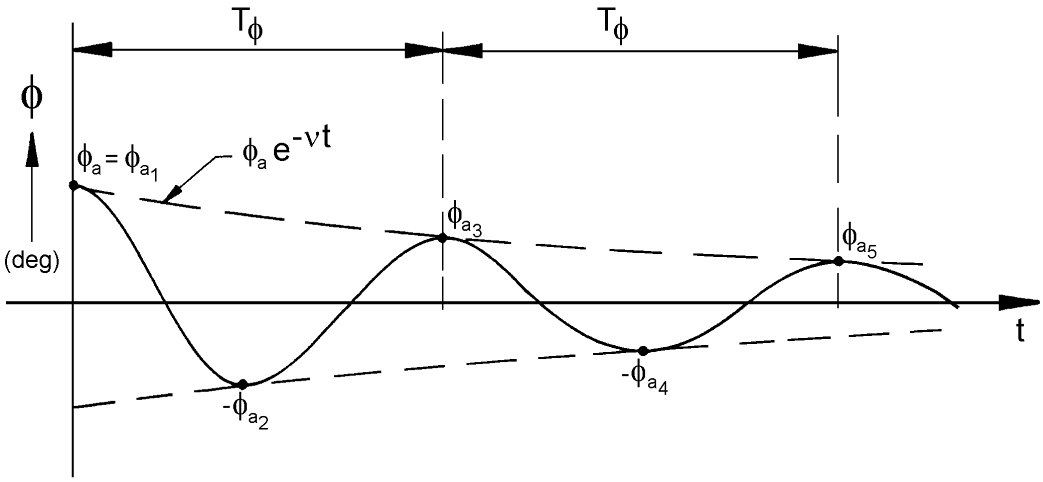

Figure 5.2: Time History of a Roll Decay Test

The results of free decay tests can be presented in different ways:

- Generally they are presented by plotting the non-dimensional damping coefficient, obtained from two successive positive or negative maximum roll angles ϕ_{a_i} and $\phi_{a_{i+2}}$, by:

$$\kappa = \frac{1}{2\pi} \cdot \ln \left\{ \frac{\phi_{a_i}}{\phi_{a_{i+2}}} \right\} \quad \text{versus} \quad \overline{\phi_a} = \left| \frac{\phi_{a_i} + \phi_{a_{i+2}}}{2} \right|$$

- To avoid spreading in the successively determined κ-values, caused by a possible zero-shift of the measuring signal, double amplitudes can be used instead:

$$\kappa = \frac{1}{2\pi} \cdot \ln \left\{ \frac{\phi_{a_i} - \phi_{a_{i+1}}}{\phi_{a_{i+2}} - \phi_{a_{i+3}}} \right\} \quad \text{versus} \quad \overline{\phi_a} = \left| \frac{\phi_{a_i} - \phi_{a_{i+1}} + \phi_{a_{i+2}} - \phi_{a_{i+3}}}{4} \right|$$

- Sometimes the results of free rolling tests are presented by:

$$\frac{\Delta \overline{\phi_a}}{\overline{\phi_a}} \quad \text{versus} \quad \overline{\phi_a}$$

with the absolute value of the average of two successive positive or negative maximum roll angles, given by:

$$\overline{\phi_a} = \left| \frac{\phi_{a_i} + \phi_{a_{i+1}}}{2} \right|$$

and the absolute value of the difference of the average of two successive positive or negative maximum roll angles, given by:

$$\Delta \overline{\phi_a} = \left| \phi_{a_i} - \phi_{a_{i+1}} \right|$$

Then the total non-dimensional roll damping coefficient becomes:

$$\kappa = \frac{1}{2\pi} \cdot \ln \left\{ \frac{2 + \frac{\Delta\overline{\phi_a}}{\overline{\phi_a}}}{2 - \frac{\Delta\overline{\phi_a}}{\overline{\phi_a}}} \right\}$$

The decay coefficient κ can therefore be estimated from the decaying oscillation by determining the ratio between any pair of successive (double) amplitudes. When the damping is very small and the oscillation decays very slowly, several estimates of the decay can be obtained from a single record. It is obvious that for a linear system a constant κ-value should be found in relation to $\overline{\phi_a}$.
Note that these decay tests provide no information about the relation between the potential coefficients and the frequency of oscillation. Indeed, this is impossible since decay tests are carried out at only one frequency: the natural frequency. These experiments deliver no information on the relation with the frequency of oscillation.
The method is not really practical when ν is much greater than about 0.2 and is in any case strictly valid for small values of ν only. Luckily, this is generally the case.
Be aware that this damping coefficient is determined by assuming an uncoupled roll motion (no other motions involved). Strictly, this damping coefficient is not valid for the actual coupled motions of a ship which will be moving in all directions simultaneously.

The successively found values for κ, plotted on base of the average roll amplitude, will often have a non-linear behavior as illustrated in the next figure.

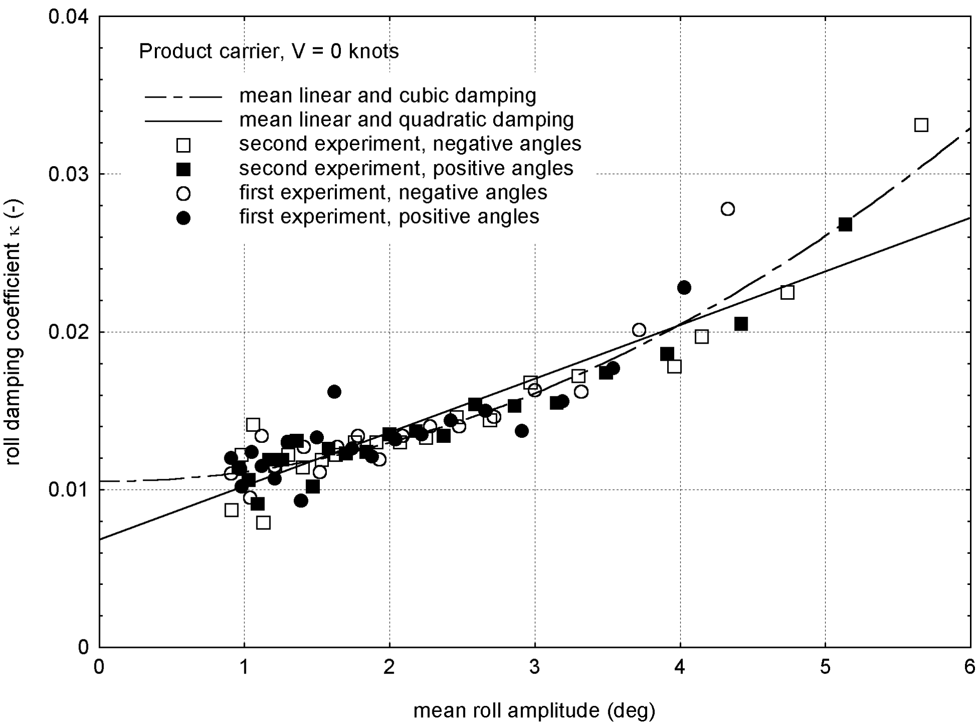

Figure 5.3: Roll Damping Coefficient

For a behavior like this, it will be found:

$$\kappa = \kappa_1 + \kappa_2 \cdot \phi_a$$

while sometimes even a cubic roll damping coefficient, $\kappa_3 \cdot \phi_a^2$, has to be added to this formula.
This non-linear behavior holds that during frequency domain calculations, the damping term is depending on the - so far unknown - solution for the transfer function of roll: ϕ_a/ζ_a. With a known wave amplitude, ζ_a, this problem can be solved in an iterative manner. A less accurate method is to use a fixed ϕ_a.

5.2.2 Empirical Formula for Barges

From model experiments with rectangular barges - with its center of gravity, G, in the water line - it is found by [Journée, 1991]:

$$\kappa = \kappa_1 + \kappa_2 \cdot \phi_a$$

with:

$$\begin{aligned}\kappa_1 &= 0.0013 \cdot \left(\frac{B}{d}\right)^2 \\ \kappa_2 &= 0.50\end{aligned}$$

in which B is the breadth and d is the draft of the barge.

5.2.3 Empirical Method of Miller

According to [Miller, 1974], the non-dimensional total roll damping coefficient, κ, can be obtained by:

$$\kappa = \kappa_1 + \kappa_2 \cdot \sqrt{\phi_a}$$

with:

$$\begin{aligned}\kappa_1 &= C_V \cdot 0.00085 \cdot \frac{L}{B} \cdot \sqrt{\frac{L}{\overline{GM}}} \cdot \left\{ \left(\frac{F_n}{C_b}\right) + \left(\frac{F_n}{C_b}\right)^2 + 2 \cdot \left(\frac{F_n}{C_b}\right)^3 \right\} \\ \kappa_2 &= 19.25 \cdot \left\{ A_{bk} \cdot \sqrt{\frac{l_{bk}}{r_b}} + 0.0024 \cdot L \cdot B \right\} \cdot \frac{r_b^3}{L \cdot B^3 \cdot d \cdot C_b}\end{aligned}$$

in which:

A_{bk}	=	$l_{bk} \cdot h_{bk}$ = one sided area of bilge keel (m^2)
l_{bk}	=	length of bilge keel (m)
h_{bk}	=	height of bilge keel (m)
r_b	=	distance center line of water plane to turn of bilge (m) (first point at which turn of bilge starts, relative to water plane)
L	=	length of ship (m)
B	=	breadth of ship (m)
d	=	draft of ship (m)
C_b	=	block coefficient (-)
$\overline{GM}$	=	initial metacentric height (m)
F_n	=	Froude number (-)
ϕ_a	=	amplitude of roll (rad)
C_V	=	correction factor on κ_1 for speed effect (-) (in the original formulation of Miller: $C_V = 1.0$)

Generally, $C_V = 1.0$ but for slender ships, like frigates, a suitable value for C_V seems to be:

$$C_V = 4.85 - 3.00 \cdot \sqrt{\overline{GM}_{Full\ Scale}}$$

5.2.4 Semi-Empirical Method of Ikeda

Because the viscous part of the roll damping acts upon the viscosity of the fluid significantly, it is not possible to calculate the total roll damping coefficient in a pure theoretical way. Besides this, also experiments showed a non-linear behavior of viscous parts of the roll damping.

Sometimes, for applications in frequency domain, an equivalent linear roll damping coefficient, $B^{(1)}_{44_V}$, has to be determined. This coefficient can be obtained by stipulating that an equivalent linear roll damping dissipates an identical amount of energy as the non-linear roll damping. This results for a linearized quadratic roll damping coefficient, $B^{(2)}_{44_V}$, into:

$$B^{(1)}_{44_V} \cdot \int_0^{T_\phi} \dot{\phi} \cdot \dot{\phi} dt = B^{(2)}_{44_V} \cdot \int_0^{T_\phi} |\dot{\phi}|\dot{\phi} \cdot \dot{\phi} dt$$

or:

$$B^{(1)}_{44_V} = \frac{8}{3\pi} \phi_a \omega \cdot B^{(2)}_{44_V}$$

For the estimation of the non-potential parts of the roll damping, use has been made of work published by [Ikeda et al., 1978]. A few subordinate parts have been modified and this empiric method is called here the "Ikeda method".

The Ikeda method estimates the following linear components of the roll damping coefficient of a ship:

B_{44_S}	=	a correction on the potential roll damping coefficient due to forward speed
B_{44_F}	=	the frictional roll damping coefficient
B_{44_E}	=	the eddy making roll damping coefficient
B_{44_L}	=	the lift roll damping coefficient
B_{44_K}	=	the bilge keel roll damping coefficient

So, the additional - mainly viscous - roll damping coefficient B_{44_V} is given by:

$$B_{44_V} = B_{44_S} + B_{44_F} + B_{44_E} + B_{44_L} + B_{44_K}$$

Ikeda, Himeno and Tanaka claim fairly good agreements between their prediction method and experimental results. They conclude that the method can be used safely for ordinary ship forms, which conclusion has been confirmed by the author too. But for unusual ship forms, for very full ship forms and for ships with a very large breadth to draft ratio the method is not always accurate sufficiently.

For numerical reasons three restrictions have been made here during the calculations:

- if, local, $\sigma_s > 0.999$ then $\sigma_s = 0.999$
- if, local, $\overline{OG} < -D_s \sigma_s$ then $\overline{OG} = -D_s \sigma_s$
- if a calculated component of the viscous roll damping coefficient becomes less than zero, this component will be set to zero.

Notation

In this description of the Ikeda method the notation of the authors (Ikeda, Himeno and Tanaka) is maintained as far as possible:

ρ	=	density of water	σ_s	=	sectional area coefficient
ν	=	kinematic viscosity of water	H_0	=	sectional half breadth to draft ratio
g	=	acceleration of gravity	a_1	=	sectional Lewis coefficient
V	=	forward ship speed	a_3	=	sectional Lewis coefficient
Rn	=	Reynolds number	M_s	=	sectional Lewis scale factor
ω	=	circular roll frequency	r_f	=	average distance between roll axis and hull surface
ϕ_a	=	roll amplitude			
L	=	length of the ship	L_O	=	distance point of taking representative angle of attack to roll axis approximated by $L_O = 0.3D$
B	=	breadth			
D	=	amidships draft			
C_M	=	amidships section coefficient	L_R	=	distance of centre of action of lift force in roll motion to roll axis approximated by $L_R = 0.5D$
C_B	=	block coefficient			
S_L	=	$L \cdot D \approx$ lateral area			
S_f	=	wetted hull surface area	h_k	=	height of the bilge keels
$\overline{OG}$	=	distance of centre of gravity above still water level, so positive upwards (sign convention deviates from the paper of Ikeda)	L_k	=	length of the bilge keels
			r_k	=	distance between roll axis and bilge keel
			f_k	=	correction for increase of flow velocity at the bilge
B_s	=	sectional breadth water line	C_p	=	pressure coefficient
D_s	=	sectional draft	l_m	=	lever of the moment
A_s	=	sectional area	r_b	=	local radius of the bilge circle

Effect of Forward Speed, B_{44_S}

Ikeda obtained an empirical formula for the three-dimensional forward speed correction on the zero speed potential damping by making use of the general characteristics of a doublet flow model. The rolling ship has been represented by two doublets: one at the stern and one at the bow of the ship.
With this, semi-theoretically the forward speed effect on the linear potential damping coefficient has been approximated as a fraction of the potential damping coefficient by:

$$B_{44_S} = B_{44} \cdot \left\{ 0.5 \left[A_2 + 1 + (A_2 - 1)\tanh\left(20(\Omega - 0.3)\right) + (2A_1 - A_2 - 1)e^{-150(\Omega-0.25)^2} \right] - 1.0 \right\}$$

with:

$$\begin{aligned} B_{44} &= \text{potential roll damping coefficient of the ship (about } G) \\ \Omega &= \omega \frac{V}{g} = \text{non-dimensional circular roll frequency} \\ \xi_D &= \omega^2 \frac{D}{g} = \text{non-dimensional circular roll frequency squared} \\ A_1 &= 1.0 + \xi_D^{-1.2} e^{-2\xi_D} = \text{ maximum value of } B_{44} \text{ at } \Omega = 0.25 \\ A_2 &= 0.5 + \xi_D^{-1.0} e^{-2\xi_D} = \text{ minimum value of } B_{44} \text{ at large } \Omega \end{aligned}$$

Frictional Roll Damping, B_{44_F}

Kato deduced semi-empirical formulas for the frictional roll damping from experimental results of circular cylinders, wholly immersed in the fluid. An effective Reynolds number of the roll motion was defined by:

$$Rn = \frac{0.512 \cdot (r_f \phi_a)^2 \omega}{\nu}$$

In here, for ship forms the average distance between the roll axis and the hull surface can be approximated by:

$$r_f = \frac{(0.887 + 0.145 \cdot C_B)\frac{S_f}{L} + 2\overline{OG}}{\pi}$$

with a wetted hull surface area S_f, approximated by:

$$S_f = L(1.7D + C_B B)$$

The relation between the density, kinematic viscosity and temperture of fresh water and sea water are given in figure 5.4.

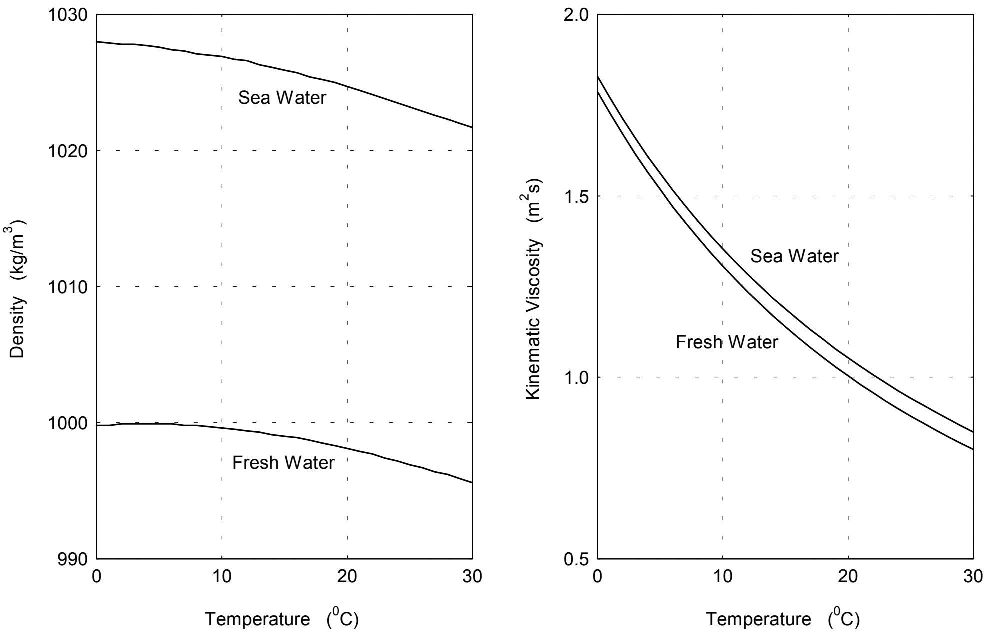

Figure 5.4: Relation Between Density, Kinematic Viscosity and Temperature of Water

When eliminating the temperature of water, the kinematic viscosity can be expressed into the density of water by the following relation in the kg-m-s system:

$$\begin{aligned}
\nu \cdot 10^6 &= 1.442 + 0.3924 \cdot (\rho - 1000) + 0.07424 \cdot (\rho - 1000)^2 \text{ m}^2/\text{s} \quad \text{(fresh water)} \\
\nu \cdot 10^6 &= 1.063 + 0.1039 \cdot (\rho - 1025) + 0.02602 \cdot (\rho - 1025)^2 \text{ m}^2/\text{s} \quad \text{(salt water)}
\end{aligned}$$

as given in figure 5.5.

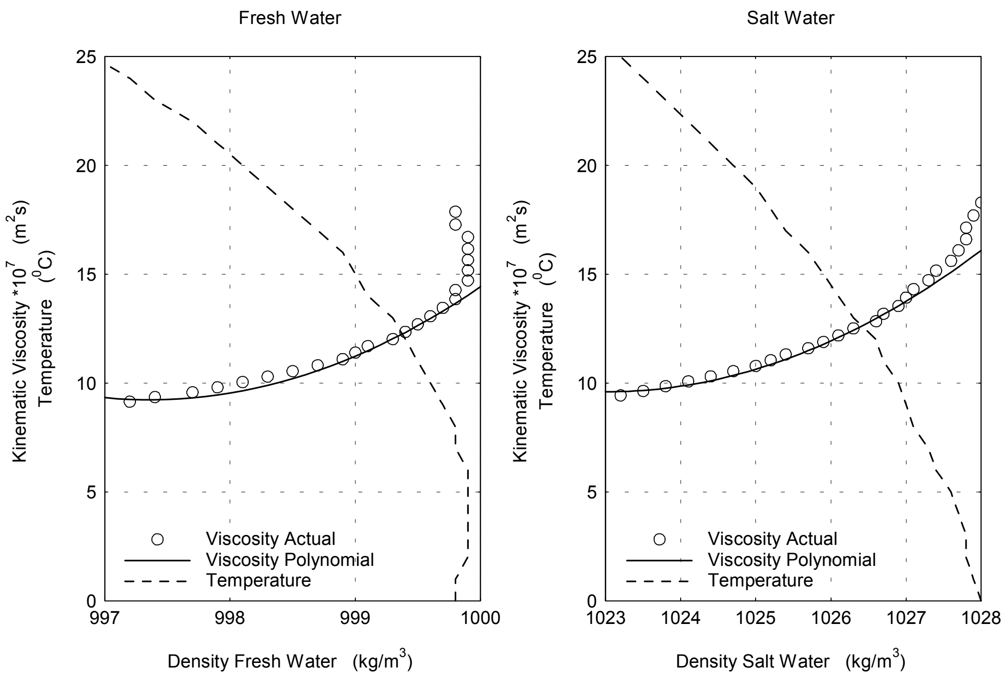

Figure 5.5: Kinematic Viscosity as a Function of Density

Kato expressed the skin friction coefficient as:

$$C_f = 1.328 \cdot Rn^{-0.5} + 0.014 \cdot Rn^{-0.114}$$

The first part in this expression represents the laminar flow case. The second part has been ignored by Ikeda, but has been included here.
Using this, the quadratic roll damping coefficient due to skin friction at zero forward ship speed is expressed as:

$$B^{(2)}_{44_{F_0}} = \frac{1}{2}\rho r_f^3 S_f C_f$$

This frictional roll damping component increases slightly with forward speed. Semi-theoretically, Tamiya deduced a modification coefficient for the effect of forward speed on the friction component.
Accurate enough from a practical point of view, this results into the following formula for the speed dependent frictional damping coefficient:

$$B^{(2)}_{44_F} = \frac{1}{2}\rho r_f^3 S_f C_f \cdot \left(1.0 + 4.1 \cdot \frac{V}{\omega L}\right)$$

Then, the equivalent linear roll damping coefficient due to skin friction is expressed as:

$$B_{44_F} = \frac{8}{3\pi}\phi_a\omega \cdot \frac{1}{2}\rho r_f^3 S_f C_f \cdot \left(1.0 + 4.1 \cdot \frac{V}{\omega L}\right)$$

Ikeda confirmed the use of his formula for the three-dimensional turbulent boundary layer over the hull of an oscillating ellipsoid in roll motion.

Eddy Making Damping, B_{44_E}

At zero forward speed the eddy making roll damping for the naked hull is mainly caused by vortices, generated by a two-dimensional separation. From a number of experiments with two-dimensional cylinders it was found that for a naked hull this component of the roll moment is proportional to the roll frequency squared and the roll amplitude squared. This means that the corresponding quadratic roll damping coefficient does not depend on the period parameter but on the hull form only.
When using a simple form for the pressure distribution on the hull surface it appears that the pressure coefficient C_p is a function of the ratio γ of the maximum relative velocity U_{max} to the mean velocity U_{mean} on the hull surface:

$$\gamma = \frac{U_{max}}{U_{mean}}$$

The C_p-γ relation was obtained from experimental roll damping data of two-dimensional models. These experimental results are fitted by:

$$C_p = 0.35 \cdot e^{-\gamma} - 2.0 \cdot e^{-0.187 \cdot \gamma} + 1.5$$

The value of γ around a cross section is approximated by the potential flow theory for a rotating Lewis form cylinder in an infinite fluid.
An estimation of the local maximum distance between the roll axis and the hull surface, r_{max}, has to be made.
Values of $r_{max}(\psi)$ have to be calculated for:

$$\psi = \psi_1 = 0.0 \qquad \text{and} \qquad \psi = \psi_2 = \frac{0.5}{\cos\left(\frac{a_1(1+a_3)}{4a_3}\right)}$$

The values of $r_{max}(\psi)$ follow from:

$$\begin{aligned} r_{max}(\psi) &= M_s \left\{ \left((1+a_1)\sin(\psi) - a_3 \sin(3\psi)\right)^2 + \right. \\ & \qquad \left. \left((1-a_1)\cos(\psi) + a_3\cos(3\psi)\right)^2 \right\}^{1/2} \end{aligned}$$

With these two results a value r_{max} and a value ψ follow from the conditions:
- for $r_{max}(\psi_1) > r_{max}(\psi_2)$: $r_{max} = r_{max}(\psi_1)$ and $\psi = \psi_1$
- for $r_{max}(\psi_1) < r_{max}(\psi_2)$: $r_{max} = r_{max}(\psi_2)$ and $\psi = \psi_2$

The relative velocity ratio γ on a cross section is obtained by:

$$\gamma = \frac{f_3\sqrt{\pi}}{2D_s\left(\sigma_s + \frac{\overline{OG}}{D_s}\right)\sqrt{H_0\sigma_s}} \cdot \left(r_{max} + \frac{2M_s}{H}\sqrt{a^2+b^2}\right)$$

with:

$$\begin{aligned} H_0 &= \frac{B_s}{2D_s} \\ \sigma_s &= \frac{A_s}{B_sD_s} \\ M_s &= \frac{B_s}{2(1+a_1+a_3)} = \frac{D_s}{1-a_1+a_3} \end{aligned}$$

$$
\begin{aligned}
H &= 1 + a_1^2 + 9a_3^2 + 2a_1(1-3a_3)\cos(2\psi) - 6a_3\cos(4\psi) \\
a &= -2a_3\cos(5\psi) + a_1(1-a_3)\cos(3\psi) + \left((6-3a_1)a_3^2 + (a_1^2-3a_1)a_3 + a_1^2\right)\cos(\psi) \\
b &= -2a_3\sin(5\psi) + a_1(1-a_3)\sin(3\psi) + \left((6+3a_1)a_3^2 + (a_1^2+3a_1)a_3 + a_1^2\right)\sin(\psi) \\
f_3 &= 1 + 4e^{-1.65\cdot 10^5(1-\sigma_s)^2}
\end{aligned}
$$

With this a quadratic sectional eddy making damping coefficient for zero forward speed follows from:

$$
\begin{aligned}
B_{44_{E_0}}^{(2)}{}' &= \frac{1}{2}\rho D_s^4 \left(\frac{r_{max}}{D_s}\right)^2 C_p \cdot \\
& \left(\left(1-\frac{f_1 r_b}{D_s}\right)\left(1+\frac{\overline{OG}}{D_s}-\frac{f_1 r_b}{D_s}\right) + f_2\left(H_0 - \frac{f_1 r_b}{D_s}\right)^2\right)
\end{aligned}
$$

with:

$$
\begin{aligned}
f_1 &= 0.5\cdot(1+\tanh(20\sigma_s - 14)) \\
f_2 &= 0.5\cdot 1\,(-\cos(\pi\sigma_s)) - 1.5\cdot\left(1-e^{5-5\sigma_s}\right)\sin^2(\pi\sigma_s)
\end{aligned}
$$

The approximations of the local radius of the bilge circle r_b are given as:

$$
\begin{aligned}
r_b &= 2D_s\sqrt{\frac{H_0(\sigma_s-1)}{\pi-4}} && \text{for: } r_b < D_s \text{ and } r_b < \frac{B_s}{2} \\
r_b &= D_s && \text{for: } H_0 > 1 \text{ and } r_b > D_s \\
r_b &= \frac{B_s}{2} && \text{for: } H_0 < 1 \text{ and } r_b > H_0 D_s
\end{aligned}
$$

For three-dimensional ship forms the zero forward speed eddy making quadratic roll damping coefficient is found by an integration over the ship length:

$$
B_{44_{E_0}}^{(2)} = \int_L B_{44_{E_0}}^{(2)}{}' dx_b
$$

This eddy making roll damping decreases rapidly with the forward speed to a non-linear correction for the lift force on a ship with a small angle of attack. Ikeda has analyzed this forward speed effect by experiments and the result has been given in an empirical formula. With this the equivalent linear eddy making damping coefficient at forward speed is given by:

$$
B_{44_E} = \frac{8}{3\pi}\cdot\phi_a\omega B_{44e_0}^{(2)}\cdot\frac{1}{1+K^2}
$$

with:

$$
K = \frac{V}{0.04\cdot\omega L}
$$

Lift Damping, B_{44_L}

The roll damping coefficient due to the lift force is described by a modified formula of Yumuro:

$$
B_{44_L} = \frac{1}{2}\rho S_L V k_N L_O L_R \left(1.0 + 1.4\cdot\frac{\overline{OG}}{L_R} + 0.7\cdot\frac{\overline{OG}^2}{L_O L_R}\right)
$$

The slope of the lift curve C_L/α is defined by:

$$\begin{aligned} k_N &= \frac{C_L}{\alpha} \\ &= \frac{2\pi D}{L} + \chi \left(4.1 \cdot \frac{B}{L} - 0.045 \right) \end{aligned}$$

in which the coefficient χ is given by Ikeda in relation to the amidships section coefficient C_M:

$$\begin{array}{rl} C_M < 0.92: & \chi = 0.00 \\ 0.92 < C_M < 0.97: & \chi = 0.10 \\ 0.97 < C_M < 0.99: & \chi = 0.30 \end{array}$$

These data are fitted here by:

$$\chi = 106 \cdot (C_M - 0.91)^2 - 700 \cdot (C_M - 0.91)^3$$

with the restrictions:
-if $C_M < 0.91$ then $\chi = 0.00$
-if $C_M > 1.00$ then $\chi = 0.35$

Bilge Keel Damping, B_{44_K}

The quadratic bilge keel roll damping coefficient is divided into two components:
- a component $B^{(2)}_{44_{K_N}}$ due to the normal force on the bilge keels
- a component $B^{(2)}_{44_{K_S}}$ due to the pressure on the hull surface, created by the bilge keels.

The coefficient of the normal force component $B^{(2)}_{44_{K_N}}$ of the bilge keel damping can be deduced from experimental results of oscillating flat plates. The drag coefficient C_D depends on the period parameter or the Keulegan-Carpenter number. Ikeda measured this non-linear drag also by carrying out free rolling experiments with an ellipsoid with and without bilge keels.
This resulted in a quadratic sectional damping coefficient:

$$B^{(2)\,\prime}_{44_{K_N}} = \rho r_k^3 h_k f_k^2 \cdot C_D$$

with:

$$\begin{aligned} C_D &= 22.5 \cdot \frac{h_k}{\pi r_k \phi_a f_k} + 2.40 \\ f_k &= 1.0 + 0.3 \cdot e^{-160 \cdot (1.0 - \sigma_s)} \end{aligned}$$

The approximation of the local distance between the roll axis and the bilge keel r_k is given as:

$$r_k = D_s \sqrt{\left(H_0 - 0.293 \cdot \frac{r_b}{D_s} \right)^2 + \left(1.0 + \frac{\overline{OG}}{D_s} - 0.293 \cdot \frac{r_b}{D_s} \right)^2}$$

The approximation of the local radius of the bilge circle r_b in here is given before.

Assuming a pressure distribution on the hull caused by the bilge keels, a quadratic sectional roll damping coefficient can be defined:

$$B^{(2)\ \prime}_{44_{K_S}} = \frac{1}{2}\rho r_k^2 f_k^2 \int_0^{h_k} C_p l_m dh$$

Ikeda carried out experiments to measure the pressure on the hull surface created by bilge keels. He found that the coefficient C_p^+ the pressure on the front face of the bilge keel does not depend on the period parameter, while the coefficient C_p^- of the pressure on the back face of the bilge keel and the length of the negative pressure region depend on the period parameter.
Ikeda defines an equivalent length of a constant negative pressure region S_0 over the height of the bilge keels, which is fitted to the following empirical formula:

$$S_0 = 0.3 \cdot \pi f_k r_k \phi_a + 1.95 \cdot h_k$$

The pressure coefficients on the front face of the bilge keel, C_p^+, and on the back face of the bilge keel, C_p^-, are given by:

$$C_p^+ = 1.20 \qquad \text{and} \qquad C_p^- = -22.5 \cdot \frac{h_k}{\pi f_k r_k \phi_a} - 1.20$$

and the sectional pressure moment is given by:

$$\int_0^{h_k} C_p l_m \cdot dh = D_s^2 \left(-A \cdot C_p^- + B \cdot C_p^+\right)$$

with:

$$\begin{aligned}
A &= (m_3 + m_4)m_8 - m_7^2 \\
B &= \frac{m_4^3}{3(H_0 - 0.215 \cdot m_1)} + \frac{(1-m_1)^2(2m_3 - m_2)}{6(1 - 0.215 \cdot m_1)} + m_1(m_3 m_5 + m_4 m_6) \\
m_1 &= \frac{r_b}{D_s} \\
m_2 &= \frac{-\overline{OG}}{D_s} \\
m_3 &= 1.0 - m_1 - m_2 \\
m_4 &= H_0 - m_1 \\
m_5 &= \frac{0.414 \cdot H_0 + 0.0651 \cdot m_1^2 - (0.382 \cdot H_0 + 0.0106)m_1}{(H_0 - 0.215 \cdot m_1)(1 - 0.215 \cdot m_1)} \\
m_6 &= \frac{0.414 \cdot H_0 + 0.0651 \cdot m_1^2 - (0.382 + 0.0106 \cdot H_0)m_1}{(H_0 - 0.215 \cdot m_1)(1 - 0.215 \cdot m_1)} \\
m_7 &= \frac{S_0}{D_s} - 0.25 \cdot \pi m_1 \qquad \text{for: } S_0 > 0.25 \cdot \pi r_b \\
&= 0.0 \qquad \text{for: } S_0 < 0.25 \cdot \pi r_b \\
m_8 &= m_7 + 0.414 \cdot m_1 \qquad \text{for: } S_0 > 0.25 \cdot \pi r_b \\
&= m_7 + 1.414 \cdot m_1 \left(1 - \cos\left(\frac{S_0}{r_b}\right)\right) \qquad \text{for: } S_0 < 0.25 \cdot \pi r_b
\end{aligned}$$

The equivalent linear total bilge keel damping coefficient can be obtained now by integrating the two sectional roll damping coefficients over the length of the bilge keels and linearizing the result:

$$B_{44_K} = \frac{8}{3\pi}\phi_a\omega \int\limits_{L_k} \left(B^{(2)}_{44_{K_N}}{}' + B^{(2)}_{44_{K_S}}{}' \right) dx_b$$

Experiments of Ikeda have shown that the effect of forward ship speed on this roll damping coefficient can be ignored.

Calculated Roll Damping Components

In figure 5.6, an example is given of the several roll damping components, as derived with Ikeda's method, for the S-175 container ship design.

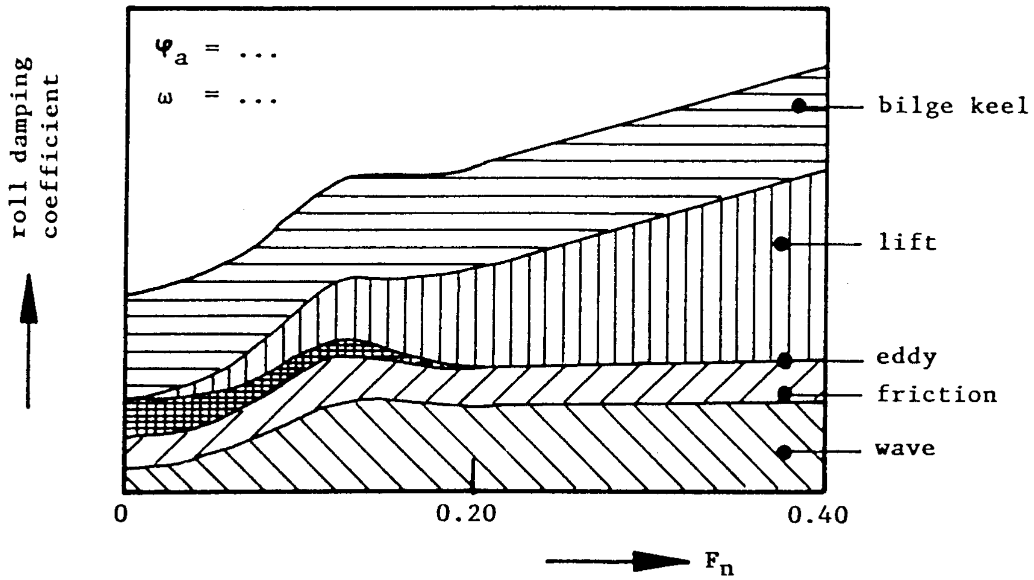

Figure 5.6: Roll Damping Coefficients of Ikeda

It may be noted that for full scale ships, because of the higher Reynolds number, the frictional part of the roll damping is expected to be smaller than showed above.

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Chapter 6

Hydromechanical Loads

With an approach as mentioned before, a description will be given here of the determination of the hydromechanical forces and moments for all six modes of motions.
In the "Ordinary Strip Theory", as published by [Korvin-Kroukovsky and Jacobs, 1957] and others, the uncoupled two-dimensional potential hydromechanical loads in the direction j are defined by:

$$X'_{h_j} = \frac{D}{Dt}\left\{M'_{jj} \cdot \dot{\zeta}^*_{h_j}\right\} + N'_{jj} \cdot \dot{\zeta}^*_{h_j} + X'_{RS_j} \tag{6.1}$$

In the "Modified Strip Theory", as published for instance by [Tasai, 1969] and others, these loads become:

$$X'_{h_j} = \frac{D}{Dt}\left\{\left(M'_{jj} - \frac{i}{\omega_e}N'_{jj}\right) \cdot \dot{\zeta}^*_{h_j}\right\} + X'_{RS_j}$$

In these definitions of the two-dimensional hydromechanical loads, $\zeta^*_{h_j}$ is the harmonic oscillatory motion, M_{jj} and N_{jj} are the two-dimensional potential mass and damping and the non-diffraction part X_{RS_j} is the two-dimensional quasi-static restoring spring term.
At the following pages, the hydromechanical loads are calculated in the $G(x_b, y_b, z_b)$ axes system with the centre of gravity G in the still water level, so $\overline{OG} = 0$.
Some of the terms in the hydromechanical loads are outlined there. The "Modified Strip Theory" includes these outlined terms. When ignoring these outlined terms the "Ordinary Strip Theory" is presented.

6.1 Hydromechanical Forces for Surge

The hydromechanical forces for surge are found by an integration over the ship length of the two-dimensional values:

$$X_{h_1} = \int_L X'_{h_1} \cdot dx_b$$

When assuming that the cross sectional hydromechanical force hold at a plane through the local centroid of the cross section b, parallel to (x_b, y_b), equivalent longitudinal motions

[0] J.M.J. Journée, *"Theoretical Manual of SEAWAY, Release 4.19"*, Report 1216a, February 2001, Ship Hydromechanics Laboratory, Delft University of Technology, Mekelweg 2, 2628 CD Delft, The Netherlands. For updates see web site: http://dutw189.wbmt.tudelft.nl/~johan or http://www.shipmotions.nl.

of the water particles, relative to the cross section of an oscillating ship in still water, are defined by:

$$
\begin{aligned}
\zeta_{h_1}^{*} &= -x + \overline{bG} \cdot \theta \\
\dot{\zeta}_{h_1}^{*} &= -\dot{x} + \overline{bG} \cdot \dot{\theta} - V \frac{\partial \overline{bG}}{\partial x_b} \cdot \theta \\
&\approx -\dot{x} + \overline{bG} \cdot \dot{\theta} \\
\ddot{\zeta}_{h_1}^{*} &= -\ddot{x} + \overline{bG} \cdot \ddot{\theta} - 2V \frac{\partial \overline{bG}}{\partial x_b} \cdot \dot{\theta} + V^2 \frac{\partial^2 \overline{bG}}{\partial x_b^2} \cdot \theta \\
&\approx -\ddot{x} + \overline{bG} \cdot \ddot{\theta}
\end{aligned}
$$

In here, $\overline{bG}$ is the vertical distance of the centre of gravity of the ship G above the centroid b of the local submerged sectional area.
According to the "Ordinary Strip Theory" the two-dimensional potential hydromechanical force on a surging cross section in still water is defined by:

$$
\begin{aligned}
X'_{h_1} &= \frac{D}{Dt} \left\{ M'_{11} \cdot \dot{\zeta}_{h_1}^{*} \right\} + N'_{11} \cdot \dot{\zeta}_{h_1}^{*} \\
&= M'_{11} \cdot \ddot{\zeta}_{h_1}^{*} + \left(N'_{11} - V \cdot \frac{dM'_{11}}{dx_b} \right) \cdot \dot{\zeta}_{h_1}^{*}
\end{aligned}
$$

According to the "Modified Strip Theory" this hydromechanical force becomes:

$$
\begin{aligned}
X'_{h_1} &= \frac{D}{Dt} \left\{ \left(M'_{11} - \frac{i}{\omega_e} \cdot N'_{11} \right) \cdot \dot{\zeta}_{h_1}^{*} \right\} \\
&= \left(M'_{11} + \frac{V}{\omega_e^2} \cdot \frac{dN'_{11}}{dx_b} \right) \cdot \ddot{\zeta}_{h_1}^{*} + \left(N'_{11} - V \cdot \frac{dM'_{11}}{dx_b} \right) \cdot \dot{\zeta}_{h_1}^{*}
\end{aligned}
$$

This results into the following coupled surge equation:

$$
\begin{aligned}
\rho \nabla \cdot \ddot{x} - X_{h_1} = \quad & (\rho \nabla + a_{11}) \cdot \ddot{x} + b_{11} \cdot \dot{x} + c_{11} \cdot x \\
& + a_{13} \cdot \ddot{z} + b_{13} \cdot \dot{z} + c_{13} \cdot z \\
& + a_{15} \cdot \ddot{\theta} + b_{15} \cdot \dot{\theta} + c_{15} \cdot \theta \\
= \quad & X_{w_1}
\end{aligned}
$$

with:

$$
\begin{aligned}
a_{11} &= + \int_L M'_{11} \cdot dx_b \\
&\quad + \boxed{\frac{V}{\omega_e^2} \int_L \frac{dN'_{11}}{dx_b} \cdot dx_b} \\
b_{11} &= + \int_L \left(N'_{11} - V \cdot \frac{dM'_{11}}{dx_b} \right) \cdot dx_b + b_{11_V} \\
c_{11} &= 0 \\
a_{13} &= 0
\end{aligned}
$$

$$
\begin{aligned}
b_{13} &= 0 \\
c_{13} &= 0 \\
a_{15} &= -\int_L M'_{11} \cdot \overline{bG} \cdot dx_b \\
&\quad + \boxed{\frac{-V}{\omega_e^2} \int_L \frac{dN'_{11}}{dx_b} \cdot \overline{bG} \cdot dx_b} \\
b_{15} &= -\int_L \left(N'_{11} - V \cdot \frac{dM'_{11}}{dx_b} \right) \cdot \overline{bG} \cdot dx_b - b_{11_V} \cdot \overline{BG} \\
c_{15} &= 0
\end{aligned}
$$

The "Modified Strip Theory" includes the outlined terms. When ignoring the outlined terms the "Ordinary Strip Theory" is presented.
A small viscous surge damping coefficient b_{11_V}, derived from the still water resistance approximation of [Troost, 1955], has been added here.
After simplification, the expressions for the total hydromechanical coefficients in the coupled surge equation become:

$$
\begin{aligned}
a_{11} &= +\int_L M'_{11} \cdot dx_b \\
b_{11} &= +\int_L N'_{11} \cdot dx_b + b_{11_V} \\
c_{11} &= 0 \\
a_{13} &= 0 \\
b_{13} &= 0 \\
c_{13} &= 0 \\
a_{15} &= -\int_L M'_{11} \cdot \overline{bG} \cdot dx_b \\
&\quad + \boxed{\frac{-V}{\omega_e^2} \int_L \frac{dN'_{11}}{dx_b} \cdot \overline{bG} \cdot dx_b} \\
b_{15} &= -\int_L \left(N'_{11} - V \cdot \frac{dM'_{11}}{dx_b} \right) \cdot \overline{bG} \cdot dx_b - b_{11_V} \cdot \overline{BG} \\
c_{15} &= 0
\end{aligned}
$$

6.2 Hydromechanical Forces for Sway

The hydromechanical forces for sway are found by an integration over the ship length of the two-dimensional values:

$$X_{h_2} = \int_L X'_{h_2} \cdot dx_b$$

The lateral and roll motions of the water particles, relative to the cross section of an oscillating ship in still water, are defined by:

$$\begin{array}{lll} \zeta^*_{h_2} & = & -y - x_b \cdot \psi - \overline{OG} \cdot \phi \qquad \zeta^*_{h_4} = -\phi \\ \dot{\zeta}^*_{h_2} & = & -\dot{y} - x_b \cdot \dot{\psi} + V \cdot \psi - \overline{OG} \cdot \dot{\phi} \qquad \dot{\zeta}^*_{h_4} = -\dot{\phi} \\ \ddot{\zeta}^*_{h_2} & = & -\ddot{y} - x_b \cdot \ddot{\psi} + 2V \cdot \dot{\psi} - \overline{OG} \cdot \ddot{\phi} \qquad \ddot{\zeta}^*_{h_4} = -\ddot{\phi} \end{array}$$

According to the "Ordinary Strip Theory" the two-dimensional potential hydromechanical force on a swaying cross section in still water is defined by:

$$\begin{array}{lll} X'_{h_2} & = & \frac{D}{Dt}\left\{M'_{22} \cdot \dot{\zeta}^*_{h_2}\right\} + N'_{22} \cdot \dot{\zeta}^*_{h_2} \\ & & +\frac{D}{Dt}\left\{M'_{24} \cdot \dot{\zeta}^*_{h_4}\right\} + N'_{24} \cdot \dot{\zeta}^*_{h_4} \\ & = & M'_{22} \cdot \ddot{\zeta}^*_{h_2} + \left(N'_{22} - V \cdot \frac{dM'_{22}}{dx_b}\right) \cdot \dot{\zeta}^*_{h_2} \\ & & +M'_{24} \cdot \ddot{\zeta}^*_{h_4} + \left(N'_{24} - V \cdot \frac{dM'_{24}}{dx_b}\right) \cdot \dot{\zeta}^*_{h_4} \end{array}$$

According to the "Modified Strip Theory" this hydromechanical force becomes:

$$\begin{array}{lll} X'_{h_2} & = & \frac{D}{Dt}\left\{\left(M'_{22} - \frac{i}{\omega_e} \cdot N'_{22}\right) \cdot \dot{\zeta}^*_{h_2}\right\} \\ & & +\frac{D}{Dt}\left\{\left(M'_{24} - \frac{i}{\omega_e} \cdot N'_{24}\right) \cdot \dot{\zeta}^*_{h_4}\right\} \\ & = & \left(M'_{22} + \frac{V}{\omega_e^2} \cdot \frac{dN'_{22}}{dx_b}\right) \cdot \ddot{\zeta}^*_{h_2} + \left(N'_{22} - V \cdot \frac{dM'_{22}}{dx_b}\right) \cdot \dot{\zeta}^*_{h_2} \\ & & +\left(M'_{24} + \frac{V}{\omega_e^2} \cdot \frac{dN'_{24}}{dx_b}\right) \cdot \ddot{\zeta}^*_{h_4} + \left(N'_{24} - V \cdot \frac{dM'_{24}}{dx_b}\right) \cdot \dot{\zeta}^*_{h_4} \end{array}$$

This results into the following coupled sway equation:

$$\begin{array}{lll} \rho\nabla \cdot \ddot{y} - X_{h_2} & = & (\rho\nabla + a_{22}) \cdot \ddot{y} + b_{22} \cdot \dot{y} + c_{22} \cdot y \\ & & +a_{24} \cdot \ddot{\phi} + b_{24} \cdot \dot{\phi} + c_{24} \cdot \phi \\ & & +a_{26} \cdot \ddot{\psi} + b_{26} \cdot \dot{\psi} + c_{26} \cdot \psi \\ & = & X_{w_2} \end{array}$$

with:

$$\begin{array}{lll} a_{22} & = & +\int\limits_L M'_{22} \cdot dx_b \\ & & +\boxed{\frac{V}{\omega_e^2} \int\limits_L \frac{dN'_{22}}{dx_b} \cdot dx_b} \\ b_{22} & = & +\int\limits_L \left(N'_{22} - V \cdot \frac{dM'_{22}}{dx_b}\right) \cdot dx_b \end{array}$$

$$
\begin{aligned}
c_{22} &= 0 \\
a_{24} &= +\int\limits_L M'_{24} \cdot dx_b + \overline{OG} \int\limits_L M'_{22} \cdot dx_b \\
&\quad + \boxed{\frac{V}{\omega_e^2} \int\limits_L \frac{dN'_{24}}{dx_b} \cdot dx_b + \frac{V}{\omega_e^2} \cdot \overline{OG} \int\limits_L \frac{dN'_{22}}{dx_b} \cdot dx_b} \\
b_{24} &= +\int\limits_L \left(N'_{24} - V \cdot \frac{dM'_{24}}{dx_b} \right) \cdot dx_b + \overline{OG} \int\limits_L \left(N'_{22} - V \cdot \frac{dM'_{22}}{dx_b} \right) \cdot dx_b \\
c_{24} &= 0 \\
a_{26} &= +\int\limits_L M'_{22} \cdot x_b \cdot dx_b + \frac{V}{\omega_e^2} \int\limits_L \left(N'_{22} - V \cdot \frac{dM'_{22}}{dx_b} \right) \cdot dx_b \\
&\quad + \boxed{\frac{V}{\omega_e^2} \int\limits_L N'_{22} \cdot dx_b + \frac{V}{\omega_e^2} \int\limits_L \frac{dN'_{22}}{dx_b} \cdot x_b \cdot dx_b} \\
b_{26} &= +\int\limits_L \left(N'_{22} - V \cdot \frac{dM'_{22}}{dx_b} \right) \cdot x_b \cdot dx_b - 2V \int\limits_L M'_{22} \cdot dx_b \\
&\quad + \boxed{\frac{V^2}{\omega_e^2} \int\limits_L \frac{dN'_{22}}{dx_b} \cdot dx_b} \\
c_{26} &= 0
\end{aligned}
$$

The "Modified Strip Theory" includes the outlined terms. When ignoring the outlined terms the "Ordinary Strip Theory" is presented.
After simplification, the expressions for the total hydromechanical coefficients in the coupled sway equation become:

$$
\begin{aligned}
a_{22} &= +\int\limits_L M'_{22} \cdot dx_b \\
b_{22} &= +\int\limits_L N'_{22} \cdot dx_b \\
c_{22} &= 0 \\
a_{24} &= +\int\limits_L M'_{24} \cdot dx_b + \overline{OG} \int\limits_L M'_{22} \cdot dx_b \\
b_{24} &= +\int\limits_L N'_{24} \cdot dx_b + \overline{OG} \int\limits_L N'_{22} \cdot dx_b \\
c_{24} &= 0 \\
a_{26} &= +\int\limits_L M'_{22} \cdot x_b \cdot dx_b + \frac{V}{\omega_e^2} \int\limits_L N'_{22} \cdot dx_b \\
b_{26} &= +\int\limits_L N'_{22} \cdot x_b \cdot dx_b - V \int\limits_L M'_{22} \cdot dx_b
\end{aligned}
$$

$$c_{26} = 0$$

So no terms are added for the "Modified Strip Theory".

6.3 Hydromechanical Forces for Heave

The hydromechanical forces for heave are found by an integration over the ship length of the two-dimensional values:

$$X_{h_3} = \int_L X'_{h_3} \cdot dx_b$$

The vertical motions of the water particles, relative to the cross section of an oscillating ship in still water, are defined by:

$$\begin{aligned}
\zeta^*_{h_3} &= -z \ + x_b \cdot \theta \\
\dot{\zeta}^*_{h_3} &= -\dot{z} \ + x_b \cdot \dot{\theta} \ - V \cdot \theta \\
\ddot{\zeta}^*_{h_3} &= -\ddot{z} \ + x_b \cdot \ddot{\theta} \ - 2V \cdot \dot{\theta}
\end{aligned}$$

According to the "Ordinary Strip Theory" the two-dimensional potential hydromechanical force on a heaving cross section in still water is defined by:

$$\begin{aligned}
X'_{h_3} &= \frac{D}{Dt}\left\{M'_{33} \cdot \dot{\zeta}^*_{h_3}\right\} + N'_{33} \cdot \dot{\zeta}^*_{h_3} + 2\rho g \cdot y_w \cdot \zeta^*_{h_3} \\
&= M'_{33} \cdot \ddot{\zeta}^*_{h_3} + \left(N'_{33} - V \cdot \frac{dM'_{33}}{dx_b}\right) \cdot \dot{\zeta}^*_{h_3} + 2\rho g \cdot y_w \cdot \zeta^*_{h_3}
\end{aligned}$$

According to the "Modified Strip Theory" this hydromechanical force becomes:

$$\begin{aligned}
X'_{h_3} &= \frac{D}{Dt}\left\{\left(M'_{33} - \frac{i}{\omega_e} \cdot N'_{33}\right) \cdot \dot{\zeta}^*_{h_3}\right\} + 2\rho g \cdot y_w \cdot \zeta^*_{h_3} \\
&= \left(M'_{33} + \frac{V}{\omega_e^2} \cdot \frac{dN'_{33}}{dx_b}\right) \cdot \ddot{\zeta}^*_{h_3} + \left(N'_{33} - V \cdot \frac{dM'_{33}}{dx_b}\right) \cdot \dot{\zeta}^*_{h_3} + 2\rho g \cdot y_w \cdot \zeta^*_{h_3}
\end{aligned}$$

This results into the following coupled heave equation:

$$\begin{aligned}
\rho\nabla \cdot \ddot{z} - X_{h_3} = \ & a_{31} \cdot \ddot{x} + b_{31} \cdot \dot{x} + c_{31} \cdot x \\
& + (\rho\nabla + a_{33}) \cdot \ddot{z} + b_{33} \cdot \dot{z} + c_{33} \cdot z \\
& + a_{35} \cdot \ddot{\theta} + b_{35} \cdot \dot{\theta} + c_{35} \cdot \theta \\
= \ & X_{w_3}
\end{aligned}$$

with:

$$\begin{aligned}
a_{31} &= 0 \\
b_{31} &= 0 \\
c_{31} &= 0 \\
a_{33} &= +\int_L M'_{33} \cdot dx_b
\end{aligned}$$

$$
\begin{aligned}
& +\boxed{\frac{V}{\omega_e^2}\int\limits_L \frac{dN'_{33}}{dx_b}\cdot dx_b} \\
b_{33} &= +\int\limits_L \left(N'_{33} - V\cdot\frac{dM'_{33}}{dx_b}\right)\cdot dx_b \\
c_{33} &= +2\rho g \int\limits_L y_w \cdot dx_b \\
a_{35} &= -\int\limits_L M'_{33}\cdot x_b \cdot dx_b - \frac{V}{\omega_e^2}\int\limits_L \left(N'_{33} - V\cdot\frac{dM'_{33}}{dx_b}\right)\cdot dx_b \\
& +\boxed{\frac{-V}{\omega_e^2}\int\limits_L N'_{33}\cdot dx_b - \frac{V}{\omega_e^2}\int\limits_L \frac{dN'_{33}}{dx_b}\cdot x_b\cdot dx_b} \\
b_{35} &= -\int\limits_L \left(N'_{33} - V\cdot\frac{dM'_{33}}{dx_b}\right)\cdot x_b\cdot dx_b + 2V\int\limits_L M'_{33}\cdot dx_b \\
& +\boxed{\frac{V^2}{\omega_e^2}\int\limits_L \frac{dN'_{33}}{dx_b}\cdot dx_b} \\
c_{35} &= -2\rho g \int\limits_L y_w \cdot x_b\cdot dx_b
\end{aligned}
$$

The "Modified Strip Theory Method" includes the outlined terms. When ignoring the outlined terms the "Ordinary Strip Theory" is presented.

After simplification, the expressions for the total hydromechanical coefficients in the coupled heave equation become:

$$
\begin{aligned}
a_{31} &= 0 \\
b_{31} &= 0 \\
c_{31} &= 0 \\
a_{33} &= +\int\limits_L M'_{33}\cdot dx_b \\
b_{33} &= +\int\limits_L N'_{33}\cdot dx_b \\
c_{33} &= +2\rho g \int\limits_L y_w\cdot dx_b \\
a_{35} &= -\int\limits_L M'_{33}\cdot x_b\cdot dx_b - \frac{V}{\omega_e^2}\int\limits_L N'_{33}\cdot dx_b \\
b_{35} &= -\int\limits_L N'_{33}\cdot x_b\cdot dx_b + V\int\limits_L M'_{33}\cdot dx_b
\end{aligned}
$$

$$c_{35} \quad = \quad -2\rho g \int\limits_L y_w \cdot x_b \cdot dx_b$$

So no terms are added for the "Modified Strip Theory".

6.4 Hydromechanical Moments for Roll

The hydromechanical moments for roll are found by an integration over the ship length of the two-dimensional values:

$$X_{h_4} = \int\limits_L X'_{h_4} \cdot dx_b$$

The roll and lateral motions of the water particles, relative to the cross section of an oscillating ship in still water, are defined by:

$$\begin{array}{llcl}
\zeta^*_{h_4} = -\phi & \zeta^*_{h_2} & = & -y \, - x_b \cdot \psi \, - \overline{OG} \cdot \phi \\
\dot{\zeta}^*_{h_4} = -\dot{\phi} & \dot{\zeta}^*_{h_2} & = & -\dot{y} \, - x_b \cdot \dot{\psi} \, + V \cdot \psi \, - \overline{OG} \cdot \dot{\phi} \\
\ddot{\zeta}^*_{h_4} = -\ddot{\phi} & \ddot{\zeta}^*_{h_2} & = & -\ddot{y} \, - x_b \cdot \ddot{\psi} \, + 2V \cdot \dot{\psi} \, - \overline{OG} \cdot \ddot{\phi}
\end{array}$$

According to the "Ordinary Strip Theory" the two-dimensional potential hydromechanical moment on a rolling cross section in still water is defined by:

$$\begin{array}{rcl}
X'_{h_4} & = & \frac{D}{Dt}\{\left\{M'_{44} \cdot \dot{\zeta}^*_{h_4}\right\} + N'_{44} \cdot \dot{\zeta}^*_{h_4} + 2\rho g \cdot \left(\frac{y_w^3}{3} - \frac{A_s}{2} \cdot \overline{bG}\right) \cdot \zeta^*_{h_4} \\
& & +\frac{D}{Dt}\left\{M'_{42} \cdot \dot{\zeta}^*_{h_2}\right\} + N'_{42} \cdot \dot{\zeta}^*_{h_2} \\
& = & M'_{44} \cdot \ddot{\zeta}^*_{h_4} + \left(N'_{44} - V \cdot \frac{dM'_{44}}{dx_b}\right) \cdot \dot{\zeta}^*_{h_4} + 2\rho g \cdot \left(\frac{y_w^3}{3} - \frac{A_s}{2} \cdot \overline{bG}\right) \cdot \zeta^*_{h_4} \\
& & +M'_{42} \cdot \ddot{\zeta}^*_{h_2} + \left(N'_{42} - V \cdot \frac{dM'_{42}}{dx_b}\right) \cdot \dot{\zeta}^*_{h_2}
\end{array}$$

or the "Modified Strip Theory" this hydromechanical moment becomes:

$$\begin{array}{rcl}
X'_{h_4} & = & \frac{D}{Dt}\left\{\left(M'_{44} - \frac{i}{\omega_e} \cdot N'_{44}\right) \cdot \dot{\zeta}^*_{h_4}\right\} + 2\rho g \cdot \left(\frac{y_w^3}{3} - \frac{A_s}{2} \cdot \overline{bG}\right) \cdot \zeta^*_{h_4} \\
& & +\frac{D}{Dt}\left\{\left(M'_{42} - \frac{i}{\omega_e} \cdot N'_{42}\right) \cdot \dot{\zeta}^*_{h_2}\right\} \\
& = & \left(M'_{44} + \frac{V}{\omega_e^2} \cdot \frac{dN'_{44}}{dx_b}\right) \cdot \ddot{\zeta}^*_{h_4} + \left(N'_{44} - V \cdot \frac{dM'_{44}}{dx_b}\right) \cdot \dot{\zeta}^*_{h_4} \\
& & +2\rho g \cdot \left(\frac{y_w^3}{3} - \frac{A_s}{2} \cdot \overline{bG}\right) \cdot \zeta^*_{h_4} \\
& & +\left(M'_{42} + \frac{V}{\omega_e^2} \cdot \frac{dN'_{42}}{dx_b}\right) \cdot \ddot{\zeta}^*_{h_2} + \left(N'_{42} - V \cdot \frac{dM'_{42}}{dx_b}\right) \cdot \dot{\zeta}^*_{h_2}
\end{array}$$

This results into the following coupled roll equation:

$$\begin{array}{rcr}
I_{xx} \cdot \ddot{\phi} - I_{xz} \cdot \ddot{\psi} - X_{h_4} & = & a_{42} \, \cdot \ddot{y} + b_{42} \cdot \dot{y} + c_{42} \cdot y \\
& & +(I_{xx} + a_{44}) \cdot \ddot{\phi} + b_{44} \cdot \dot{\phi} + c_{44} \cdot \phi \\
& & +(-I_{xz} + a_{46}) \cdot \ddot{\psi} + b_{46} \cdot \dot{\psi} + c_{46} \cdot \psi \\
& = & X_{w_4}
\end{array}$$

with:

$$
\begin{aligned}
a_{42} &= +\int_L M'_{42} \cdot dx_b + \overline{OG} \cdot a_{22} \\
&\quad + \boxed{\frac{V}{\omega_e^2} \int_L \frac{dN'_{42}}{dx_b} \cdot dx_b} \\
b_{42} &= +\int_L \left(N'_{42} - V \cdot \frac{dM'_{42}}{dx_b} \right) \cdot dx_b + \overline{OG} \cdot b_{22} \\
c_{42} &= 0 + \overline{OG} \cdot c_{22} \\
a_{44} &= +\int_L M'_{44} \cdot dx_b + \overline{OG} \int_L M'_{42} \cdot dx_b + \overline{OG} \cdot a_{24} \\
&\quad + \boxed{\frac{V}{\omega_e^2} \int_L \frac{dN'_{44}}{dx_b} \cdot dx_b + \frac{V}{\omega_e^2} \cdot \overline{OG} \int_L \frac{dN'_{42}}{dx_b} \cdot dx_b} \\
b_{44} &= +\int_L \left(N'_{24} - V \cdot \frac{dM'_{44}}{dx_b} \right) \cdot dx_b + \overline{OG} \int_L \left(N'_{42} - V \cdot \frac{dM'_{42}}{dx_b} \right) \cdot dx_b + b_{44_V} + \overline{OG} \cdot b_{24} \\
c_{44} &= +2\rho g \int_L \left(\frac{y_w^3}{3} - \frac{A_s}{2} \cdot \overline{bG} \right) dx_b \\
&= +\rho g \nabla \cdot \overline{GM} \\
a_{46} &= +\int_L M'_{42} \cdot x_b \cdot dx_b + \frac{V}{\omega_e^2} \int_L \left(N'_{42} - V \cdot \frac{dM'_{42}}{dx_b} \right) \cdot dx_b + \overline{OG} \cdot a_{26} \\
&\quad + \boxed{\frac{V}{\omega_e^2} \int_L N'_{42} \cdot dx_b + \frac{V}{\omega_e^2} \int_L \frac{dN'_{42}}{dx_b} \cdot x_b \cdot dx_b} \\
b_{46} &= +\int_L \left(N'_{42} - V \cdot \frac{dM'_{42}}{dx_b} \right) \cdot x_b \cdot dx_b - 2V \int_L M'_{42} \cdot dx_b + \overline{OG} \cdot b_{26} \\
&\quad + \boxed{\frac{V^2}{\omega_e^2} \int_L \frac{dN'_{42}}{dx_b} \cdot dx_b} \\
c_{46} &= 0 + \overline{OG} \cdot c_{26}
\end{aligned}
$$

The "Modified Strip Theory" includes the outlined terms. When ignoring the outlined terms the "Ordinary Strip Theory Method" is presented.
A viscous roll damping coefficient b_{44_V}, derived for instance with the empiric method of [Ikeda et al., 1978], has been added here.
After simplification, the expressions for the total hydromechanical coefficients in the coupled roll equation become:

$$
a_{42} = +\int_L M'_{42} \cdot dx_b + \overline{OG} \cdot a_{22}
$$

$$\begin{aligned}
b_{42} &= +\int_L N'_{42} \cdot dx_b + \overline{OG} \cdot b_{22} \\
c_{42} &= 0 \\
a_{44} &= +\int_L M'_{44} \cdot dx_b + \overline{OG} \int_L M'_{42} \cdot dx_b + \overline{OG} \cdot a_{24} \\
b_{44} &= +\int_L N'_{44} \cdot dx_b + \overline{OG} \int_L N'_{42} \cdot dx_b + b_{44_V} + \overline{OG} \cdot b_{24} \\
c_{44} &= +\rho g \nabla \cdot \overline{GM} \\
a_{46} &= +\int_L M'_{42} \cdot x_b \cdot dx_b + \frac{V}{\omega_e^2} \int_L N'_{42} \cdot dx_b + \overline{OG} \cdot a_{26} \\
b_{46} &= +\int_L N'_{42} \cdot x_b \cdot dx_b - V \int_L M'_{42} \cdot dx_b + \overline{OG} \cdot b_{26} \\
c_{46} &= 0
\end{aligned}$$

So no terms are added for the "Modified Strip Theory".

6.5 Hydromechanical Moments for Pitch

The hydromechanical moments for pitch are found by an integration over the ship length of the two-dimensional contributions of surge and heave into the pitch moment:

$$X_{h_5} = \int_L X'_{h_5} \cdot dx_b \qquad \text{with:} \quad X'_{h_5} = -X'_{h_1} \cdot \overline{bG} - X'_{h_3} \cdot x_b$$

According to the "Ordinary Strip Theory" the two-dimensional potential hydromechanical moment on a pitching cross section in still water is defined by surge and heave contributions:

$$\begin{aligned}
X'_{h_5} = & -M'_{11} \cdot \overline{bG} \cdot \ddot{\zeta}^*_{h_1} - \left(N'_{11} - V \cdot \frac{dM'_{11}}{dx_b} \right) \cdot \overline{bG} \cdot \dot{\zeta}^*_{h_1} \\
& -M'_{33} \cdot x_b \cdot \ddot{\zeta}^*_{h_3} - \left(N'_{33} - V \cdot \frac{dM'_{33}}{dx_b} \right) \cdot x_b \cdot \dot{\zeta}^*_{h_3} - 2\rho g \cdot y_w \cdot x_b \cdot \zeta^*_{h_3}
\end{aligned}$$

According to the "Modified Strip Theory" this hydromechanical moment becomes:

$$\begin{aligned}
X'_{h_5} = & -\left(M'_{11} + \frac{V}{\omega_e^2} \cdot \frac{dN'_{11}}{dx_b} \right) \cdot \overline{bG} \cdot \ddot{\zeta}^*_{h_1} - \left(N'_{11} - V \cdot \frac{dM'_{11}}{dx_b} \right) \cdot \overline{bG} \cdot \dot{\zeta}^*_{h_1} \\
& -\left(M'_{33} + \frac{V}{\omega_e^2} \cdot \frac{dN'_{33}}{dx_b} \right) \cdot x_b \cdot \ddot{\zeta}^*_{h_3} - \left(N'_{33} - V \cdot \frac{dM'_{33}}{dx_b} \right) \cdot x_b \cdot \dot{\zeta}^*_{h_3} - 2\rho g \cdot y_w \cdot x_b \cdot \zeta^*_{h_3}
\end{aligned}$$

This results into the following coupled pitch equation:

$$\begin{aligned}
I_{yy} \cdot \ddot{\theta} - X_{h_5} = & \quad a_{51} \cdot \ddot{x} + b_{51} \cdot \dot{x} + c_{51} \cdot x \\
& + a_{53} \cdot \ddot{z} + b_{53} \cdot \dot{z} + c_{53} \cdot z \\
& + (I_{yy} + a_{55}) \cdot \ddot{\theta} + b_{55} \cdot \dot{\theta} + c_{55} \cdot \theta \\
= & \quad X_{w_5}
\end{aligned}$$

with:

$$
\begin{aligned}
a_{51} &= -\int\limits_L M'_{11} \cdot \overline{bG} \cdot dx_b \\
&\quad \boxed{+ \frac{-V}{\omega_e^2} \int\limits_L \frac{dN'_{11}}{dx_b} \cdot \overline{bG} \cdot dx_b} \\
b_{51} &= -\int\limits_L \left(N'_{11} - V \cdot \frac{dM'_{11}}{dx_b} \right) \cdot \overline{bG} \cdot dx_b - b_{11_V} \cdot \overline{BG} \\
c_{51} &= 0 \\
a_{53} &= -\int\limits_L M'_{33} \cdot x_b \cdot dx_b \\
&\quad \boxed{+ \frac{-V}{\omega_e^2} \int\limits_L \frac{dN'_{33}}{dx_b} \cdot x_b \cdot dx_b} \\
b_{53} &= -\int\limits_L \left(N'_{33} - V \cdot \frac{dM'_{33}}{dx_b} \right) \cdot x_b \cdot dx_b \\
c_{53} &= -2\rho g \int\limits_L y_w \cdot x_b \cdot dx_b \\
a_{55} &= +\int\limits_L M'_{11} \cdot \overline{bG}^2 \cdot dx_b \\
&\quad \boxed{+ \frac{V}{\omega_e^2} \int\limits_L \frac{dN'_{11}}{dx_b} \cdot \overline{bG}^2 \cdot dx_b} \\
&\quad + \int\limits_L M'_{33} \cdot x_b^2 \cdot dx_b + \frac{V}{\omega_e^2} \int\limits_L \left(N'_{33} - V \cdot \frac{dM'_{33}}{dx_b} \right) \cdot x_b \cdot dx_b \\
&\quad \boxed{+ \frac{V}{\omega_e^2} \int\limits_L N'_{33} \cdot x_b \cdot dx_b + \frac{V}{\omega_e^2} \int\limits_L \frac{dN'_{33}}{dx_b} \cdot x_b^2 \cdot dx_b} \\
b_{55} &= +\int\limits_L \left(N'_{11} - V \cdot \frac{dM'_{11}}{dx_b} \right) \cdot \overline{bG}^2 \cdot dx_b + b_{11_V} \cdot \overline{BG}^2 \\
&\quad + \int\limits_L \left(N'_{33} - V \cdot \frac{dM'_{33}}{dx_b} \right) \cdot x_b^2 \cdot dx_b - 2V \int\limits_L M'_{33} \cdot x_b \cdot dx_b \\
&\quad \boxed{+ \frac{-V^2}{\omega_e^2} \int\limits_L \frac{dN'_{33}}{dx_b} \cdot x_b \cdot dx_b} \\
c_{55} &= +2\rho g \int\limits_L y_w \cdot x_b^2 \cdot dx_b
\end{aligned}
$$

The "Modified Strip Theory" includes the outlined terms. When ignoring the outlined terms the "Ordinary Strip Theory" is presented.
After simplification, the expressions for the total hydromechanical coefficients in the coupled pitch equation become:

$$
\begin{aligned}
a_{51} &= -\int_L M'_{11} \cdot \overline{bG} \cdot dx_b \\
&\quad + \boxed{\frac{-V}{\omega_e^2} \int_L \frac{dN'_{11}}{dx_b} \cdot \overline{bG} \cdot dx_b} \\
b_{51} &= -\int_L \left(N'_{11} - V \cdot \frac{dM'_{11}}{dx_b} \right) \cdot \overline{bG} \cdot dx_b - b_{11_V} \cdot \overline{BG} \\
c_{51} &= 0 \\
a_{53} &= -\int_L M'_{33} \cdot x_b \cdot dx_b \\
&\quad + \boxed{\frac{V}{\omega_e^2} \int_L N'_{33} \cdot dx_b} \\
b_{53} &= -\int_L N'_{33} \cdot x_b \cdot dx_b - V \int_L M'_{33} \cdot dx_b \\
c_{53} &= -2\rho g \int_L y_w \cdot x_b \cdot dx_b \\
a_{55} &= +\int_L M'_{11} \cdot \overline{bG}^2 \cdot dx_b \\
&\quad + \boxed{\frac{V}{\omega_e^2} \int_L \frac{dN'_{11}}{dx_b} \cdot \overline{bG}^2 \cdot dx_b} \\
&\quad + \int_L M'_{33} \cdot x_b^2 \cdot dx_b + \frac{V}{\omega_e^2} \int_L N'_{33} \cdot x_b \cdot dx_b + \frac{V^2}{\omega_e^2} \int_L M'_{33} \cdot dx_b \\
&\quad + \boxed{\frac{-V}{\omega_e^2} \int_L N'_{33} \cdot x_b \cdot dx_b} \\
b_{55} &= +\int_L \left(N'_{11} - V \cdot \frac{dM'_{11}}{dx_b} \right) \cdot \overline{bG}^2 \cdot dx_b + b_{11_V} \cdot \overline{BG}^2 + \int_L N'_{33} \cdot x_b^2 \cdot dx_b \\
&\quad + \boxed{\frac{V^2}{\omega_e^2} \int_L N'_{33} \cdot dx_b} \\
c_{55} &= +2\rho g \int_L y_w \cdot x_b^2 \cdot dx_b
\end{aligned}
$$

6.6 Hydromechanical Moments for Yaw

The hydromechanical moments for yaw are found by an integration over the ship length of the two-dimensional contributions of sway into the yaw moment:

$$X_{h_6} = \int_L X'_{h_6} \cdot dx_b \qquad \text{with:} \quad X'_{h_6} = +X'_{h_2} \cdot x_b$$

According to the "Ordinary Strip Theory" the two-dimensional potential hydromechanical force on a yawing cross section in still water is defined by sway contributions:

$$\begin{aligned} X'_{h_6} &= \frac{D}{Dt}\left\{M'_{22} \cdot x_b \cdot \dot{\zeta}^*_{h_2}\right\} + N'_{22} \cdot x_b \cdot \dot{\zeta}^*_{h_2} \\ &\quad + \frac{D}{Dt}\left\{M'_{24} \cdot x_b \cdot \dot{\zeta}^*_{h_4}\right\} + N'_{24} \cdot x_b \cdot \dot{\zeta}^*_{h_4} \\ &= M'_{22} \cdot x_b \cdot \ddot{\zeta}^*_{h_2} + \left(N'_{22} - V \cdot \frac{dM'_{22}}{dx_b}\right) \cdot x_b \cdot \dot{\zeta}^*_{h_2} \\ &\quad + M'_{24} \cdot x_b \cdot \ddot{\zeta}^*_{h_4} + \left(N'_{24} - V \cdot \frac{dM'_{24}}{dx_b}\right) \cdot x_b \cdot \dot{\zeta}^*_{h_4} \end{aligned}$$

According to the "Modified Strip Theory" this hydromechanical force becomes:

$$\begin{aligned} X'_{h_6} &= \frac{D}{Dt}\left\{\left(M'_{22} - \frac{i}{\omega_e} \cdot N'_{22}\right) \cdot x_b \cdot \dot{\zeta}^*_{h_2}\right\} \\ &\quad + \frac{D}{Dt}\left\{\left(M'_{24} - \frac{i}{\omega_e} \cdot N'_{24}\right) \cdot x_b \cdot \dot{\zeta}^*_{h_4}\right\} \\ &= \left(M'_{22} + \frac{V}{\omega_e^2} \cdot \frac{dN'_{22}}{dx_b}\right) \cdot x_b \cdot \ddot{\zeta}^*_{h_2} + \left(N'_{22} - V \cdot \frac{dM'_{22}}{dx_b}\right) \cdot x_b \cdot \dot{\zeta}^*_{h_2} \\ &\quad + \left(M'_{24} + \frac{V}{\omega_e^2} \cdot \frac{dN'_{24}}{dx_b}\right) \cdot x_b \cdot \ddot{\zeta}^*_{h_4} + \left(N'_{24} - V \cdot \frac{dM'_{24}}{dx_b}\right) \cdot x_b \cdot \dot{\zeta}^*_{h_4} \end{aligned}$$

This results into the following coupled yaw equation:

$$\begin{aligned} I_{zz} \cdot \ddot{\psi} - I_{zx} \cdot \ddot{\phi} - X_{h_6} &= a_{62} \cdot \ddot{y} + b_{62} \cdot \dot{y} + c_{62} \cdot y \\ &\quad +(-I_{zx} + a_{64}) \cdot \ddot{\phi} + b_{64} \cdot \dot{\phi} + c_{64} \cdot \phi \\ &\quad +(+I_{zz} + a_{66}) \cdot \ddot{\psi} + b_{66} \cdot \dot{\psi} + c_{66} \cdot \psi \\ &= X_{w_6} \end{aligned}$$

with:

$$\begin{aligned} a_{62} &= +\int_L M'_{22} \cdot x_b \cdot dx_b \\ &\quad + \boxed{\frac{V}{\omega_e^2} \int_L \frac{dN'_{22}}{dx_b} \cdot x_b \cdot dx_b} \\ b_{62} &= +\int_L \left(N'_{22} - V \cdot \frac{dM'_{22}}{dx_b}\right) \cdot x_b \cdot dx_b \end{aligned}$$

$$
\begin{aligned}
c_{62} &= 0 \\
a_{64} &= +\int\limits_L M'_{24} \cdot x_b \cdot dx_b \ + \overline{OG} \int\limits_L M'_{22} \cdot x_b \cdot dx_b \\
&\quad \boxed{+\frac{V}{\omega_e^2} \int\limits_L \frac{dN'_{24}}{dx_b} \cdot x_b \cdot dx_b + \frac{V}{\omega_e^2} \cdot \overline{OG} \int\limits_L \frac{dN'_{22}}{dx_b} \cdot x_b \cdot dx_b} \\
b_{64} &= +\int\limits_L \left(N'_{24} - V \cdot \frac{dM'_{24}}{dx_b} \right) \cdot x_b \cdot dx_b + \overline{OG} \int\limits_L \left(N'_{22} - V \cdot \frac{dM'_{22}}{dx_b} \right) \cdot x_b \cdot dx_b \\
c_{64} &= 0 \\
a_{66} &= +\int\limits_L M'_{22} \cdot x_b^2 \cdot dx_b + \frac{V}{\omega_e^2} \int\limits_L \left(N'_{22} - V \cdot \frac{dM'_{22}}{dx_b} \right) \cdot x_b \cdot dx_b \\
&\quad \boxed{+\frac{V}{\omega_e^2} \int\limits_L N'_{22} \cdot x_b \cdot dx_b + \frac{V}{\omega_e^2} \int\limits_L \frac{dN'_{22}}{dx_b} \cdot x_b^2 \cdot dx_b} \\
b_{66} &= +\int\limits_L \left(N'_{22} - V \cdot \frac{dM'_{22}}{dx_b} \right) \cdot x_b^2 \cdot dx_b - 2V \int\limits_L M'_{22} \cdot x_b \cdot dx_b \\
&\quad \boxed{+\frac{V^2}{\omega_e^2} \int\limits_L \frac{dN'_{22}}{dx_b} \cdot x_b \cdot dx_b} \\
c_{66} &= 0
\end{aligned}
$$

The "Modified Strip Theory" includes the outlined terms. When ignoring the outlined terms the "Ordinary Strip Theory" is presented.

After simplification, the expressions for the total hydromechanical coefficients in the coupled yaw equation become:

$$
\begin{aligned}
a_{62} &= +\int\limits_L M'_{22} \cdot x_b \cdot dx_b \\
&\quad \boxed{+\frac{-V}{\omega_e^2} \int\limits_L N'_{22} \cdot dx_b} \\
b_{62} &= +\int\limits_L N'_{22} \cdot x_b \cdot dx_b + V \int\limits_L M'_{22} \cdot dx_b \\
c_{62} &= 0 \\
a_{64} &= +\int\limits_L M'_{24} \cdot x_b \cdot dx_b + \overline{OG} \int\limits_L M'_{22} \cdot x_b \cdot dx_b \\
&\quad \boxed{+\frac{-V}{\omega_e^2} \int\limits_L N'_{24} \cdot dx_b - \frac{V}{\omega_e^2} \cdot \overline{OG} \int\limits_L N'_{22} \cdot dx_b}
\end{aligned}
$$

$$
\begin{aligned}
b_{64} &= +\int_L N'_{24} \cdot x_b \cdot dx_b + V \int_L M'_{24} \cdot dx_b + \overline{OG} \int_L N'_{22} \cdot x_b \cdot dx_b + V \cdot \overline{OG} \int_L M'_{22} \cdot dx_b \\
c_{64} &= 0 \\
a_{66} &= +\int_L M'_{22} \cdot x_b^2 \cdot dx_b + \frac{V}{\omega_e^2} \int_L N'_{22} \cdot x_b \cdot dx_b + \frac{V^2}{\omega_e^2} \int_L M'_{22} \cdot dx_b \\
&\quad + \boxed{\frac{-V}{\omega_e^2} \int_L N'_{22} \cdot x_b \cdot dx_b} \\
b_{66} &= +\int_L N'_{22} \cdot x_b^2 \cdot dx_b \\
&\quad + \boxed{\frac{-V^2}{\omega_e^2} \int_L N'_{22} \cdot dx_b} \\
c_{66} &= 0
\end{aligned}
$$

.

Chapter 7

Exciting Wave Loads

The first order wave potential in a fluid - with any arbitrary water depth h - is given by:

$$\Phi_w = \frac{-g}{\omega} \cdot \frac{\cosh k(h + z_b)}{\cosh(kh)} \cdot \zeta_a \sin(\omega t - kx_b \cos\mu - ky_b \sin\mu)$$

in an axes system with the centre of gravity in the waterline.
The velocities and accelerations in the direction j of the water particles have to be defined. The local relative orbital velocities of the water particles in a certain direction follow from the derivative in that direction of the wave potential. The orbital accelerations of the water particles can be obtained from these velocities by:

$$\ddot{\zeta}'_{wj} = \frac{D}{Dt}\left\{\dot{\zeta}'_{wj}\right\} \qquad \text{with: } \frac{D}{Dt} = \left\{\frac{\partial}{\partial t} - V \cdot \frac{\partial}{\partial x_b}\right\} \qquad \text{for: } j = 1, 2, 3, 4$$

With this, the relative velocities and accelerations in the different directions can be found:

- Surge direction:

$$\begin{aligned}
\dot{\zeta}'_{w1} &= \frac{\partial \Phi_w}{\partial x_b} \\
&= +\omega \cdot \frac{\cosh k(h + z_b)}{\sinh(kh)} \cdot \cos\mu \cdot \zeta_a \cos(\omega t - kx_b \cos\mu - ky_b \sin\mu) \\
\ddot{\zeta}'_{w1} &= \frac{\partial \dot{\zeta}'_{w1}}{\partial t} \\
&= -\omega^2 \cdot \frac{\cosh k(h + z_b)}{\sinh(kh)} \cdot \cos\mu \cdot \zeta_a \sin(\omega t - kx_b \cos\mu - ky_b \sin\mu)
\end{aligned}$$

- Sway direction:

$$\begin{aligned}
\dot{\zeta}'_{w2} &= \frac{\partial \Phi_w}{\partial y_b} \\
&= +\omega \cdot \frac{\cosh k(h + z_b)}{\sinh(kh)} \cdot \sin\mu \cdot \zeta_a \cos(\omega t - kx_b \cos\mu - ky_b \sin\mu)
\end{aligned}$$

[0] J.M.J. Journée, *"Theoretical Manual of SEAWAY, Release 4.19"*, Report 1216a, February 2001, Ship Hydromechanics Laboratory, Delft University of Technology, Mekelweg 2, 2628 CD Delft, The Netherlands. For updates see web site: http://dutw189.wbmt.tudelft.nl/~johan or http://www.shipmotions.nl.

$$
\begin{aligned}
\ddot{\zeta}'_{w2} &= \frac{\partial \dot{\zeta}'_{w2}}{\partial t} \\
\ddot{\zeta}'_{w2} &= -\omega^2 \cdot \frac{\cosh k(h+z_b)}{\sinh(kh)} \cdot \sin\mu \cdot \zeta_a \sin(\omega t - kx_b\cos\mu - ky_b\sin\mu)
\end{aligned}
$$

- Heave direction:

$$
\begin{aligned}
\dot{\zeta}'_{w3} &= \frac{\partial \Phi_w}{\partial z_b} \\
&= -\omega \cdot \frac{\sinh k(h+z_b)}{\sinh(kh)} \cdot \zeta_a \sin(\omega t - kx_b\cos\mu - ky_b\sin\mu) \\
\ddot{\zeta}'_{w3} &= \frac{\partial \dot{\zeta}'_{w3}}{\partial t} \\
&= -\omega^2 \cdot \frac{\sinh k(h+z_b)}{\sinh(kh)} \cdot \zeta_a \cos(\omega t - kx_b\cos\mu - ky_b\sin\mu)
\end{aligned}
$$

- Roll direction:

$$
\begin{aligned}
\dot{\zeta}'_{w4} &= \frac{\partial \dot{\zeta}'_{w2}}{\partial z_b} - \frac{\partial \dot{\zeta}'_{w3}}{\partial y_b} = 0 \\
\ddot{\zeta}'_{w4} &= 0
\end{aligned}
$$

of which the zero solution is obvious, because the potential fluid is free of rotation.

The pressure in the fluid follows from the linearized equation of Bernoulli:

$$
\begin{aligned}
p &= -\rho g z_b + \rho g \cdot \frac{\cosh k(h+z_b)}{\cosh(kh)} \cdot \zeta_a \cos(\omega t - kx_b\cos\mu - ky_b\sin\mu) \\
&= p_0 + \frac{\partial p}{\partial x_b} \cdot dx_b + \frac{\partial p}{\partial y_b} \cdot dy_b + \frac{\partial p}{\partial z_b} \cdot dz_b
\end{aligned}
$$

with the following expressions for the pressure gradients:

$$
\begin{aligned}
\frac{\partial p}{\partial x_b} &= +\rho k g \cos\mu \cdot \frac{\cosh k(h+z_b)}{\cosh(kh)} \cdot \zeta_a \sin(\omega t - kx_b\cos\mu - ky_b\sin\mu) \\
&= +\rho \omega^2 \cos\mu \cdot \frac{\cosh k(h+z_b)}{\sinh(kh)} \cdot \zeta_a \sin(\omega t - kx_b\cos\mu - ky_b\sin\mu) \\
\frac{\partial p}{\partial y_b} &= +\rho k g \sin\mu \cdot \frac{\cosh k(h+z_b)}{\cosh(kh)} \cdot \zeta_a \sin(\omega t - kx_b\cos\mu - ky_b\sin\mu) \\
&= +\rho \omega^2 \sin\mu \cdot \frac{\cosh k(h+z_b)}{\sinh(kh)} \cdot \zeta_a \sin(\omega t - kx_b\cos\mu - ky_b\sin\mu) \\
\frac{\partial p}{\partial z_b} &= -\rho g + \rho k g \cdot \frac{\sinh k(h+z_b)}{\cosh(kh)} \cdot \zeta_a \cos(\omega t - kx_b\cos\mu - ky_b\sin\mu) \\
&= -\rho g + \rho \omega \cdot \frac{\sinh k(h+z_b)}{\sinh(kh)} \cdot \zeta_a \cos(\omega t - kx_b\cos\mu - ky_b\sin\mu)
\end{aligned}
$$

These pressure gradients can be expressed in the orbital accelerations too:

$$\begin{aligned}
\frac{\partial p}{\partial x_b} &= -\rho \cdot \ddot{\zeta}'_{w1} \\
\frac{\partial p}{\partial y_b} &= -\rho \cdot \ddot{\zeta}'_{w2} \\
\frac{\partial p}{\partial z_b} &= -\rho \cdot (g + \ddot{\zeta}'_{w3})
\end{aligned}$$

7.1 Classical Approach

First the classical approach to obtain the wave loads - according to the relative motion principle - is given here.

7.1.1 Exciting Wave Forces for Surge

The exciting wave forces for surge on a ship are found by an integration over the ship length of the two-dimensional values:

$$X_{w_1} = \int\limits_L X'_{w_1} \cdot dx_b$$

According to the "Ordinary Strip Theory" the exciting wave forces for surge on a restrained cross section of a ship in waves are defined by:

$$\begin{aligned}
X'_{w_1} &= \frac{D}{Dt}\left\{M'_{11} \cdot \dot{\zeta}^*_{w_1}\right\} + N'_{11} \cdot \dot{\zeta}^*_{w_1} + X'_{FK_1} \\
&= M'_{11} \cdot \ddot{\zeta}^*_{w_1} + \left(N'_{11} - V \cdot \frac{dM'_{11}}{dx_b}\right) \cdot \dot{\zeta}^*_{w_1} + X'_{FK_1}
\end{aligned}$$

According to the "Modified Strip Theory" these forces become:

$$\begin{aligned}
X'_{w_1} &= \frac{D}{Dt}\left\{\left(M'_{11} - \frac{i}{\omega_e} \cdot N'_{11}\right) \cdot \dot{\zeta}^*_{w_1}\right\} + X'_{FK_1} \\
&= \left(M'_{11} + \frac{V}{\omega_e^2} \cdot \frac{dN'_{11}}{dx_b}\right) \cdot \ddot{\zeta}^*_{w_1} + \left(N'_{11} - V \cdot \frac{dM'_{11}}{dx_b}\right) \cdot \dot{\zeta}^*_{w_1} + X'_{FK_1}
\end{aligned}$$

The Froude-Krilov force in the surge direction - so the longitudinal force due to the pressure in the undisturbed fluid - is given by:

$$\begin{aligned}
X_{FK_1} &= -\int\limits_{-T}^{\zeta} \int\limits_{-y_b}^{+y_b} \frac{\partial p}{\partial x_b} \cdot dy_b \cdot dz_b \\
&= \rho \int\limits_{-T}^{\zeta} \int\limits_{-y_b}^{+y_b} \ddot{\zeta}'_{w_1} \cdot dy_b \cdot dz_b
\end{aligned}$$

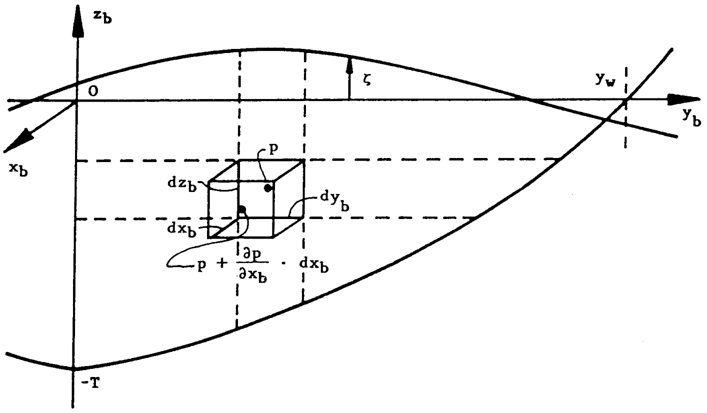

Figure 7.1: Wave Pressure Distribution on a Cross Section for Surge

After neglecting the second order terms, this can be written as:

$$X_{FK_1} = \rho A_{ch} \cdot (-kg\cos\mu) \cdot \zeta_a \sin(\omega_e t - kx_b\cos\mu)$$

with:

$$A_{ch} = 2\int_{-T}^{0} \frac{\sin(-ky_b\sin\mu)}{-ky_b\sin\mu} \cdot \frac{\cosh k(h+z_b)}{\cosh(kh)} \cdot y_b \cdot dz_b$$

When expanding the Froude-Krilov force in deep water with $\lambda \gg 2\pi \cdot y_w$ and $\lambda \gg 2\pi \cdot T$ in series, it is found:

$$X_{FK_1} = \rho \cdot \left(A + k \cdot S_y + \frac{1}{2}k^2 \cdot I_y + ...\right) \cdot (-kg\cos\mu) \cdot \zeta_a \sin(\omega_e t - kx_b\cos\mu)$$

with:

$$A = 2\int_{-T}^{0} y_b \cdot dz_b \qquad S_y = 2\int_{-T}^{0} y_b \cdot z_b \cdot dz_b \qquad I_y = 2\int_{-T}^{0} y_b \cdot z_b^2 \cdot dz_b$$

The acceleration term $kg\cos\mu \cdot \zeta_a$ in here is the amplitude of the longitudinal component of the relative orbital acceleration in deep water at $z_b = 0$.
The dominating first term in this series consists of a mass and this acceleration.
This mass term ρA is used to obtain from the total Froude-Krilov force an equivalent longitudinal component of the orbital acceleration of the water particles:

$$X_{FK_1} = \rho A \cdot \ddot{\zeta}^*_{w_1}$$

This holds that the equivalent longitudinal components of the orbital acceleration and velocity are equal to the values at $z_b = 0$ in a wave with a reduced amplitude $\zeta^*_{a_1}$:

$$\begin{aligned}
\ddot{\zeta}^*_{w_1} &= -kg\cos\mu \cdot \zeta^*_{a_1} \sin(\omega_e t - kx_b\cos\mu) \\
\dot{\zeta}^*_{w_1} &= \frac{+kg\cos\mu}{\omega} \cdot \zeta^*_{a_1} \cos(\omega_e t - kx_b\cos\mu)
\end{aligned}$$

with:

$$\zeta^*_{a_1} = \frac{A_{ch}}{A} \cdot \zeta_a$$

This equivalent acceleration and velocity will be used in the diffraction part of the wave force for surge.

From the previous follows the total wave loads for surge:

$$\begin{aligned} X_{w_1} &= +\int_L M'_{11} \cdot \ddot{\zeta}^*_{w_1} \cdot dx_b \\ &\quad + \boxed{\frac{V}{\omega \cdot \omega_e} \int_L \frac{dN'_{11}}{dx_b} \cdot \ddot{\zeta}^*_{w_1} \cdot dx_b} \\ &\quad + \int_L \left(\boxed{\frac{\omega}{\omega_e}} \cdot N'_{11} - V \cdot \frac{dM'_{11}}{dx_b} \right) \cdot \dot{\zeta}^*_{w_1} \cdot dx_b \\ &\quad + \int_L X'_{FK_1} \cdot dx_b \end{aligned}$$

The "Modified Strip Theory" includes the outlined terms. When ignoring the outlined terms the "Ordinary Strip Theory" is presented.

7.1.2 Exciting Wave Forces for Sway

The exciting wave forces for sway on a ship are found by an integration over the ship length of the two-dimensional values:

$$X_{w_2} = \int_L X'_{w_2} \cdot dx_b$$

According to the "Ordinary Strip Theory" the exciting wave forces for sway on a restrained cross section of a ship in waves are defined by:

$$\begin{aligned} X'_{w_2} &= \frac{D}{Dt} \left\{ M'_{22} \cdot \dot{\zeta}^*_{w_2} \right\} + N'_{22} \cdot \dot{\zeta}^*_{w_2} + X'_{FK_2} \\ &= M'_{22} \cdot \ddot{\zeta}^*_{w_2} + \left(N'_{22} - V \cdot \frac{dM'_{22}}{dx_b} \right) \cdot \dot{\zeta}^*_{w_2} + X'_{FK_2} \end{aligned}$$

According to the "Modified Strip Theory" these forces become:

$$\begin{aligned} X'_{w_2} &= \frac{D}{Dt} \left\{ \left(M'_{22} - \frac{i}{\omega_e} \cdot N'_{22} \right) \cdot \dot{\zeta}^*_{w_2} \right\} + X'_{FK_2} \\ &= \left(M'_{22} + \frac{V}{\omega_e^2} \cdot \frac{dN'_{22}}{dx_b} \right) \cdot \ddot{\zeta}^*_{w_2} + \left(N'_{22} - V \cdot \frac{dM'_{22}}{dx_b} \right) \cdot \dot{\zeta}^*_{w_2} + X'_{FK_2} \end{aligned}$$

The Froude-Krilov force in the sway direction - so the lateral force due to the pressure in the undisturbed fluid - is given by:

$$X_{FK_2} = -\int_{-T}^{\zeta} \int_{-y_b}^{+y_b} \frac{\partial p}{\partial y_b} \cdot dy_b \cdot dz_b$$

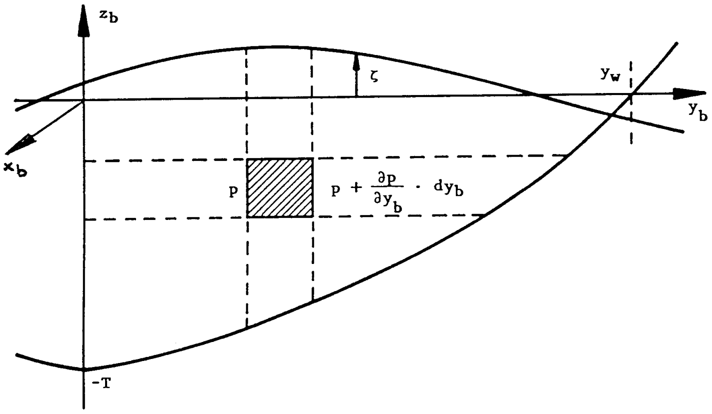

Figure 7.2: Wave Pressure Distribution on a Cross Section for Sway

$$= \rho \int_{-T}^{\zeta} \int_{-y_b}^{+y_b} \ddot{\zeta}_{w_2}' \cdot dy_b \cdot dz_b$$

After neglecting the second order terms, this can be written as:

$$X_{FK_2} = \rho A_{ch} \cdot (-kg \sin \mu) \cdot \zeta_a \sin(\omega_e t - k x_b \cos \mu)$$

with:

$$A_{ch} = 2 \int_{-T}^{0} \frac{\sin(-k y_b \sin \mu)}{-k y_b \sin \mu} \cdot \frac{\cosh k(h + z_b)}{\cosh(kh)} \cdot y_b \cdot dz_b$$

When expanding the Froude-Krilov force in deep water with $\lambda \gg 2\pi \cdot y_w$ and $\lambda \gg 2\pi \cdot T$ in series, it is found:

$$X_{FK_2} = \rho \cdot \left(A + k \cdot S_y + \frac{1}{2} k^2 \cdot I_y + ... \right) \cdot (-kg \sin \mu) \cdot \zeta_a \sin(\omega_e t - k x_b \cos \mu)$$

with:

$$A = 2 \int_{-T}^{0} y_b \cdot dz_b \qquad S_y = 2 \int_{-T}^{0} y_b \cdot z_b \cdot dz_b \qquad I_y = 2 \int_{-T}^{0} y_b \cdot z_b^2 \cdot dz_b$$

The acceleration term $kg \sin \mu \cdot \zeta_a$ in here is the amplitude of the lateral component of the relative orbital acceleration in deep water at $z_b = 0$.
The dominating first term in this series consists of a mass and this acceleration.
This mass term ρA is used to obtain from the total Froude-Krilov force an equivalent lateral component of the orbital acceleration of the water particles:

$$X_{FK_2} = \rho A \cdot \ddot{\zeta}_{w_2}^*$$

This holds that the equivalent lateral components of the orbital acceleration and velocity are equal to the values at $z_b = 0$ in a wave with a reduced amplitude $\zeta_{a_2}^*$:

$$\begin{aligned} \ddot{\zeta}_{w_2}^* &= -kg \sin \mu \cdot \zeta_{a_2}^* \sin(\omega_e t - k x_b \cos \mu) \\ \dot{\zeta}_{w_2}^* &= \frac{+kg \sin \mu}{\omega} \cdot \zeta_{a_2}^* \cos(\omega_e t - k x_b \cos \mu) \end{aligned}$$

with:

$$\zeta^*_{a_2} = \frac{A_{ch}}{A} \cdot \zeta_a$$

This equivalent acceleration and velocity will be used in the diffraction part of the wave force for sway.
From the previous follows the total wave loads for sway:

$$\begin{aligned}
X_{w_2} &= +\int_L M'_{22} \cdot \ddot{\zeta}^*_{w_2} \cdot dx_b \\
&\quad \boxed{+\left|\frac{V}{\omega \cdot \omega_e} \int_L \frac{dN'_{22}}{dx_b} \cdot \ddot{\zeta}^*_{w_2} \cdot dx_b\right|} \\
&\quad +\int_L \left(\boxed{\frac{\omega}{\omega_e}} \cdot N'_{22} - V \cdot \frac{dM'_{22}}{dx_b}\right) \cdot \dot{\zeta}^*_{w_2} \cdot dx_b \\
&\quad +\int_L X'_{FK_2} \cdot dx_b
\end{aligned}$$

The "Modified Strip Theory" includes the outlined terms. When ignoring the outlined terms the "Ordinary Strip Theory" is presented.

7.1.3 Exciting Wave Forces for Heave

The exciting wave forces for heave on a ship are found by an integration over the ship length of the two-dimensional values:

$$X_{w_3} = \int_L X'_{w_3} \cdot dx_b$$

According to the "Ordinary Strip Theory" the exciting wave forces for heave on a restrained cross section of a ship in waves are defined by:

$$\begin{aligned}
X'_{w_3} &= \frac{D}{Dt}\left\{M'_{33} \cdot \dot{\zeta}^*_{w_3}\right\} + N'_{33} \cdot \dot{\zeta}^*_{w_3} + X'_{FK_3} \\
&= M'_{33} \cdot \ddot{\zeta}^*_{w_3} + \left(N'_{33} - V \cdot \frac{dM'_{33}}{dx_b}\right) \cdot \dot{\zeta}^*_{w_3} + X'_{FK_3}
\end{aligned}$$

According to the "Modified Strip Theory" these forces become:

$$\begin{aligned}
X'_{w_3} &= \frac{D}{Dt}\left\{\left(M'_{33} - \frac{i}{\omega_e} \cdot N'_{33}\right) \cdot \dot{\zeta}^*_{w_3}\right\} + X'_{FK_3} \\
&= \left(M'_{33} + \frac{V}{\omega_e^2} \cdot \frac{dN'_{33}}{dx_b}\right) \cdot \ddot{\zeta}^*_{w_3} + \left(N'_{33} - V \cdot \frac{dM'_{33}}{dx_b}\right) \cdot \dot{\zeta}^*_{w_3} + X'_{FK_3}
\end{aligned}$$

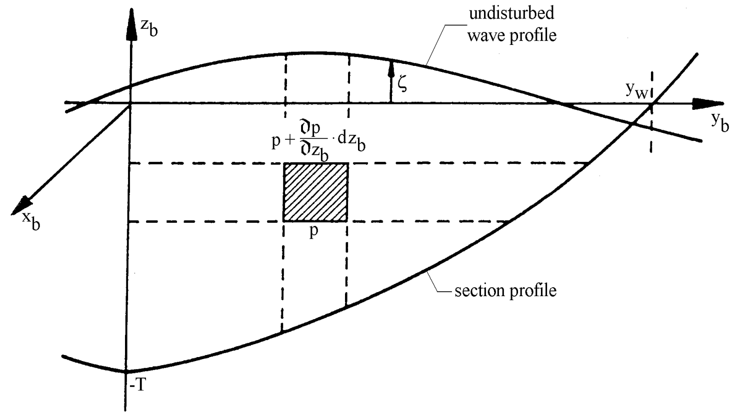

Figure 7.3: Wave Pressure Distriubution on a Cross Section for Heave

The Froude-Krilov force in the heave direction - so the vertical force due to the pressure in the undisturbed fluid - is given by:

$$\begin{aligned} X'_{FK_3} &= -\int_{-T}^{\zeta}\int_{-y_b}^{+y_b} \frac{\partial p}{\partial z_b} \cdot dy_b \cdot dz_b \\ &= \rho \int_{-T}^{\zeta}\int_{-y_b}^{+y_b} (g + \ddot{\zeta}'_{w3}) \cdot dy_b \cdot dz_b \end{aligned}$$

After neglecting the second order terms, this force can be written as:

$$X'_{FK_3} = 2\rho g y_w \cdot C_3 \cdot \zeta_a \cos(\omega_e t - k x_b \cos\mu)$$

with:

$$C_3 = \frac{\sin(-k y_w \sin\mu)}{-k y_w \sin\mu} - \frac{k}{y_w} \int_{-T}^{0} \frac{\sin(-k y_b \sin\mu)}{-k y_b \sin\mu} \cdot \frac{\sinh k(h + z_b)}{\cosh(kh)} \cdot y_b \cdot dz_b$$

When expanding the Froude-Krilov force in **deep water** with $\lambda \gg 2\pi \cdot T$ and in long waves with $\lambda \gg 2\pi \cdot y_w$ in series, it is found:

$$\begin{aligned} C_3 &= \frac{\sin(-k y_w \sin\mu)}{-k y_w \sin\mu} - \frac{k}{y_w} \int_{-T}^{0} \frac{\sin(-k y_b \sin\mu)}{-k y_b \sin\mu} \cdot \frac{\sinh k(h + z_b)}{\cosh(kh)} \cdot y_b \cdot dz_b \\ &\approx 1 - \frac{k}{y_w} \int_{-T}^{0} e^{k z_b} \cdot y_b \cdot dz_b \\ &= 1 - \frac{k}{y_w} \int_{-T}^{0} \left(1 + k \cdot z_b + \frac{k^2}{2} \cdot z_b^2 + ...\right) \cdot y_b \cdot dz_b \end{aligned}$$

$$
\begin{aligned}
&= 1 - k \cdot \left(\frac{A}{2y_w} + k \cdot \frac{S_y}{2y_w} + \frac{k^2}{2} \cdot \frac{I_y}{2y_w} + ... \right) \\
&= 1 - k \cdot \left(\frac{A}{2y_w} + ... \right) \\
&= 1 - kT_3^* \\
&\approx \exp\left\{ -kT_3^* \right\}
\end{aligned}
$$

with:

$$
A = 2 \int_{-T}^{0} y_b \cdot dz_b \qquad S_y = 2 \int_{-T}^{0} y_b \cdot z_b \cdot dz_b \qquad I_y = 2 \int_{-T}^{0} y_b \cdot z_b^2 \cdot dz_b
$$

T_3^* can be considered as the draft at which the pressure in the vertical direction is equal to the average vertical pressure on the cross section in the fluid and can be obtained by.

$$
T_3^* = \frac{-\ln C_3}{k}
$$

This holds that the equivalent vertical components of the orbital acceleration and velocity are equal to the values at $z_b = -T_3^*$:

$$
\begin{aligned}
\ddot{\zeta}_{w_3}^* &= -kg \cdot e^{-kT_3^*} \cdot \zeta_{a_3} \cdot \cos(\omega_e t - kx_b \cos\mu) \\
&= -\omega^2 \cdot \left(e^{-kT_3^*} \cdot \zeta_{a_3} \right) \cdot \cos(\omega_e t - kx_b \cos\mu)
\end{aligned}
$$

$$
\begin{aligned}
\dot{\zeta}_{w_3}^* &= \frac{-kg}{\omega} \cdot e^{-kT_3^*} \cdot \zeta_{a_3} \cdot \sin(\omega_e t - kx_b \cos\mu) \\
&= -\omega \cdot \left(e^{-kT_3^*} \cdot \zeta_{a_3} \right) \cdot \sin(\omega_e t - kx_b \cos\mu)
\end{aligned}
$$

$$
\begin{aligned}
\dot{\zeta}_{w_3}^* &= \frac{kg}{\omega^2} \cdot e^{-kT_3^*} \cdot \zeta_{a_3} \cdot \cos(\omega_e t - kx_b \cos\mu) \\
&= \left(e^{-kT_3^*} \cdot \zeta_{a_3} \right) \cdot \cos(\omega_e t - kx_b \cos\mu)
\end{aligned}
$$

When expanding the Froude-Krilov force in **shallow water** with $kh \to 0$ and in long waves with $\lambda \gg 2\pi \cdot y_w$ in series, it is found:

$$
\begin{aligned}
C_3 &= \frac{\sin(-ky_w \sin\mu)}{-ky_w \sin\mu} - \frac{k}{y_w} \int_{-T}^{0} \frac{\sin(-ky_b \sin\mu)}{-ky_b \sin\mu} \cdot \frac{\sinh k(h + z_b)}{\cosh(kh)} \cdot y_b \cdot dz_b \\
&\approx 1 - \frac{k}{y_w} \int_{-T}^{0} \frac{\sinh k(h + z_b)}{\cosh(kh)} \cdot y_b \cdot dz_b \\
&= 1 - \frac{k}{y_w} \int_{-T}^{0} \frac{k(h + z_b) + k^3 \cdot (......) + ...}{\cosh(kh)} \cdot y_b \cdot dz_b \\
&= 1 - \frac{kh}{\cosh(kh)} \cdot \frac{k}{y_w} \cdot \left(\int_{-T}^{0} y_b \cdot dz_b + \frac{1}{h} \int_{-T}^{0} y_b \cdot z_b \cdot dz_b + ... \right)
\end{aligned}
$$

$$
\begin{aligned}
&= 1 - \frac{kh}{\cosh(kh)} \cdot k \cdot \left(\frac{A}{2y_w} + \frac{S_y}{h \cdot 2y_w} + ... \right) \\
&= 1 - \frac{kh}{\cosh(kh)} \cdot k \cdot \left(1 + \frac{z_B}{h} + ... \right) \cdot \frac{A}{2y_w} \\
&= 1 - \frac{kh}{\cosh(kh)} \cdot \left(1 + \frac{z_B}{h} \right) \cdot k \cdot \left(\frac{A}{2y_w} + ... \right) \\
&= 1 - \frac{kh}{\cosh(kh)} \left(1 + \frac{z_B}{h} \right) \cdot kT_3^* \\
&\approx \exp \left\{ - \frac{kh}{\cosh(kh)} \left(1 + \frac{z_B}{h} \right) \cdot kT_3^* \right\}
\end{aligned}
$$

with:

$$
A = 2 \int_{-T}^{0} y_b \cdot dz_b \qquad S_y = 2 \int_{-T}^{0} y_b \cdot z_b \cdot dz_b = z_B \cdot A
$$

So in shallow water, T_3^* can be obtained by.

$$
T_3^* = \frac{- \ln C_3}{\frac{kh}{\cosh(kh)} \left(1 + \frac{z_B}{h} \right) \cdot k}
$$

This holds that the equivalent vertical components of the orbital acceleration and velocity are equal to the values at $z_b = -T_3^*$:

$$
\begin{aligned}
\ddot{\zeta}_{w_3}^* &= -kg \cdot \frac{\sinh k(h - T_3^*)}{\cosh(kh)} \cdot \zeta_{a_3} \cdot \cos(\omega_e t - kx_b \cos \mu) \\
&= -\omega^2 \cdot \left(\frac{\sinh k(h - T_3^*)}{\sinh(kh)} \cdot \zeta_{a_3} \right) \cdot \cos(\omega_e t - kx_b \cos \mu) \\
\dot{\zeta}_{w_3}^* &= \frac{-kg}{\omega} \cdot \frac{\sinh k(h - T_3^*)}{\cosh(kh)} \cdot \zeta_{a_3} \cdot \sin(\omega_e t - kx_b \cos \mu) \\
&= -\omega \cdot \left(\frac{\sinh k(h - T_3^*)}{\sinh(kh)} \cdot \zeta_{a_3} \right) \cdot \sin(\omega_e t - kx_b \cos \mu) \\
\zeta_{w_3}^* &= \frac{kg}{\omega^2} \cdot \frac{\sinh k(h - T_3^*)}{\cosh(kh)} \cdot \zeta_{a_3} \cdot \cos(\omega_e t - kx_b \cos \mu) \\
&= \left(\frac{\sinh k(h - T_3^*)}{\sinh(kh)} \cdot \zeta_{a_3} \right) \cdot \cos(\omega_e t - kx_b \cos \mu)
\end{aligned}
$$

It may be noted that this shallow water definition for T_3^* is valid in deep water too, because:

$$
\frac{kh}{\cosh(kh)} \rightarrow 1 \qquad \text{and} \qquad \frac{z_B}{h} \rightarrow 0 \qquad \text{for:} \quad h \rightarrow \infty
$$

These equivalent accelerations and velocities will be used to determine the diffraction part of the wave forces for heave.

From the previous follows the total wave loads for heave:

$$\begin{aligned} X'_{w_3} &= +\int_L M'_{33} \cdot \ddot{\zeta}^*_{w_3} \cdot dx_b \\ &\quad \boxed{+\frac{V}{\omega \cdot \omega_e} \int_L \frac{dN'_{33}}{dx_b} \cdot \ddot{\zeta}^*_{w_3} \cdot dx_b} \\ &\quad +\int_L \left(\boxed{\frac{\omega}{\omega_e}} \cdot N'_{33} - V \cdot \frac{dM'_{33}}{dx_b} \right) \cdot \dot{\zeta}^*_{w_3} \cdot dx_b \\ &\quad +\int_L X'_{FK_3} \cdot dx_b \end{aligned}$$

The "Modified Strip Theory" includes the outlined terms. When ignoring the outlined terms the "Ordinary Strip Theory" is presented.

7.1.4 Exciting Wave Moments for Roll

The exciting wave moments for roll on a ship are found by an integration over the ship length of the two-dimensional values:

$$X_{w_4} = \int_L X'_{w_4} \cdot dx_b$$

According to the "Ordinary Strip Theory" the exciting wave moments for roll on a restrained cross section of a ship in waves are defined by:

$$\begin{aligned} X'_{w_4} &= +X'_{FK_4} + \frac{D}{Dt}\left\{ M'_{42} \cdot \dot{\zeta}^*_{w_2} \right\} + N'_{42} \cdot \dot{\zeta}^*_{w_2} + \overline{OG} \cdot X'_{w_2} \\ &= X'_{FK_4} + M'_{42} \cdot \ddot{\zeta}^*_{w_2} + \left(N'_{42} - V \cdot \frac{dM'_{42}}{dx_b} \right) \cdot \dot{\zeta}^*_{w_2} + \overline{OG} \cdot X'_{w_2} \end{aligned}$$

According to the "Modified Strip Theory" these moments become:

$$\begin{aligned} X'_{w_4} &= +X'_{FK_4} + \frac{D}{Dt}\left\{ \left(M'_{42} - \frac{i}{\omega_e} \cdot N'_{42} \right) \cdot \dot{\zeta}^*_{w_2} \right\} + \overline{OG} \cdot X'_{w_2} \\ &= X'_{FK_4} + \left(M'_{42} + \frac{V}{\omega_e^2} \cdot \frac{dN'_{42}}{dx_b} \right) \cdot \ddot{\zeta}^*_{w_2} + \left(N'_{42} - V \cdot \frac{dM'_{42}}{dx_b} \right) \cdot \dot{\zeta}^*_{w_2} + \overline{OG} \cdot X'_{w_2} \end{aligned}$$

The Froude-Krilov moment in the roll direction - so the roll moment due to the pressure in the undisturbed fluid - is given by:

$$\begin{aligned} X_{FK_4} &= -\int_{-T}^{\zeta} \int_{-y_b}^{+y_b} \left(-\frac{\partial p}{\partial y_b} \cdot z_b + \frac{\partial p}{\partial z_b} \cdot y_b \right) \cdot dy_b \cdot dz_b \\ &= \rho \int_{-T}^{\zeta} \int_{-y_b}^{+y_b} \left(-\ddot{\zeta}'_{w_2} \cdot z_b + (g + \ddot{\zeta}'_{w_3}) \cdot y_b \right) \cdot dy_b \cdot dz_b \end{aligned}$$

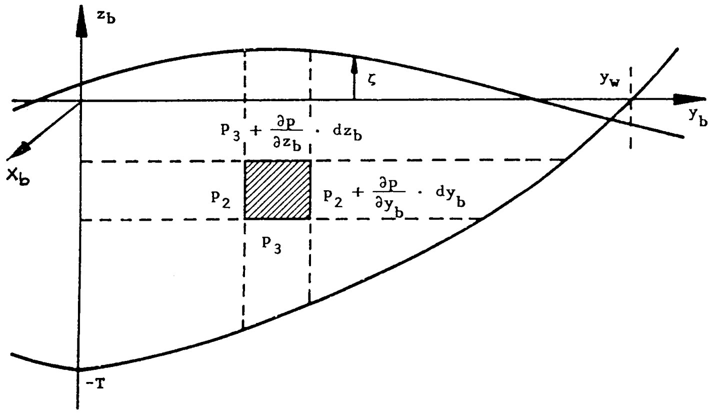

Figure 7.4: Wave Pressure Distriubution on a Cross Section for Roll

After neglecting the second order terms, this can be written as:

$$X_{FK_4} = \rho \cdot \left(-\frac{C_{Cy}}{k} - \frac{C_{Sy}}{k} + C_{Iz} \right) \cdot (-k^2 g \sin\mu) \cdot \zeta_a \sin(\omega_e t - k x_b \cos\mu)$$

with:

$$\begin{aligned}
C_{Cy} &= 2 \cdot \frac{\frac{\sin(-k y_w \sin\mu)}{-k y_w \sin\mu} - \cos(-k y_w \sin\mu)}{(-k y_w \sin\mu)^2} \cdot y_w^3 \\
C_{Sy} &= 2 \int\limits_{-T}^{0} \frac{\sin(-k y_b \sin\mu)}{-k y_b \sin\mu} \cdot \frac{\cosh k(h+z_b)}{\cosh(kh)} \cdot y_b \cdot z_b \cdot dz_b \\
C_{Iz} &= 2 \int\limits_{-T}^{0} \frac{\frac{\sin(-k y_b \sin\mu)}{-k y_b \sin\mu} - \cos(-k y_b \sin\mu)}{(-k y_b \sin\mu)^2} \cdot \frac{\cosh k(h+z_b)}{\cosh(kh)} \cdot y_b^3 \cdot dz_b
\end{aligned}$$

For deep water, the cosine-hyperbolic expressions in here reduce to exponentials.

From the previous follows the total wave loads for roll:

$$\begin{aligned}
X_{w_4} = & \int\limits_L X'_{FK_4} \cdot dx_b \\
& + \int\limits_L M'_{42} \cdot \ddot{\zeta}^*_{w_2} \cdot dx_b \\
& + \boxed{\frac{V}{\omega \cdot \omega_e} \int\limits_L \frac{dN'_{42}}{dx_b} \cdot \ddot{\zeta}^*_{w_2} \cdot dx_b} \\
& + \int\limits_L \left(\boxed{\frac{\omega}{\omega_e}} \cdot N'_{42} - V \cdot \frac{dM'_{42}}{dx_b} \right) \cdot \dot{\zeta}^*_{w_2} \cdot dx_b \\
& + \overline{OG} \cdot X_{w_2}
\end{aligned}$$

The "Modified Strip Theory" includes the outlined terms. When ignoring the outlined terms the "Ordinary Strip Theory" is presented.

7.1.5 Exciting Wave Moments for Pitch

The exciting wave moments for pitch are found by an integration over the ship length of the two-dimensional contributions of surge and heave into the pitch moment:

$$X_{w_5} = \int_L X'_{w_5} \cdot dx_b$$

with:

$$X'_{w_5} = -X'_{w_1} \cdot \overline{bG} - X'_{w_3} \cdot x_b$$

In here, $\overline{bG}$ is the vertical distance of the centre of gravity of the ship G above the centroid b of the local submerged sectional area.
From this follows the total wave loads for pitch:

$$\begin{aligned} X_{w_5} \quad = \quad & -\int_L M'_{11} \cdot \overline{bG} \cdot \ddot{\zeta}^*_{w_1} \cdot dx_b \\ & + \boxed{\frac{-V}{\omega \cdot \omega_e} \int_L \frac{dN'_{11}}{dx_b} \cdot \overline{bG} \cdot \ddot{\zeta}^*_{w_1} \cdot dx_b} \\ & - \int_L \left(\boxed{\frac{\omega}{\omega_e}} \cdot N'_{11} - V \cdot \frac{dM'_{11}}{dx_b} \right) \cdot \overline{bG} \cdot \dot{\zeta}^*_{w_1} \cdot dx_b \\ & - \int_L X'_{FK_1} \cdot \overline{bG} \cdot dx_b \\ & - \int_L M'_{33} \cdot x_b \cdot \ddot{\zeta}^*_{w_3} \cdot dx_b \\ & + \boxed{\frac{-V}{\omega \cdot \omega_e} \int_L \frac{dN'_{33}}{dx_b} \cdot x_b \cdot \ddot{\zeta}^*_{w_3} \cdot dx_b} \\ & - \int_L \left(\boxed{\frac{\omega}{\omega_e}} \cdot N'_{33} - V \cdot \frac{dM'_{33}}{dx_b} \right) \cdot x_b \cdot \dot{\zeta}^*_{w_3} \cdot dx_b \\ & - \int_L X'_{FK_3} \cdot x_b \cdot dx_b \end{aligned}$$

The "Modified Strip Theory" includes the outlined terms. When ignoring the outlined terms the "Ordinary Strip Theory" is presented.

7.1.6 Exciting Wave Moments for Yaw

The exciting wave moments for yaw are found by an integration over the ship length of the two-dimensional contributions of sway into the yaw moment:

$$X_{w_6} = \int_L X'_{w_6} \cdot dx_b$$

with:

$$X'_{w_6} = +X'_{w_2} \cdot x_b$$

From this follows the total wave loads for yaw:

$$\begin{aligned}
X_{w_6} \quad = \quad & + \int\limits_L M'_{22} \cdot x_b \cdot \ddot{\zeta}^*_{w_2} \cdot dx_b \\
& + \boxed{\frac{V}{\omega \cdot \omega_e} \int\limits_L \frac{dN'_{22}}{dx_b} \cdot x_b \cdot \ddot{\zeta}^*_{w_2} \cdot dx_b} \\
& + \int\limits_L \boxed{\frac{\omega}{\omega_e}} \cdot N'_{22} - V \cdot \frac{dM'_{22}}{dx_b} \cdot x_b \cdot \dot{\zeta}^*_{w_2} \cdot dx_b \\
& + \int\limits_L X'_{FK_2} \cdot x_b \cdot dx_b
\end{aligned}$$

The "Modified Strip Theory" includes the outlined terms. When ignoring the outlined terms the "Ordinary Strip Theory" is presented.

7.2 Equivalent Motions of Water Particles

In the classic relative motion theory, the average (or equivalent) motions of the water particles around the cross section are calculated from the pressure distribution in the undisturbed waves on this cross section. An alternative approach - based on diffraction of waves - to determine the equivalent accelerations and velocities of the water particles around the cross section, as given by [Journée and van 't Veer, 1995], is described now.

7.2.1 Hydromechanical Loads

Suppose an infinite long cylinder in the still water surface of a fluid. The cylinder is forced to carry out a simple harmonic oscillation about its initial position with a frequency of oscillation ω and a small amplitude of displacement x_{ja}:

$$x_j = x_{ja} \cos \omega t \qquad \text{for: } j = 2, 3, 4$$

The 2-D hydrodynamic loads $X_{hi}^{'}$ in the sway, heave and roll directions i, exercised by the fluid on a cross section of the cylinder, can be obtained from the 2-D velocity potentials and the linearized equations of Bernoulli.. The velocity potentials have been obtained by using the work of [Ursell, 1949] and N-parameter conformal mapping. These hydrodynamic loads are:

$$X_{hi}^{'} = 2\rho g y_{wl} \frac{g\zeta_{ja}}{\pi} \left[A_{ij} \cos\left(\omega t + \varepsilon_{\phi x j}\right) + B_{ij} \sin\left(\omega t + \varepsilon_{\phi x j}\right)\right]$$

in where j is the mode of oscillation and i is the direction of the load. The phase lag $\varepsilon_{\phi xj}$ is defined as the phase lag between the velocity potential of the fluid Φ and the forced motion x_j. The radiated damping waves have an amplitude ζ_{ja} and y_{wl} is half the breadth of the cross section at the waterline. The potential coefficients A_{ij} and B_{ij} and the phase lags $\varepsilon_{\phi xj}$, expressed in terms of conformal mapping coefficients, are given in a previous chapter. These loads $X_{hi}^{'}$ can be expressed in terms of in-phase and out-phase components with the harmonic oscillations:

$$X_{hi}^{'} = \frac{\rho a_{ij}}{\omega^2 x_{ja}} \left(\frac{g\zeta_{ja}}{\pi}\right)^2 \left[\left(A_{ij}Q_{0j} + B_{ij}P_{0j}\right) \cos \omega t + \left(A_{ij}P_{0j} - B_{ij}Q_{0j}\right) \sin \omega t\right]$$

with $a_{22} = 2$, $a_{24} = 4/y_{wl}$, $a_{33} = 2$, $a_{44} = 8$, $a_{42} = 4y_{wl}$ and for the terms P_{0j} and Q_{0j}:

$$\begin{aligned} P_{0j} &= -\frac{x_{ja}}{\zeta_{ja}} \pi \frac{\omega^2}{g} y_{wl} \sin \varepsilon_{\phi xj} \\ Q_{0j} &= +\frac{x_{ja}}{\zeta_{ja}} \pi \frac{\omega^2}{g} y_{wl} \cos \varepsilon_{\phi xj} \end{aligned}$$

The phase lag $\varepsilon_{\phi xj}$ between he velocity potentials and the forced motion is incorporated in the coefficients P_{0j} and Q_{0j} and can be obtained by using:

$$\varepsilon_{\phi xj} = \arctan\left(\frac{-P_{0j}}{+Q_{0j}}\right)$$

This equation will be used further on for obtaining wave load phases.

Generally, these hydrodynamic loads are expressed in terms of potential mass and damping coefficients:

$$\begin{aligned} X'_{hi} &= -M_{ij}\ddot{x}_j - N_{ij}\dot{x}_j \\ &= M_{ij}\omega^2 x_{ja}\cos\omega t + N_{ij}\omega x_{ja}\sin\omega t \end{aligned}$$

with:

$$\begin{aligned} M_{ij} &= \rho b_{ij}\frac{A_{ij}Q_{0j} + B_{ij}P_{0j}}{P_{0j}^2 + Q_{0j}^2} \\ N_{ij} &= \rho b_{ij}\frac{A_{ij}P_{0j} - B_{ij}Q_{0j}}{P_{0j}^2 + Q_{0j}^2}\cdot\omega \end{aligned}$$

with $b_{22} = 2y_{wl}^2$, $b_{24} = y_{wl}^3$, $b_{33} = 2y_{wl}^2$, $b_{44} = 2y_{wl}^4$ and $b_{42} = 2y_{wl}^3$. Note that the phase lag information $\varepsilon_{\phi xj}$ is vanished here.
[Tasai, 1965] has used the following potential damping coupling coefficients in his formulation of the hydrodynamic loads for roll:

$$N'_{42} = \frac{N'_{44}}{l'_w} \qquad \text{and} \qquad N'_{24} = N'_{22}\cdot l'_w$$

in which l'_w is the lever of the rolling moment.
Because $N'_{42} = N'_{24}$, one may write for the roll damping coefficient:

$$N'_{44} = \frac{\left(N'_{24}\right)^2}{N'_{22}} = \frac{\left(N'_{42}\right)^2}{N'_{22}}$$

This relation - which has been confirmed by numerical calculations with SEAWAY - will be used further on for obtaining the wave loads for roll from those for sway.

7.2.2 Energy Considerations

The wave velocity, c_{wave}, and the group velocity, c_{group}, of regular waves are defined by:

$$c_{wave} = \frac{\omega}{k} \qquad \text{and} \qquad c_{group} = \frac{c_{wave}}{2}\cdot\frac{2kh}{\sinh\left[2kh\right]}$$

Consider a cross section which is harmonic oscillating with a frequency $\omega = 2\pi/T$ and an amplitude x_{ja} in the direction j in previously still water by an oscillatory force X'_{hj} in the same direction j:

$$\begin{aligned} x_j &= x_{ja}\cos\omega t \qquad\qquad \text{for: } j = 2,3,4 \\ X'_{hj} &= X'_{hja}\cos\left(\omega t + \varepsilon'_{hj}\right) \\ &= X'_{hja}\cos\varepsilon'_{hj}\cos\omega t - X'_{hja}\sin\varepsilon'_{hj}\sin\omega t \end{aligned}$$

The energy required for this oscillation should be equal to the energy radiated by the damping waves:

$$\frac{1}{T}\int_0^T X'_{hj} \cdot \dot{x}_j \, dt = \frac{1}{T}\int_0^T N'_{jj}\, \dot{x}_j \cdot \dot{x}_j \, dt = 2 \cdot \frac{1}{2}\rho g \zeta_a^2 \cdot c_{group}$$

or:

$$\frac{1}{2} X'_{hja} \omega x_{ja} \sin \varepsilon'_{hj} = \frac{1}{2} N'_{jj} \omega^2 x_{ja}^2 = \rho g \zeta_a^2 \cdot c_{group}$$

From the first part of this equation follows

$$\frac{X'_{hja} \sin \varepsilon'_{hj}}{\zeta_a} = N'_{jj} \omega \frac{x_{ja}}{\zeta_a}$$

From the second part of this equation follows the amplitude ratio of the oscillatory motions and the radiated waves:

$$\frac{x_{ja}}{\zeta_a} = \frac{1}{\omega}\sqrt{\frac{2\rho g \cdot c_{group}}{N'_{jj}}}$$

Combining these two last equations provides for the out-phase part - so the damping part - of the oscillatory force:

$$\frac{X'_{hja} \sin \varepsilon'_{hj}}{\zeta_a} = \sqrt{2\rho g \cdot c_{group} \cdot N'_{jj}} \qquad \text{for: } j = 2, 3, 4$$

In here, $X'_{hja} \sin \varepsilon'_{hj}$ is the in-phase with the velocity part of the exciting force or moment.

7.2.3 Wave Loads

Consider now the opposite case: the cross section is restrained and is subject to regular incoming beam waves with an amplitude ζ_a. Let x_{wj} represents the equivalent (or average) oscillation of the water particles with respect to the restrained cross section. The resulting wave force is caused by these motions, which will be in phase with its velocity (damping waves). Then the energy consumed by this oscillation is equal to the energy supplied by the incoming waves.

$$\begin{aligned} x_{wj} &= x_{wja} \cos\left(\omega t + \varepsilon'_{wj}\right) \qquad \text{for: } j = 2, 3, 4 \\ X'_{wj} &= X'_{wja} \cos\left(\omega t + \varepsilon'_{wj}\right) \end{aligned}$$

in which ε'_{wj} is the phase lag with respect to the wave surface elevation at the center of the cross section.

This leads for the amplitude of the exciting wave force to:

$$\frac{X'_{wja}}{\zeta_a} = \sqrt{2\rho g \cdot c_{group} \cdot N'_{jj}} \qquad \text{for: } j = 2, 3, 4$$

which is in principle the same equation as the previous one for the out-phase part of the oscillatory force in still water.

However, for the phase lag of the wave force, ε'_{wj}, an approximation has to be found.

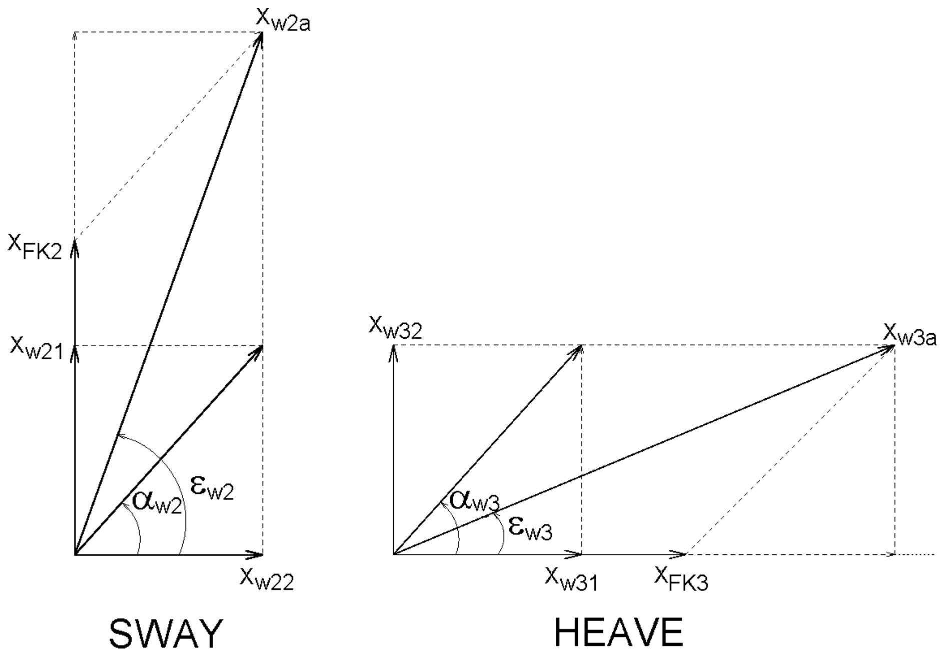

Figure 7.5: Vector Diagrams of Wave Components for Sway and Heave

Heave Mode

The vertical wave force on a restrained cross section in waves is:

$$\begin{aligned} X_{w3}^{'} &= X_{w3a}^{'} \cos\left(\omega t + \varepsilon_{w3}^{'}\right) \\ &= X_{w3a}^{'} \cos\varepsilon_{w3}^{'} \cos\omega t - X_{w3a}^{'} \sin\varepsilon_{w3}^{'} \sin\omega t \end{aligned}$$

of which the amplitude is equal to:

$$X_{w3a}^{'} = \zeta_a \sqrt{2\rho g \cdot c_{group} \cdot N_{33}^{'}}$$

For the phase lag of this wave force, $\varepsilon_{w3}^{'}$, an approximation has to be found.
The phase lag of a radiated wave, $\varepsilon_{w_R 3}^{'}$, at the intersection of the ship's hull with the waterline, $y_b = y_{wl}$, is

$$\varepsilon_{w_R 3}^{'} = k y_{wl}$$

The phase lag of the wave force, $\varepsilon_{w3}^{'}$, has been approximated by this phase:

$$\varepsilon_{w3}^{'} = \varepsilon_{w_R 3}^{'} = k y_{wl}$$

Then, the in-phase and out-phase parts of the wave loads are:

$$\begin{aligned} X_{FK_3}^{'} + X_{w31}^{'} &= +X_{w3a}^{'} \cos\varepsilon_{w3}^{'} = + \left(\zeta_a \sqrt{2\rho g \cdot c_{group} \cdot N_{33}^{'}}\right) \cos\varepsilon_{w3}^{'} \\ X_{w32}^{'} &= -X_{w3a}^{'} \sin\varepsilon_{w3}^{'} \ = - \left(\zeta_a \sqrt{2\rho g \cdot c_{group} \cdot N_{33}^{'}}\right) \sin\varepsilon_{w3}^{'} \end{aligned}$$

from which the diffraction terms, X'_{w31} and X'_{w32} follow.
These diffraction terms can also be written as:

$$\begin{aligned} X'_{w31} &= M'_{33} \cdot \bar{a}'_3 \\ X'_{w32} &= M'_{33} \cdot \bar{v}'_3 \end{aligned}$$

in which $\bar{a}'_3$ and $\bar{v}'_3$ are the equivalent amplitudes of the acceleration and the velocity of the water particles around the cross section.
Herewith, the equivalent acceleration and velocity amplitudes of the water particles are:

$$\begin{aligned} \bar{a}'_3 &= \frac{X'_{w31}}{M'_{33}} \\ \bar{v}'_3 &= \frac{X'_{w32}}{N'_{33}} \end{aligned}$$

Sway Mode

The horizontal wave force on a restrained cross section in beam waves is:

$$\begin{aligned} X'_{w2} &= X'_{w2a} \cos\left(\omega t + \varepsilon'_{w2}\right) \\ &= X'_{w2a} \cos \varepsilon'_{w2} \cos \omega t - X'_{w2a} \sin \varepsilon'_{w2} \sin \omega t \end{aligned}$$

of which the amplitude is equal to:

$$X'_{w2a} = \zeta_a \sqrt{2\rho g \cdot c_{group} \cdot N'_{22}}$$

For the phase lag of this wave force, ε'_{w2}, an approximation has to be found.
The phase lag of an incoming undisturbed wave, $\varepsilon'_{w_I 2}$, at the intersection of the ship's hull with the waterline, $y_b = y_{wl}$, is

$$\begin{aligned} \varepsilon'_{w_I 2} &= -k y_{wl} \sin \mu \\ \text{if } \sin \mu < 0 \text{ then: } \varepsilon'_{w_I 2} &= \varepsilon'_{W_I 2} + \pi \end{aligned}$$

In very short waves - so at high wave frequencies $\omega \rightarrow \infty$ - the ship's hull behaves like a vertical wall and all waves will be diffracted. Then, the phase lag of the wave force, ε'_{w2}, is equal to:

$$\varepsilon'_{w2}\left(\omega \rightarrow \infty\right) = -\varepsilon'_{w_I 2}$$

The acceleration and velocity amplitudes of the water particles in the undisturbed surface of the incoming waves are:

$$\begin{aligned} \left(a'_2\right)_{\text{still water surface}} &= -kg \sin \mu \\ \left(v'_2\right)_{\text{still water surface}} &= \frac{-a'_2}{\omega} = \frac{kg \sin \mu}{\omega} \end{aligned}$$

In very long waves - so at low wave frequencies $\omega \rightarrow 0$ - the wave force is dominated by the Froude-Krylov force and the amplitudes of the water particle motions do not change very much over the draft of the section. Apparently, the phase lag of the wave force, $\varepsilon_{w2}^{'}$, can be approximated by:

$$\varepsilon_{w2}^{'}(\omega \rightarrow 0) = -\arctan\left[\frac{X_{FK_2}^{\prime} + M_{22}^{\prime} \cdot (-kg\sin\mu)}{N_{22}^{\prime} \cdot \left(\frac{kg\sin\mu}{\omega}\right)}\right]$$

When plotted against ω, the two curves $\varepsilon_{w2}^{'}(\omega \rightarrow 0)$ and $\varepsilon_{w2}^{'}(\omega \rightarrow \infty)$ will intersect each other. The phase lag of the wave force, $\varepsilon_{w2}^{'}$, can now be approximated by the lowest of these two values:

$$\begin{array}{rcl} \varepsilon_{w2}^{'} & = & \varepsilon_{w2}^{'}(\omega \rightarrow \infty) \\ \text{if } \varepsilon_{w2}^{'}(\omega \rightarrow 0) > \varepsilon_{w2}^{'}(\omega \rightarrow \infty) \text{ then: } \varepsilon_{w2}^{'} & = & \varepsilon_{w2}^{'}(\omega \rightarrow 0) \end{array}$$

Because $\varepsilon_{w2}^{'}(\omega \rightarrow \infty)$ goes to zero in the low frequency region and $\varepsilon_{w2}^{'}(\omega \rightarrow 0)$ can have values between 0 and 2π, one simple precaution has to be taken:

$$\text{if } \varepsilon_{w2}^{'}(\omega \rightarrow 0) > \frac{\pi}{2} \text{ then: } \varepsilon_{w2}^{'}(\omega \rightarrow 0) = \varepsilon_{w2}^{'}(\omega \rightarrow 0) - 2\pi$$

Now the in-phase and out-phase terms of the wave force in beam waves are:

$$\begin{array}{rcl} X_{FK_2}^{\prime} + X_{w21}^{'} & = & -X_{w2a}^{'}\sin\varepsilon_{w2}^{'} = -\left(\zeta_a\sqrt{2\rho g \cdot c_{group} \cdot N_{22}^{'}}\right)\sin\varepsilon_{w2}^{'} \\ X_{w22}^{'} & = & +X_{w2a}^{'}\cos\varepsilon_{w2}^{'} = +\left(\zeta_a\sqrt{2\rho g \cdot c_{group} \cdot N_{22}^{'}}\right)\cos\varepsilon_{w2}^{'} \end{array}$$

from which the diffraction terms, $X_{w21}^{'}$ and $X_{w22}^{'}$ follow.
These terms can also be written as:

$$\begin{array}{rcl} X_{w21}^{'} & = & M_{22}^{'} \cdot \bar{a}_2^{'} \\ X_{w22}^{'} & = & N_{22}^{'} \cdot \bar{v}_2^{'} \end{array}$$

in which $\bar{a}_2^{'}$ and $\bar{v}_2^{'}$ are the equivalent amplitudes of the acceleration and the velocity of the water particles around the cross section.
Then - when using an approximation for the influence of the wave direction - the equivalent acceleration and velocity amplitudes of the water particles are:

$$\begin{array}{rcl} \bar{a}_2^{'} & = & \dfrac{X_{w21}^{'}}{M_{22}^{'}} \cdot |\sin\mu| \\ \bar{v}_2^{'} & = & \dfrac{X_{w22}^{'}}{N_{22}^{'}} \cdot |\sin\mu| \end{array}$$

Roll Mode

The fluid is free of rotation; so the wave moment for roll consists of sway contributions only. However, the equivalent amplitudes of the acceleration and the velocity of the water particles will differ from those of sway.
From a study on potential coefficients, the following relation between sway and roll damping coefficients has been found:

$$N_{44}^{'} = \frac{\left(N_{24}^{'}\right)^2}{N_{22}^{'}} = \frac{\left(N_{42}^{'}\right)^2}{N_{22}^{'}}$$

The horizontal wave moment on a restrained cross section in beam waves is:

$$X_{w4}^{'} = X_{w4a}^{'} \cos\left(\omega t + \varepsilon_{w4}^{'}\right)$$

of which the amplitude is equal to:

$$\begin{aligned} X_{w4a}^{'} &= \zeta_a \sqrt{2\rho g \cdot c_{group} \cdot N_{44}^{'}} \\ &= \zeta_a \sqrt{2\rho g \cdot c_{group} \cdot \frac{\left(N_{24}^{'}\right)^2}{N_{22}^{'}}} \\ &= \zeta_a \sqrt{2\rho g \cdot c_{group} \cdot N_{22}^{'}} \cdot \frac{\left|N_{24}^{'}\right|}{N_{22}^{'}} \\ &= X_{w2a}^{'} \cdot \frac{\left|N_{24}^{'}\right|}{N_{22}^{'}} \end{aligned}$$

The in-phase and out-phase parts of the wave moment in beam waves are:

$$\begin{aligned} X_{FK_4}^{\prime} + X_{w41}^{'} &= \left(X_{FK_2}^{\prime} + X_{w21}^{'}\right) \cdot \frac{\left|N_{24}^{'}\right|}{N_{22}^{'}} \\ X_{w42}^{'} &= X_{w22}^{'} \cdot \frac{\left|N_{24}^{'}\right|}{N_{22}^{'}} \end{aligned}$$

from which the diffraction terms, $X_{w41}^{'}$ and $X_{w42}^{'}$ follow.
These terms can also be written as:

$$\begin{aligned} X_{41}^{'} &= M_{24}^{'} \cdot \bar{a}_{24}^{'} \\ X_{42}^{'} &= N_{24}^{'} \cdot \bar{v}_{24}^{'} \end{aligned}$$

in which $\bar{a}_{24}^{'}$ and $\bar{v}_{24}^{'}$ are the equivalent amplitudes of the acceleration and the velocity of the water particles around the cross section.
Then - when using an approximation for the influence of the wave direction - the equivalent acceleration and velocity amplitudes of the water particles are:

$$\begin{aligned} \bar{a}_{24}^{'} &= \frac{X_{w41}^{'}}{M_{24}^{'}} \cdot |\sin \mu| \\ \bar{v}_{24}^{'} &= \frac{X_{w42}^{'}}{N_{24}^{'}} \cdot |\sin \mu| \end{aligned}$$

Surge Mode

The equivalent acceleration and velocity amplitudes of the water particles around the cross section for surge have been found from:

$$\begin{aligned} \bar{a}_1^{'} &= \frac{\bar{a}_2^{'}}{\tan \mu} \\ \bar{v}_1^{'} &= \frac{-\bar{a}_1^{'}}{\omega} \end{aligned}$$

7.3 Numerical Comparison

Figures 7.6 and 7.7 give a comparison between these sway, heave and roll wave loads on a crude oil carrier in oblique waves - obtained by the classic approach and the simple diffraction approach, respectively - with the 3-D zero speed ship motions program DELFRAC of Pinkster; see [Dimitrieva, 1994].

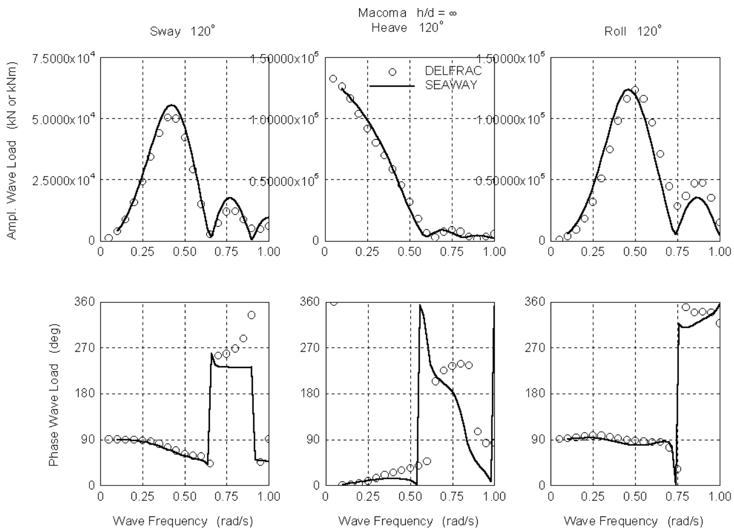

Figure 7.6: Comparison of Classic Wave Loads with DELFRAC Data

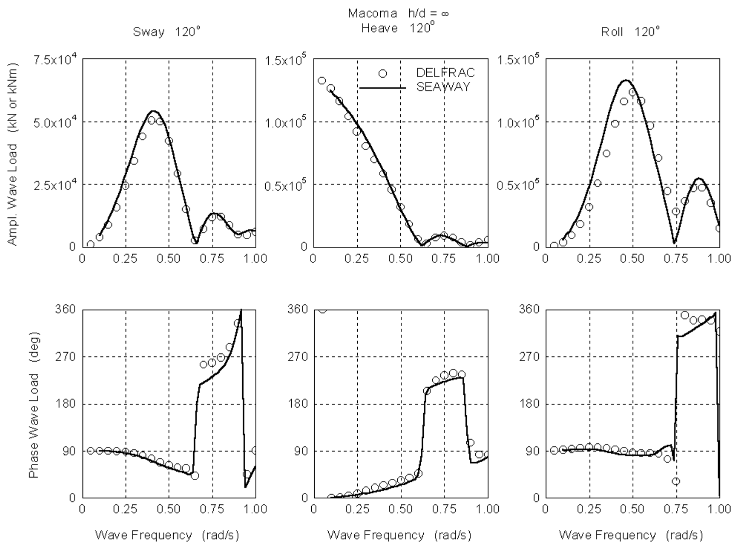

Figure 7.7: Comparison of Simple Diffraction Wave Loads with DELFRAC Data

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Chapter 8

Transfer Functions of Motions

After dividing the left and right hand terms by the wave amplitude ζ_a, two sets of six coupled equations of motion are available.
The variables in the coupled equations for the vertical plane motions are:

$$\begin{array}{llcl}
\text{Surge:} & \frac{x_a}{\zeta_a} \cdot \cos \varepsilon_{x\zeta} & \text{and} & \frac{x_a}{\zeta_a} \cdot \sin \varepsilon_{x\zeta} \\
\text{Heave:} & \frac{z_a}{\zeta_a} \cdot \cos \varepsilon_{z\zeta} & \text{and} & \frac{z_a}{\zeta_a} \cdot \sin \varepsilon_{z\zeta} \\
\text{Pitch:} & \frac{\theta_a}{\zeta_a} \cdot \cos \varepsilon_{\theta\zeta} & \text{and} & \frac{\theta_a}{\zeta_a} \cdot \sin \varepsilon_{\theta\zeta}
\end{array}$$

The variables in the coupled equations for the horizontal plane motions are:

$$\begin{array}{llcl}
\text{Sway:} & \frac{y_a}{\zeta_a} \cdot \cos \varepsilon_{y\zeta} & \text{and} & \frac{y_a}{\zeta_a} \cdot \sin \varepsilon_{y\zeta} \\
\text{Roll:} & \frac{\phi_a}{\zeta_a} \cdot \cos \varepsilon_{\phi\zeta} & \text{and} & \frac{\phi_a}{\zeta_a} \cdot \sin \varepsilon_{\phi\zeta} \\
\text{Yaw:} & \frac{\psi_a}{\zeta_a} \cdot \cos \varepsilon_{\psi\zeta} & \text{and} & \frac{\psi_a}{\zeta_a} \cdot \sin \varepsilon_{\psi\zeta}
\end{array}$$

These sets of motions have to be solved by a numerical method. A method which provides continuous good results, given by [Zwaan, 1977], has been used in the strip theory program SEAWAY-DELFT.

8.1 Centre of Gravity Motions

From the solutions of these in and out of phase terms follow the transfer functions of the motions, which is the motion amplitude to wave amplitude ratio, and the phase lags of the motions relative to the wave elevation at the ship's centre of gravity:

$$\frac{x_a}{\zeta_a} \qquad \frac{y_a}{\zeta_a} \qquad \frac{z_a}{\zeta_a} \qquad \frac{\theta_a}{\zeta_a} \qquad \frac{\phi_a}{\zeta_a} \qquad \frac{\psi_a}{\zeta_a}$$

The associated phase lags are:

$$\varepsilon_{x\zeta} \qquad \varepsilon_{y\zeta} \qquad \varepsilon_{z\zeta} \qquad \varepsilon_{\theta\zeta} \qquad \varepsilon_{\phi\zeta} \qquad \varepsilon_{\psi\zeta}$$

[0] J.M.J. Journée, *"Theoretical Manual of SEAWAY, Release 4.19"*, Report 1216a, February 2001, Ship Hydromechanics Laboratory, Delft University of Technology, Mekelweg 2, 2628 CD Delft, The Netherlands. For updates see web site: http://dutw189.wbmt.tudelft.nl/~johan or http://www.shipmotions.nl.

The transfer functions of the translations are non-dimensional.
The transfer functions of the rotations can be made non-dimensional by dividing the amplitude of the rotations by the amplitude of the wave slope $k\zeta_a$ in lieu of the wave amplitude ζ_a:

$$\frac{x_a}{\zeta_a} \qquad \frac{y_a}{\zeta_a} \qquad \frac{z_a}{\zeta_a} \qquad \frac{\theta_a}{k\zeta_a} \qquad \frac{\psi_a}{k\zeta_a} \qquad \frac{\phi_a}{k\zeta_a}$$

For motions with a spring term, three frequency regions can be distinguished:

- the low frequency region ($\omega^2 \ll c/(m+a)$), with motions dominated by the restoring spring term,
- the natural frequency region ($\omega^2 \approx c/(m+a)$), with motions dominated by the damping term and
- the high frequency region ($\omega^2 \gg c/a$), with motions dominated by the mass term.

An example for heave is given in figure 8.1.

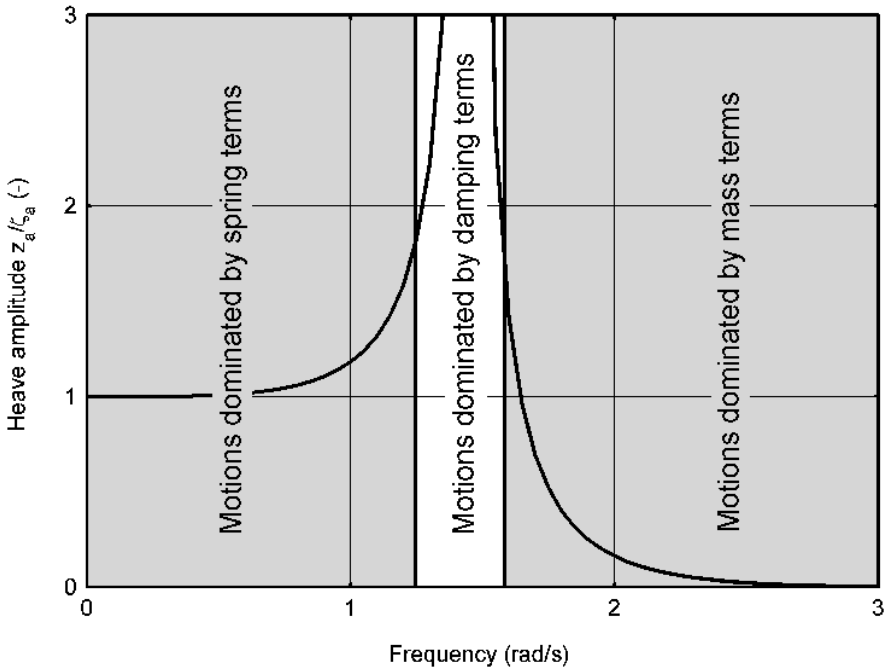

Figure 8.1: Frequency Regions and Motional Behavior

With the six centre of gravity motions, the harmonic motions in the ship-bound x_b, y_b and z_b directions - or in the earth bound x, y and z directions - in any point $P(x_b, y_b, z_b)$ on the ship can be calculated.

8.2 Absolute Displacements

Consider a point $P(x_b, y_b, z_b)$ on the ship in the $G(x_b, y_b, z_b)$ ship-bound axes system. The harmonic displacements in the ship-bound x_b, y_b and z_b directions - or in the earth bound

x, y and z directions - in a point $P(x_b, y_b, z_b)$ on the ship can be obtained from the six centre of gravity motions as presented below.
The harmonic longitudinal displacement is given by:

$$\begin{aligned} x_P &= x - y_b \cdot \psi + z_b \cdot \theta \\ &= x_{P_a} \cdot \cos(\omega_e t + \varepsilon_{x_P\zeta}) \end{aligned}$$

The harmonic lateral displacement is given by:

$$\begin{aligned} y_P &= y + x_b \cdot \psi - z_b \cdot \phi \\ &= y_{P_a} \cdot \cos(\omega_e t + \varepsilon_{y_P\zeta}) \end{aligned}$$

The harmonic vertical displacement is given by:

$$\begin{aligned} z_P &= z - x_b \cdot \theta - y_b \cdot \phi \\ &= z_{P_a} \cdot \cos(\omega_e t + \varepsilon_{z_P\zeta}) \end{aligned}$$

Some examples of calculated transfer functions of a crude oil carrier and a containership are given in the figures 8.2, 8.3 and 8.4.

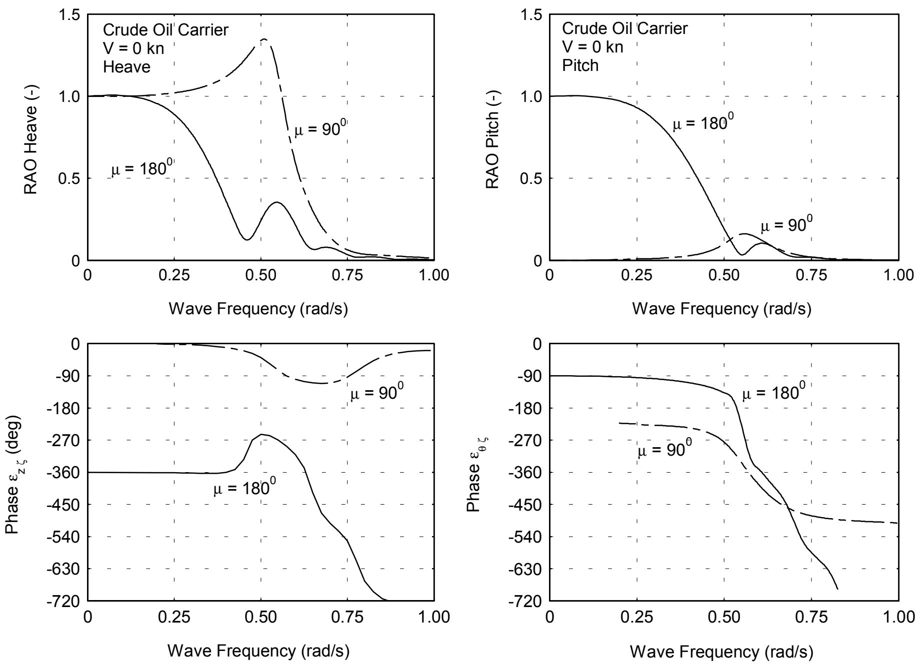

Figure 8.2: Heave and Pitch of a Crude Oil Carrier, V = 0 Knots

Notify the different speed effects for roll and pitch in figure 8.4.

8.3 Absolute Velocities

The harmonic velocities in the ship-bound x_b, y_b and z_b directions - or in the earth bound x, y and z directions - in a point $P(x_b, y_b, z_b)$ on the ship are obtained by taking the derivative of the three harmonic displacements.

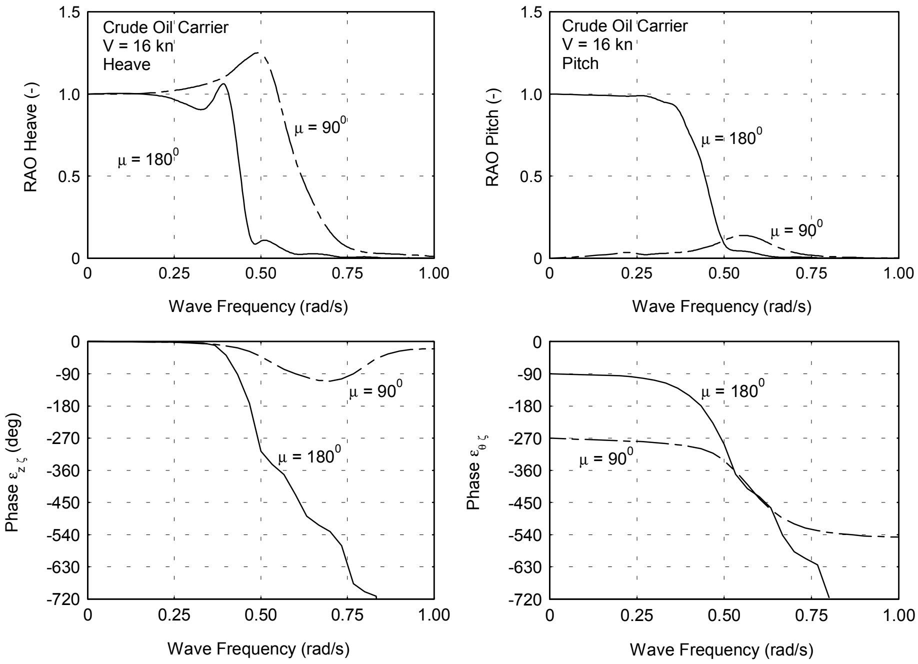

Figure 8.3: Heave and Pitch of a Crude Oil Carrier, V = 16 Knots

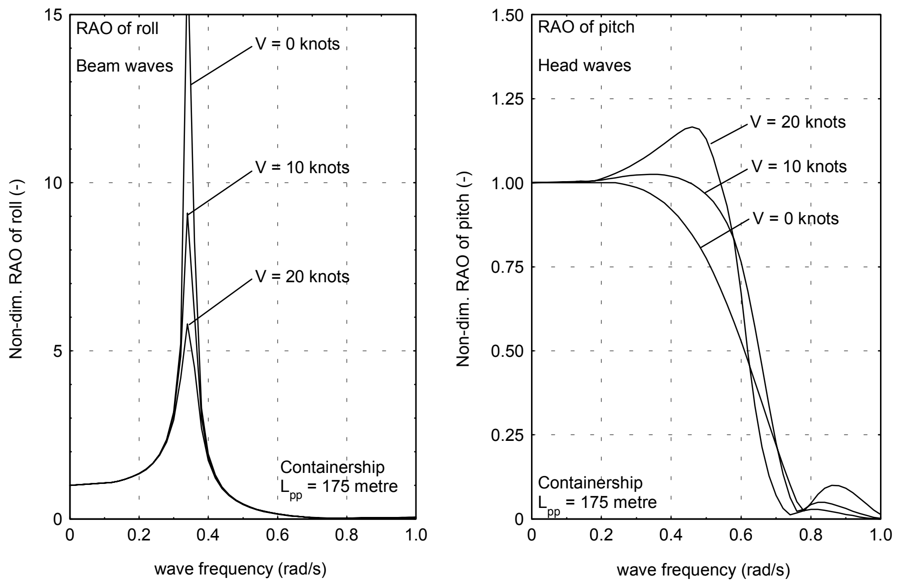

Figure 8.4: RAO's of Roll and Pitch of a Containership

The harmonic longitudinal velocity is given by:

$$\begin{aligned}
\dot{x}_P &= \dot{x} \ - y_b \cdot \dot{\psi} \ + z_b \cdot \dot{\theta} \\
&= -\omega_e \cdot x_{P_a} \cdot \sin(\omega_e t + \varepsilon_{x_p\zeta}) \\
&= \dot{x}_{P_a} \cdot \cos(\omega_e t + \varepsilon_{\dot{x}_p\zeta})
\end{aligned}$$

The harmonic lateral velocity is given by:

$$\begin{aligned}
\dot{y}_P &= \dot{y} \ + x_b \cdot \dot{\psi} \ - z_b \cdot \dot{\phi} \\
&= -\omega_e \cdot y_{P_a} \cdot \sin(\omega_e t + \varepsilon_{y_p\zeta}) \\
&= \dot{y}_{P_a} \cdot \cos(\omega_e t + \varepsilon_{\dot{y}_p\zeta})
\end{aligned}$$

The harmonic vertical velocity is given by:

$$\begin{aligned}
\dot{z}_P &= \dot{z} \ - x_b \cdot \dot{\theta} \ + y_b \cdot \dot{\phi} \\
&= -\omega_e \cdot z_{P_a} \cdot \sin(\omega_e t + \varepsilon_{z_p\zeta}) \\
&= \dot{z}_{P_a} \cdot \cos(\omega_e t + \varepsilon_{\dot{z}_p\zeta})
\end{aligned}$$

8.4 Absolute Accelerations

In the earth-bound axes system, the harmonic accelerations on the ship are obtained by taking the second derivative of the displacements. In the ship-bound axes system, a component of the acceleration of gravity has to be added to the accelerations in the horizontal plane direction.

8.4.1 Accelerations in the Earth-Bound Axes System

In the earth-bound axes system, $O(x, y, z)$, the harmonic accelerations in the x, y and z direction in a point $P(x_b, y_b, z_b)$ on the ship are obtained by taking the second derivative of the three harmonic displacements.
Thus:

- Longitudinal acceleration:

$$\begin{aligned}
\ddot{x}_P &= \ddot{x} \ - y_b \cdot \ddot{\psi} \ + z_b \cdot \ddot{\theta} \\
&= -\omega_e^2 \cdot x_{P_a} \cdot \cos(\omega_e t + \varepsilon_{x_P\zeta}) \\
&= \ddot{x}_{P_a} \cdot \cos(\omega_e t + \varepsilon_{\ddot{x}_P\zeta})
\end{aligned}$$

- Lateral acceleration:

$$\begin{aligned}
\ddot{y}_P &= \ddot{y} \ + x_b \cdot \ddot{\psi} \ - z_b \cdot \ddot{\phi} \\
&= -\omega_e^2 \cdot y_{P_a} \cdot \cos(\omega_e t + \varepsilon_{y_P\zeta}) \\
&= \ddot{y}_{P_a} \cdot \cos(\omega_e t + \varepsilon_{\ddot{y}_P\zeta})
\end{aligned}$$

- Vertical acceleration:

$$\begin{aligned}
\ddot{z}_P &= \ddot{z} \ - x_b \cdot \ddot{\theta} \ + y_b \cdot \ddot{\phi} \\
&= -\omega_e^2 \cdot z_{P_a} \cdot \cos(\omega_e t + \varepsilon_{z_P\zeta}) \\
&= \ddot{z}_{P_a} \cdot \cos(\omega_e t + \varepsilon_{\ddot{z}_P\zeta})
\end{aligned}$$

8.4.2 Accelerations in the Ship-Bound Axes System

In the ship-bound axes system, $G(x_b, y_b, z_b)$, a component of the acceleration of gravity g has to be added to the accelerations in the longitudinal and lateral direction in the earth-bound axes system. The vertical acceleration does not change. These are the accelerations that will be "felt" by for instance the cargo on the ship.
Thus:

- Longitudinal acceleration:

$$\begin{aligned} \ddot{x}_P &= \ddot{x} \; - y_b \cdot \ddot{\psi} \; + z_b \cdot \ddot{\theta} \; - g \cdot \theta \\ &= -\omega_e^2 \cdot x_{P_a} \cdot \cos(\omega_e t + \varepsilon_{x_P \zeta}) \; - g \cdot \theta_a \cdot \cos(\omega_e t + \varepsilon_{\theta \zeta}) \\ &= \ddot{x}_{P_a} \cdot \cos(\omega_e t + \varepsilon_{\ddot{x}_P \zeta}) \end{aligned}$$

- Lateral acceleration:

$$\begin{aligned} \ddot{y}_P &= \ddot{y} \; + x_b \cdot \ddot{\psi} \; - z_b \cdot \ddot{\phi} \; + g \cdot \phi \\ &= -\omega_e^2 \cdot y_{P_a} \cdot \cos(\omega_e t + \varepsilon_{y_P \zeta}) \; + g \cdot \phi_a \cdot \cos(\omega_e t + \varepsilon_{\phi \zeta}) \\ &= \ddot{y}_{P_a} \cdot \cos(\omega_e t + \varepsilon_{\ddot{y}_P \zeta}) \end{aligned}$$

- Vertical acceleration:

$$\begin{aligned} \ddot{z}_P &= \ddot{z} \; - x_b \cdot \ddot{\theta} \; + y_b \cdot \ddot{\phi} \\ &= -\omega_e^2 \cdot z_{P_a} \cdot \cos(\omega_e t + \varepsilon_{z_P \zeta}) \\ &= \ddot{z}_{P_a} \cdot \cos(\omega_e t + \varepsilon_{\ddot{z}_P \zeta}) \end{aligned}$$

8.5 Vertical Relative Displacements

The harmonic vertical relative displacement with respect to the wave surface of a point $P(x_b, y_b, z_b)$ connected to the ship can be obtained too:

$$\begin{aligned} s_P &= \zeta_P - z \; + x_b \cdot \theta \; - y_b \cdot \phi \\ &= s_{P_a} \cdot \cos(\omega_e t + \varepsilon_{s_P \zeta}) \end{aligned}$$

with:

$$\zeta_P = \zeta_a \cos(\omega_e t - k x_b \cos\mu - k y_b \sin\mu)$$

It may be noted that the sign of the relative motion is chosen here in such a way that a positive relative displacement implies a decrease of the freeboard.
An oscillating ship will produce waves and these phenomena will change the relative motion. A dynamical swell up should be taken into account, which is not included in the previous formulation.
Notify the different behavior of absolute and relative vertical motioins as given in figure 8.5.

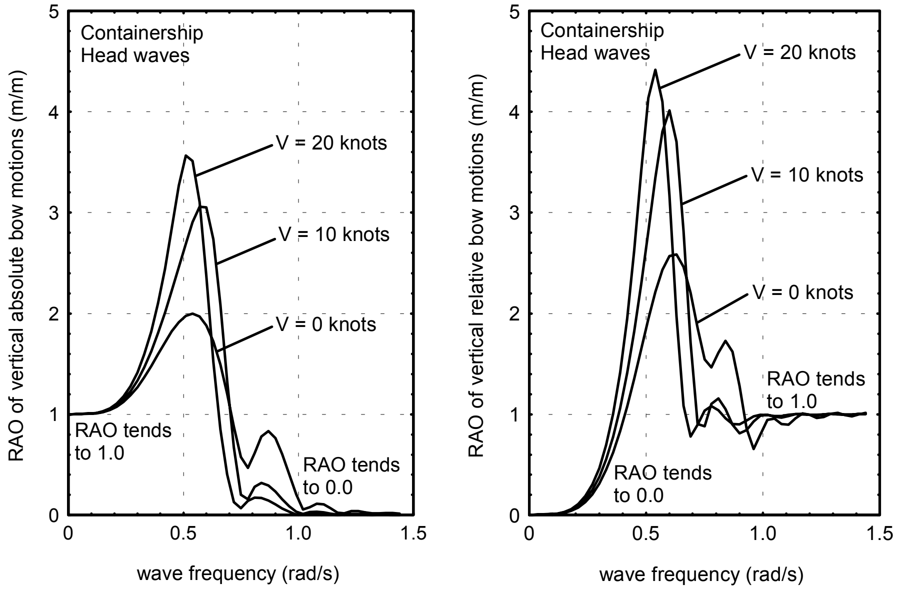

Figure 8.5: Absolute and Relative Vertical Motions at the Bow

8.6 Vertical Relative Velocities

The harmonic vertical relative velocity with respect to the wave surface of a certain point $P(x_b, y_b, z_b)$, connected to the ship, can be obtained by:

$$\begin{aligned} \dot{s}_P &= \frac{D}{Dt}\left\{\zeta_P - z + x_b \cdot \theta - y_b \cdot \phi\right\} \\ &= \dot{\zeta}_P - \dot{z} + x_b \cdot \dot{\theta} - V \cdot \theta - y_b \cdot \dot{\phi} \end{aligned}$$

in which for the vertical velocity of the water surface itself:

$$\dot{\zeta}_P = -\omega \cdot \zeta_a \sin(\omega_e t - kx_b \cos\mu - ky_b \sin\mu)$$

.

Chapter 9

Anti-Rolling Devices

Since the disappearence of sails on oceangoing ships, with their stabilising wind effect on the rolling motions, naval architects have been concerned in reducing the rolling of ships among waves. With bilge keels they performed a first successful attack on the problem of rolling, but in several cases these bilge keels did not prove to be sufficient. Since 1880, numerous other more or less successful ideas have been tested and used.

- Four types of anti-rolling devices and its contribution to the equations of motion are described here:
- bilge keels
- passive free-surface tanks
- active fin stabilisers
- active rudder stabilisers.

The active fin and rudder stabilisers are not build into the program SEAWAY yet.

9.1 Bilge Keels

Bilge keels can deliver an important contribution to an increase the damping of the rolling motions of ships. A reliable method to determine this contribution is given by [Ikeda et al., 1978], as described before.
Ikeda divides the two-dimensional quadratic bilge keel roll damping into a component due to the normal force on the bilge keels and a component due to the pressure on the hull surface, created by the bilge keels.
The normal force component of the bilge keel damping has been be deduced from experimental results of oscillating flat plates. The drag coefficient C_D depends on the period parameter or the Keulegan-Carpenter number. Ikeda measured the quadratic two-dimensional drag by carrying out free rolling experiments with an ellipsoid with and without bilge keels. Assuming a pressure distribution on the hull caused by the bilge keels, a quadratic two-dimensional roll damping can be defined. Ikeda carried out experiments to measure the

[0] J.M.J. Journée, *"Theoretical Manual of SEAWAY, Release 4.19"*, Report 1216a, February 2001, Ship Hydromechanics Laboratory, Delft University of Technology, Mekelweg 2, 2628 CD Delft, The Netherlands. For updates see web site: http://dutw189.wbmt.tudelft.nl/~johan or http://www.shipmotions.nl.

pressure on the hull surface created by bilge keels. He found that the coefficient C_p^+ of the pressure on the front face of the bilge keel does not depend on the period parameter, while the coefficient C_p^- of the pressure on the back face of the bilge keel and the length of the negative pressure region depend on the period parameter. Ikeda defines an equivalent length of a constant negative pressure region S_0 over the height of the bilge keels and a two-dimensional roll damping component can be found.
The total bilge keel damping has been obtained by integrating these two two-dimensional roll damping components over the length of the bilge keels.
Experiments of Ikeda showed that the effect of forward speed on the roll damping due to the bilge keels can be ignored.
The equivalent linear total bilge keel damping has been obtained by linearising the result, as has been shown in a separate chapter.

9.2 Passive Free-Surface Tanks

The roll damping, caused by a passive free-surface tank, is essentially based on the existence of a hydraulic jump or bore in the tank. [Verhagen and van Wijngaarden, 1965] give a theoretical approach to determine the counteracting moments by free-surface anti-rolling tanks.[Bosch and Vugts, 1966] give extended quantitative information on these moments.

9.2.1 Theoretical Approach

When a tank which contains a fluid with a free surface is forced to carry out roll oscillations, resonance frequencies can be obtained with high wave amplitudes at lower water depths. Under these circumstances a hydraulic jump or bore is formed, which travels periodically back and forth between the walls of the tank. This hydraulic jump can be a strongly non-linear phenomenon. A theory, based on gasdynamics for the shock wave in a gas flow under similar resonance circumstances, as given by [Verhagen and van Wijngaarden, 1965], has been adapted and used to describe the motions of the fluid. For low and high frequencies and the frequencies near to the natural frequency, different approaches have been used.
Observe a rectangular tank with a length l and a breadth b, which has been filled until a water level h with a fluid with a mass density ρ. The distance of the bottom of the tank above the centre of gravity of the vessel is s. Figure 9.1 shows a 2-D sketch of this tank with the axis system and notations.

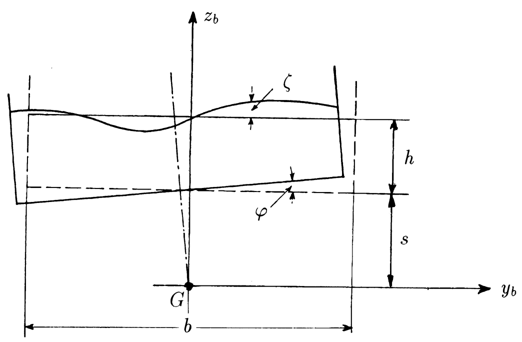

Figure 9.1: Axes System and Notations of an Oscillating Tank

The natural frequency of the surface waves in a harmonic rolling tank appears as the wave length λ equals twice the breadth b, so: $\lambda_0 = 2b$.
With the wave number and the dispersion relation:

$$k = \frac{2\pi}{\lambda} \quad \text{and} \quad \omega = \sqrt{kg \tanh\left[kh\right]}$$

it follows for the natural frequency of surface waves in the tank:

$$\omega_0 = \sqrt{\frac{\pi g}{b} \tanh\left[\frac{\pi h}{b}\right]}$$

[Verhagen and van Wijngaarden, 1965] have investigated the shallow water wave loads in a rolling rectangular container, with the centre of rotation at the bottom of the container. Their expressions for the internal wave loads are rewritten and modified to be useful for any arbitrary vertical position of the centre of rotation by [Journée, 1997]. For low and high frequencies and the frequencies near to the natural frequency, different approaches have been used. A calculation routine has been made to connect these regions.

Low and High Frequencies

The harmonic roll motion of the tank is defined by:

$$\phi = \phi_a \sin(\omega t)$$

In the axis-system of figure 9.1 and after linearisation, the vertical displacement of the tankbottom is described by:

$$z = s + y\phi$$

and after linearisation, the surface elevation of the fluid is described by:

$$z = s + h + \zeta$$

Relative to the bottom of the tank, the linearised surface elevation of the fluid is described by:

$$\xi = h + \zeta - y\phi$$

Using the shallow water theory, the continuity and momentum equations are:

$$\begin{aligned} \frac{\partial \xi}{\partial t} + v\frac{\partial \xi}{\partial y} + \xi\frac{\partial v}{\partial y} &= 0 \\ \frac{\partial v}{\partial t} + v\frac{\partial v}{\partial y} + g\frac{\partial \xi}{\partial y} + g\phi &= 0 \end{aligned}$$

In these formulations, v denotes the velocity of the fluid in the y-direction and the vertical pressure distribution is assumed to be hydrostatic. Therefore, the acceleration in the z-direction, introduced by the excitation, must be small with respect to the acceleration of gravity g, so:

$$\phi_a \omega^2 b \ll g$$

The boundary conditions for v are determined by the velocity produced in the horizontal direction by the excitation. Between the surface of the fluid and the bottom of the tank, the velocity of the fluid v varies between v_s and $v_s/\cosh kh$ with a mean velocity: v_s/kh. However, in very shallow water v does not vary between the bottom and the surface. When taking the value at the surface, it is required that:

$$v = -(s+h)\dot{\phi} \quad \text{at: } y = \pm\frac{b}{2}$$

For small values of ϕ_a, the continuity equation and the momentum equation can be given in a linearised form:

$$\begin{aligned} \frac{\partial \xi}{\partial t} + h\frac{\partial v}{\partial y} &= 0 \\ \frac{\partial v}{\partial t} + g\frac{\partial \xi}{\partial y} + g\phi &= 0 \end{aligned}$$

The solution of the surface elevation ξ in these equations, satisfying the boundary values for v, is:

$$\xi = h - \frac{b\omega_0 \left\{1 + \frac{(s+h)\omega^2}{g}\right\}}{\pi\omega \cos\left(\frac{\pi\omega}{2\omega_0}\right)} \sin\left(\frac{\pi\omega y}{b\omega_0}\right)\phi$$

Now, the roll moment follows from the quasi-static moment of the mass of the frozen liquid ρlbh and an integration of ξ over the breadth of the tank:

$$M_\phi = \rho glbh\left(s + \frac{h}{2}\right)\phi + \rho gl \int_{-b/2}^{+b/2} \xi y\, dy$$

This delivers the roll moment amplitude for low and high frequencies at small water depths:

$$\begin{aligned} M_{a\phi} &= \rho glbh\left(s+\frac{h}{2}\right)\phi_a \\ &\quad + \rho glb^3\left\{1 + \frac{(s+h)\,\omega^2}{g}\right\}\cdot\left\{2\left(\frac{\omega_0}{\pi\omega}\right)^3 \tan\left(\frac{\pi\omega}{2\omega_0}\right) - \left(\frac{\omega_0}{\pi\omega}\right)^2\right\}\phi_a \end{aligned}$$

For very low frequencies, so for the limit value $\omega \rightarrow 0$, this will result into the static moment:

$$M_\phi = \rho gl\left\{bh\left(s+\frac{h}{2}\right) + \frac{b^3}{12}\right\}\phi$$

The phase lags between the roll moments and the roll motions have not been obtained here. However, they can be set to zero for low frequencies and to -π for high frequencies:

$$\begin{aligned} \varepsilon_{M_\phi\phi} &= 0 && \text{for: } \omega \ll \omega_0 \\ \varepsilon_{M_\phi\phi} &= -\pi && \text{for: } \omega \gg \omega_0 \end{aligned}$$

Natural Frequency Region

For frequencies near to the natural frequency ω_0, the expression for the surface elevation of the fluid ξ goes to infinity. Experiments showed the appearance of a hydraulic jump or a bore at these frequencies. Obviously, then the linearised equations are not valid anymore.

Verhagen and van Wijngaarden solved the problem by using the approach in gas dynamics when a column of gas is oscillated at a small amplitude, e.g. by a piston. At frequencies near to the natural frequency at small water depths, they found a roll moment amplitude, defined by:

$$M_{a\phi} = \rho g \frac{lb^3}{12} \left(\frac{4}{\pi}\right)^4 \sqrt{\frac{2\phi_a h}{3b}} \cdot \left\{1 - \frac{\pi^2 b \left(\omega - \omega_0\right)^2}{32 g \phi_a}\right\}$$

The phase lags between the roll moment and the roll motion at small water depths are given by:

$$\begin{aligned} \varepsilon_{M_\phi \phi} &= -\frac{\pi}{2} + \alpha \qquad \text{for: } \omega < \omega_0 \\ \varepsilon_{M_\phi \phi} &= -\frac{\pi}{2} - \alpha \qquad \text{for: } \omega > \omega_0 \end{aligned}$$

with:

$$\begin{aligned} \alpha &= 2 \arcsin \left\{ \sqrt{\frac{\pi^2 b \left(\omega - \omega_0\right)^2}{24 g \phi_a}} \right\} \\ &\quad - \arcsin \left\{ \sqrt{\frac{\pi^2 b \left(\omega - \omega_0\right)^2}{96 g \phi_a - 3\pi^2 b \left(\omega - \omega_0\right)^2}} \right\} \end{aligned}$$

Because that the arguments of the square roots in the expression for $\varepsilon_{M_\phi \phi}$ have to be positive, the limits for the frequency ω are at least:

$$\omega_0 - \sqrt{\frac{24 g \phi_a}{b \pi^2}} < \omega < \omega_0 + \sqrt{\frac{24 g \phi_a}{b \pi^2}}$$

Comparison with Experimental Data

An example of the results of this theory with experimental data of an oscillating free-surface tank by [Verhagen and van Wijngaarden, 1965] is given in figure 9.2.
The roll moments have been calculated here for low and high frequencies and for frequencies near to the natural frequency of the tank. A calculation routine connects these three regions.

9.2.2 Experimental Approach

[Bosch and Vugts, 1966] have described the physical behaviour of passive free-surface tanks, used as an anti-rolling device. Extended quantitative information on the counteracting moments, caused by the water transfer in the tank, has been provided.
With their symbols, the roll motions and the exciting moments of an oscillating rectangular free-surface tank, are defined by:

$$\begin{aligned} \varphi &= \varphi_a \cos(\omega t) \\ K_t &= K_{t_a} \cos(\omega t + \varepsilon_t) \end{aligned}$$

and the dimensions of the rectangular free-surface tank are given by:

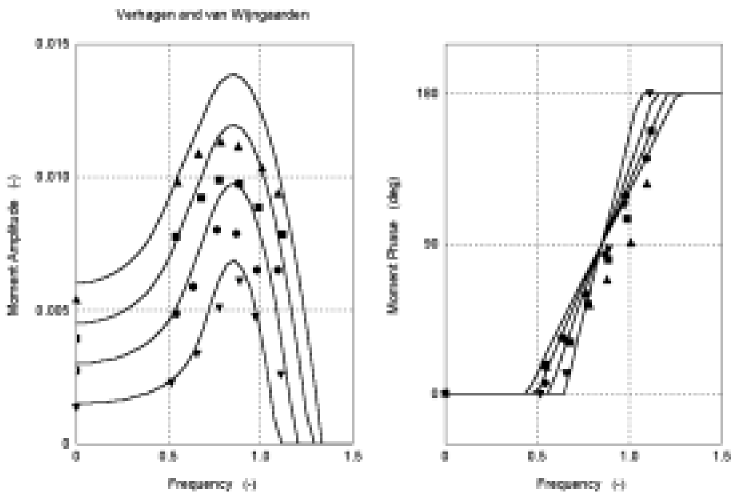

Figure 9.2: Comparison between Theoretical and Experimental Data

l	= length of the tank
b	= breadth of the tank
s	= distance of tank bottom above rotation point
h	= water depth in the tank at rest
ρ^*	= mass density of the fluid in the tank

A non-dimensional frequency range is defined by:

$$0.00 < \omega \cdot \sqrt{\frac{b}{g}} < 1.60$$

In this frequency range, [Bosch and Vugts, 1966] have presented extended experimental data of:

$$\mu_a = \frac{K_{t_a}}{\rho^* g l b^3} \qquad \text{and} \qquad \varepsilon_t$$

for:

φ_a	= 0.0333, 0.0667 and 0.1000 radians
s/b	= -0.40, -0.20, 0.00 and +0.20
h/b	= 0.02, 0.04, 0.06, 0.08 and 0.10

An example of a part of these experimental data has been shown for $s/b = -0.40$ and $\varphi_a = 0.1000$ radians in figure 9.3, taken from the report of [Bosch and Vugts, 1966].
When using these experimental data, the external roll moment due to an, with a frequency ω, oscillating free surface tank can be written as:

$$K_t = a_{4\varphi} \cdot \ddot{\varphi} + b_{4\varphi} \cdot \dot{\varphi} + c_{4\varphi} \cdot \varphi$$

with:

$$\begin{aligned} a_{4\varphi} &= 0 \\ b_{4\varphi} &= \frac{\frac{K_{t_a}}{\varphi_a} \cdot \sin \varepsilon_t}{\omega} \\ c_{4\varphi} &= \frac{K_{t_a}}{\varphi_a} \cdot \cos \varepsilon_t \end{aligned}$$

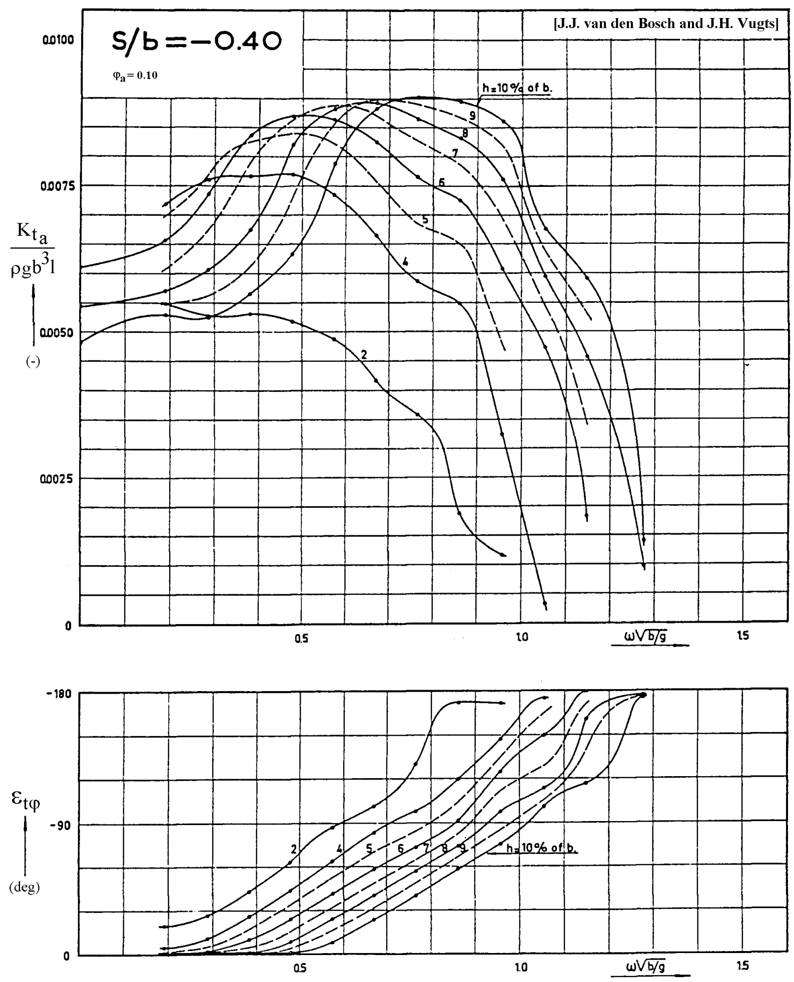

Figure 9.3: Experimental Data on Anti-Rolling Free-Surface Tanks

It is obvious that for an anti-rolling free-surface tank, build into a ship, it holds:

$$\phi_a = \varphi_a \qquad \text{and} \qquad \omega_e = \omega$$

So it can be written:

$$\begin{aligned} \phi &= \phi_a \cos(\omega_e t + \varepsilon_{\phi\zeta}) \\ K_t &= K_{t_a} \cos(\omega_e t + \varepsilon_{\phi\zeta} + \varepsilon_t) \end{aligned}$$

Then, an additional moment has to be added to the right hand side of the equations of motion for roll:

$$X_{tank_4} = a_{44_{tank}} \cdot \ddot{\phi} + b_{44_{tank}} \cdot \dot{\phi} + c_{44_{tank}} \cdot \phi$$

with:

$$\begin{aligned} a_{44_{tank}} &= 0 \\ b_{44_{tank}} &= \frac{\frac{K_{t_a}}{\phi_a} \cdot \sin \varepsilon_t}{\omega_e} \\ c_{44_{tank}} &= \frac{K_{t_a}}{\phi_a} \cdot \cos \varepsilon_t \end{aligned}$$

This holds that the anti-rolling coefficients $a_{44_{tank}}$, $b_{44_{tank}}$ and $c_{44_{tank}}$ have to be subtracted from the coefficients a_{44}, b_{44} and c_{44} in the left hand side of the equations of motion for roll.

9.2.3 Effect of Free-Surface Tanks

Figure 9.4 shows the significant reduction of the roll transfer functions and the significant roll amplitude of a trawler, being obtained by a free-surface tank.

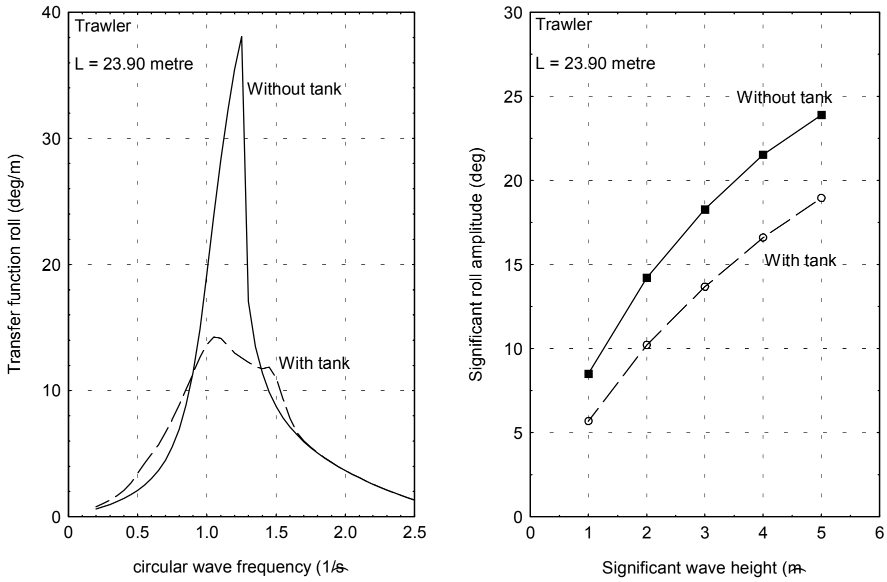

Figure 9.4: Effect of Free-Surface Tanks on Roll

9.3 Active Fin Stabilisers

To determine the effect of active fin stabilisers on ship motions, use has been made here of reports published by [Schmitke, 1978] and [Lloyd, 1989].
The ocillatory angle of the portside fin is given by:

$$\beta = \beta_a \cos(\omega_e t + \epsilon_{\beta\phi})$$

The exciting forces and moments, caused by an oscillating fin pair are given by:

$$\begin{aligned} X_{fin_2} &= a_{2_\beta} \cdot \ddot{\beta} + b_{2_\beta} \cdot \dot{\beta} + c_{2_\beta} \cdot \beta \\ X_{fin_4} &= a_{4_\beta} \cdot \ddot{\beta} + b_{4_\beta} \cdot \dot{\beta} + c_{4_\beta} \cdot \beta \\ X_{fin_6} &= a_{6_\beta} \cdot \ddot{\beta} + b_{6_\beta} \cdot \dot{\beta} + c_{6_\beta} \cdot \beta \end{aligned}$$

with:

$$\begin{aligned} a_{2_\beta} &= -2 \sin\gamma \cdot a_\beta \\ b_{2_\beta} &= -2 \sin\gamma \cdot b_\beta \end{aligned}$$

$$
\begin{aligned}
c_{2_\beta} &= -2\sin\gamma \cdot c_\beta \\
a_{4_\beta} &= +2(y_{b_{fin}}\cos\gamma + z_{b_{fin}}\sin\gamma)\cdot a_\beta \\
b_{4_\beta} &= +2(y_{b_{fin}}\cos\gamma + z_{b_{fin}}\sin\gamma)\cdot b_\beta \\
c_{4_\beta} &= +2(y_{b_{fin}}\cos\gamma + z_{b_{fin}}\sin\gamma)\cdot c_\beta \\
a_{6_\beta} &= -2x_{b_{fin}}\sin\gamma \cdot a_\beta \\
b_{6_\beta} &= -2x_{b_{fin}}\sin\gamma \cdot b_\beta \\
c_{6_\beta} &= -2x_{b_{fin}}\sin\gamma \cdot c_\beta
\end{aligned}
$$

and:

$$
\begin{aligned}
a_\beta &= \frac{1}{2}\rho s_{fin}\cdot\left(\frac{c_{fin}}{2}\right)^3\cdot\pi \\
b_\beta &= \frac{1}{2}\rho V A_{fin}\frac{c_{fin}}{2}\cdot\left(\pi + \left(\frac{\partial C_L}{\partial\alpha}\right)_{fin}\cdot C(k)\right) \\
c_\beta &= \frac{1}{2}\rho V^2 A_{fin}\cdot\left(\frac{\partial C_L}{\partial\alpha}\right)_{fin}\cdot C(k)
\end{aligned}
$$

In here:

$$
\begin{aligned}
\gamma &= \text{angle of port fin} \\
\left(\frac{\partial C_L}{\partial\alpha}\right)_{fin} &= \text{lift curve slope of fin} \\
C(k) &= \text{circulation delay function} \\
k &= \frac{\omega_e c_r}{2V} = \text{reduced frequency} \\
A_{fin} &= \text{projected fin area} \\
s_{fin} &= \text{span of fin} \\
c_{fin} &= \text{mean chord of fin} \\
x_{b_{fin}} &= x_b\text{-coordinate of the centroid of fin forces} \\
y_{b_{fin}} &= y_b\text{-coordinate of the centroid of fin forces} \\
z_{b_{fin}} &= z_b\text{-coordinate of the centroid of fin forces}
\end{aligned}
$$

The nominal lift curve slope of a fin profile in a uniform flow is approximated by:

$$
\frac{\partial C_L}{\partial\alpha} = \frac{1.80\cdot\pi\cdot(AR_E)}{1.80+\cos\Lambda\cdot\sqrt{\frac{(AR_E)^2}{\cos^4\Lambda}+4.0}}
$$

with:

$$
\begin{aligned}
\Lambda &= \text{sweep angle of fin profile} \\
(AR_E) &= \text{effective aspect ratio of fin profile}
\end{aligned}
$$

Of normal fins, the sweep angle of the fin profile is zero, so Λ=0 or $\cos\Lambda$=1.
The fin acts in the boundary layer of the ship, which will reduce the lift. This effect is translated into a reduced lift curve slope of the fin.

The velocity distribution in the hull boundary layer is estimated by the following two equations:

$$\begin{aligned} V(\delta) &= V \cdot \sqrt[7]{\frac{\delta}{\delta_{BL}}} && \text{with: } \delta < \delta_{BL} \\ \delta_{BL} &= 0.377 \cdot x_{fin} \cdot R_x^{-0.2} && \text{with: } R_x = \frac{V \cdot x}{\nu} \end{aligned}$$

in which:

$$\begin{aligned} V(\delta) &= \text{flow velocity inside boundary layer} \\ V &= \text{forward ship speed} \\ \delta &= \text{normal distance from hull} \\ \delta_{BL} &= \text{thickness of boundary layer} \\ x_{fin} &= \text{distance aft of forward perpendicular of fin} \\ R_x &= \text{local Reynolds number} \\ \nu &= \text{kinematic density of fluid} \end{aligned}$$

The kinematic viscosity of seawater can be found from the water temperature T in degrees centigrade by:

$$\nu \cdot 10^6 = \frac{1.78}{1.0 + 0.0336 \cdot T + 0.000221 \cdot T^2} \quad \text{m}^2/\text{s}$$

It is assumed here that the total lift of the fin can be found from:

$$\frac{1}{2}\rho C_L \int_0^{s_{fin}} V^2(\delta) \cdot c(\delta) \cdot d\delta = \frac{1}{2}\rho C_{L_{fin}} \cdot V^2 \cdot A_{fin}$$

where $c(\delta)$ is the chord at spanwise-location δ.
For rectangular fins, this is simply an assumption of a uniform loading.
Because:

$$c(\delta) = c_{r_{fin}} - (c_{r_{fin}} - c_{t_{fin}}) \cdot \frac{\delta}{s_{fin}}$$

in which:

$$\begin{aligned} c_{r_{fin}} &= \text{root chord of fin} \\ c_{t_{fin}} &= \text{tip chord of fin} \\ c_{fin} &= \frac{c_{r_{fin}} + c_{t_{fin}}}{2} \text{ mean chord of fin} \end{aligned}$$

the correction to the lift curve slope is:

$$E_{BL} = \frac{c_{r_{fin}}}{c_{fin}} \cdot \left(1 - \frac{2\delta_{BL}}{9 s_{fin}}\right) - \frac{c_{r_{fin}} - c_{t_{fin}}}{2 c_{fin}} \cdot \left(1 - \frac{\delta_{BL}^2}{8 s_{fin}}\right)$$

Then the corrected lift curve slope of the fin is:

$$\left(\frac{\partial C_L}{\partial \alpha}\right)_{fin} = E_{BL} \cdot \frac{1.80 \cdot \pi \cdot (AR_E)_{fin}}{1.80 + \sqrt{(AR_E)_{fin}^2 + 4.0}}$$

Generally a fin is mounted close to the hull, so the effective aspect ratio is about twice the geometric aspect ratio:

$$(AR_E)_{fin} = 2 \cdot (AR)_{fin} = 2 \cdot \frac{s_{fin}}{c_{fin}}$$

9.4 Active Rudder Stabilizers

To determine the effect of rudder stabilizers on ship motions, use has been made of reports published by [Lloyd, 1989] and [Schmitke, 1978].
The oscillatory rudder angle is given by:

$$\delta = \delta_a \cos(\omega_e t + \varepsilon_{\delta\phi})$$

with δ is positive in a counter-clockwise rotation of the rudder.
So, a positive δ results in a positive side force, a positive roll moment and a negative yaw moment.
The exciting forces and moments, caused by this oscillating rudder are given by:

$$\begin{aligned}
X_{r_2} &= a_{2_\delta} \cdot \ddot{\delta} + b_{2_\delta} \cdot \dot{\delta} + c_{2_\delta} \cdot \delta \\
X_{r_4} &= a_{4_\delta} \cdot \ddot{\delta} + b_{4_\delta} \cdot \dot{\delta} + c_{4_\delta} \cdot \delta \\
X_{r_6} &= a_{6_\delta} \cdot \ddot{\delta} + b_{6_\delta} \cdot \dot{\delta} + c_{6_\delta} \cdot \delta
\end{aligned}$$

with:

$$\begin{aligned}
a_{2_\delta} &= +a_\delta \\
b_{2_\delta} &= +b_\delta \\
c_{2_\delta} &= +c_\delta \\
a_{4_\delta} &= -z_{b_{rudder}} \cdot a_\delta \\
b_{4_\delta} &= -z_{b_{rudder}} \cdot b_\delta \\
c_{4_\delta} &= -z_{b_{rudder}} \cdot c_\delta \\
a_{6_\delta} &= +x_{b_{rudder}} \cdot a_\delta \\
b_{6_\delta} &= +x_{b_{rudder}} \cdot b_\delta \\
c_{6_\delta} &= +x_{b_{rudder}} \cdot c_\delta
\end{aligned}$$

and:

$$\begin{aligned}
a_\delta &= \frac{1}{2}\rho s_{rudder} \cdot \left(\frac{c_{rudder}}{2}\right)^3 \cdot \pi \\
b_\delta &= \frac{1}{2}\rho V_{rudder} \cdot A_{rudder} \cdot \frac{c_{rudder}}{2} \cdot \left(\pi + \left(\frac{\partial C_L}{\partial \alpha}\right)_{rudder} \cdot C(k)\right) \\
c_\delta &= \frac{1}{2}\rho V_{rudder}^2 \cdot A_{rudder} \cdot \left(\frac{\partial C_L}{\partial \alpha}\right)_{rudder} \cdot C(k)
\end{aligned}$$

In here:

$$\begin{aligned}
V_{rudder} \approx 1.125 \cdot V &= \text{equivalent flow velocity at rudder} \\
\left(\frac{\partial C_L}{\partial \alpha}\right)_{rudder} &= \text{lift curve slope of rudder} \\
C(k) &= \text{circulation delay function} \\
k = \frac{\omega_e \cdot c_{rudder}}{2V} &= \text{reduced frequency} \\
A_{rudder} &= \text{projected area of rudder}
\end{aligned}$$

$$
\begin{aligned}
s_{rudder} &= \text{span of rudder} \\
c_{rudder} &= \text{mean chord of rudder} \\
x_{b_{rudder}} &= x_b\text{-coordinate of centroid of rudder forces} \\
z_{b_{rudder}} &= z_b\text{-coordinate of centroid of rudder forces}
\end{aligned}
$$

The lift curve slope of the rudder is approximated by:

$$\left(\frac{\partial C_L}{\partial \alpha}\right)_{rudder} = \frac{1.80 \cdot \pi \cdot (AR_E)_{rudder}}{1.80 + \sqrt{(AR_E)^2_{rudder} + 4.0}}$$

Generally a rudder is not mounted close to the hull, so the effective aspect ratio is equal to the geometric aspect ratio:

$$(AR_E)_{rudder} = (AR)_{rudder} = \frac{s_{rudder}}{c_{rudder}}$$

Chapter 10

External Linear Springs

Suppose a linear spring connected to point P on the ship.

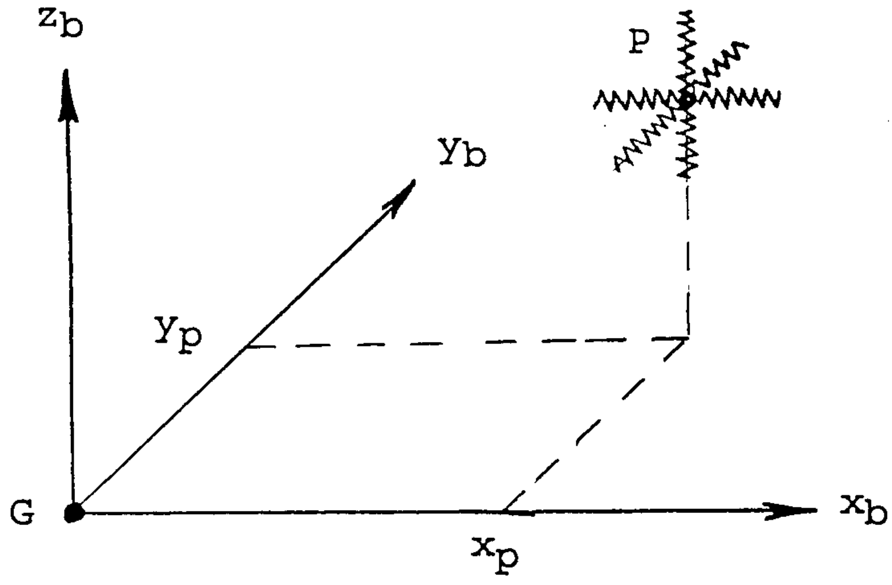

Figure 10.1: Coordinate System of Springs

The harmonic longitudinal, lateral and vertical displacements of a certain point P on the ship are given by:

$$
\begin{aligned}
x(P) &= x - y_p \cdot \psi + z_p \cdot \theta \\
y(P) &= y + x_p \cdot \psi - z_p \cdot \phi \\
z(P) &= z - x_p \cdot \theta + y_p \cdot \phi
\end{aligned}
$$

The linear spring coefficients in the three directions in a certain point P are defined by $(C_{p_x}, C_{p_y}, C_{p_z})$. The units of these coefficients are N/m or kN/m.

[0]J.M.J. Journée, *"Theoretical Manual of SEAWAY, Release 4.19"*, Report 1216a, February 2001, Ship Hydromechanics Laboratory, Delft University of Technology, Mekelweg 2, 2628 CD Delft, The Netherlands. For updates see web site: http://dutw189.wbmt.tudelft.nl/~johan or http://www.shipmotions.nl.

10.1 External Loads

The external forces and moments, caused by these linear springs, acting on the ship are given by:

$$\begin{aligned}
X_{s_1} &= -C_{p_x} \cdot (x - y_p \cdot \psi + z_p \cdot \theta) \\
X_{s_2} &= -C_{p_y} \cdot (y + x_p \cdot \psi - z_p \cdot \phi) \\
X_{s_3} &= -C_{p_z} \cdot (z - x_p \cdot \theta + y_p \cdot \phi) \\
X_{s_4} &= -X_{s_2} \cdot z_p \ + X_{s_3} \cdot y_p \\
X_{s_5} &= +X_{s_1} \cdot z_p \ - X_{s_3} \cdot x_p \\
X_{s_6} &= -X_{s_1} \cdot y_p \ + X_{s_2} \cdot x_p
\end{aligned}$$

10.2 Additional Coefficients

After a change of sign, this results into the following coefficients Δc_{ij}, which have to be added to the restoring spring coefficients c_{ij} of the hydromechanical loads in the left hand side of the equations of motions.

- Surge:

$$\begin{aligned}
\Delta c_{11} &= +C_{p_x} \\
\Delta c_{12} &= 0 \\
\Delta c_{13} &= 0 \\
\Delta c_{14} &= 0 \\
\Delta c_{15} &= +C_{p_x} \cdot z_p \\
\Delta c_{16} &= -C_{p_x} \cdot y_p
\end{aligned}$$

- Sway:

$$\begin{aligned}
\Delta c_{21} &= 0 \\
\Delta c_{22} &= +C_{p_y} \\
\Delta c_{23} &= 0 \\
\Delta c_{24} &= -C_{p_y} \cdot z_p \\
\Delta c_{25} &= 0 \\
\Delta c_{26} &= +C_{p_y} \cdot x_p
\end{aligned}$$

- Heave:

$$\begin{aligned}
\Delta c_{31} &= 0 \\
\Delta c_{32} &= 0 \\
\Delta c_{33} &= +C_{p_z} \\
\Delta c_{34} &= +C_{p_z} \cdot y_p \\
\Delta c_{35} &= -C_{p_z} \cdot x_p \\
\Delta c_{36} &= 0
\end{aligned}$$

- Roll:

$$
\begin{aligned}
\Delta c_{41} &= 0 \\
\Delta c_{42} &= -C_{p_y} \cdot z_p \\
\Delta c_{43} &= +C_{p_z} \cdot y_p \\
\Delta c_{44} &= +C_{p_y} \cdot z_p^2 + C_{p_z} \cdot y_p^2 \\
\Delta c_{45} &= -C_{p_z} \cdot x_p \cdot y_p \\
\Delta c_{46} &= -C_{p_y} \cdot x_p \cdot z_p
\end{aligned}
$$

- Pitch:

$$
\begin{aligned}
\Delta c_{51} &= +C_{p_x} \cdot z_p \\
\Delta c_{52} &= 0 \\
\Delta c_{53} &= -C_{p_z} \cdot x_p \\
\Delta c_{54} &= -C_{p_z} \cdot x_p \cdot y_p \\
\Delta c_{55} &= +C_{p_x} \cdot z_p^2 + C_{p_z} \cdot x_p^2 \\
\Delta c_{56} &= -C_{p_x} \cdot y_p \cdot z_p
\end{aligned}
$$

- Yaw:

$$
\begin{aligned}
\Delta c_{61} &= -C_{p_x} \cdot y_p \\
\Delta c_{62} &= +C_{p_y} \cdot x_p \\
\Delta c_{63} &= 0 \\
\Delta c_{64} &= -C_{p_y} \cdot x_p \cdot z_p \\
\Delta c_{65} &= -C_{p_x} \cdot y_p \cdot z_p \\
\Delta c_{66} &= +C_{p_x} \cdot y_p^2 + C_{p_y} \cdot x_p^2
\end{aligned}
$$

In case of several springs, a linear superposition of the coefficients can be used.

When using linear springs, generally 12 sets of coupled equations with the in and out of phase terms of the motions have to be solved. Because of these springs, the surge, heave and pitch motions will be coupled then with the sway, roll and yaw motions.

10.3 Linearized Mooring Coefficients

Figure 10.2 shows an example of results of static catenary line calculations, see for instance [Korkut and Hebert, 1970], for an anchored platform.
Figure 10.2-a shows the platform anchored by two anchor lines of chain at 100 m water depth. Figure 10.2-b shows the horizontal forces at the suspension points of both anchor lines as a function of the horizontal displacement of the platform. Finally, figure 10.2-c shows the relation between the total horizontal force on the platform and its horizontal displacement.
This figure shows clearly the non-linear relation between the horizontal force on the platform and its horizontal displacement.

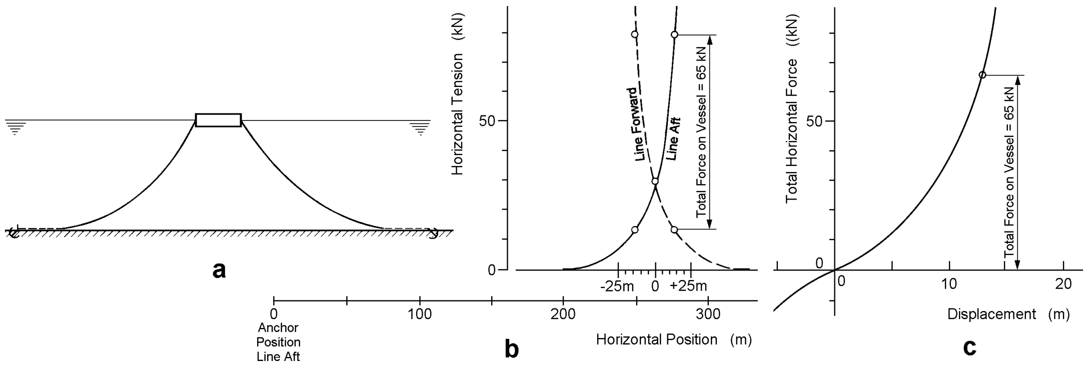

Figure 10.2: Horizontal Forces on a Floating Structure as a Function of Surge Displacements

A linear(ized) spring coefficient, to be used in frequency domain computations, can be obtained from figure 10.2-c by determining an average restoring spring coefficient, C_{p_x}, in the surge displacement region:

$$C_{p_x} = Mean \left\{ \frac{\text{Total Force}}{\text{Displacement}} \right\}$$

Chapter 11

Added Resistances due to Waves

A ship moving forward in a wave field will generate "two sets of waves": waves associated with forward speed through still water and waves associated with its vertical relative motion response to waves. Since both wave patterns dissipate energy, it is logical to conclude that a ship moving through still water will dissipate less energy than one moving through waves. The extra wave-induced loss of energy can be treated as an added propulsion resistance. Figure 11.1 shows the resistance in regular waves as a function of the time: a constant part due the calm water resistance and an oscillating part due to the motions of the ship, relative to the incoming regular waves. The time-averaged part of the increase of resistance is called: the added resistance due to waves, R_{aw}.

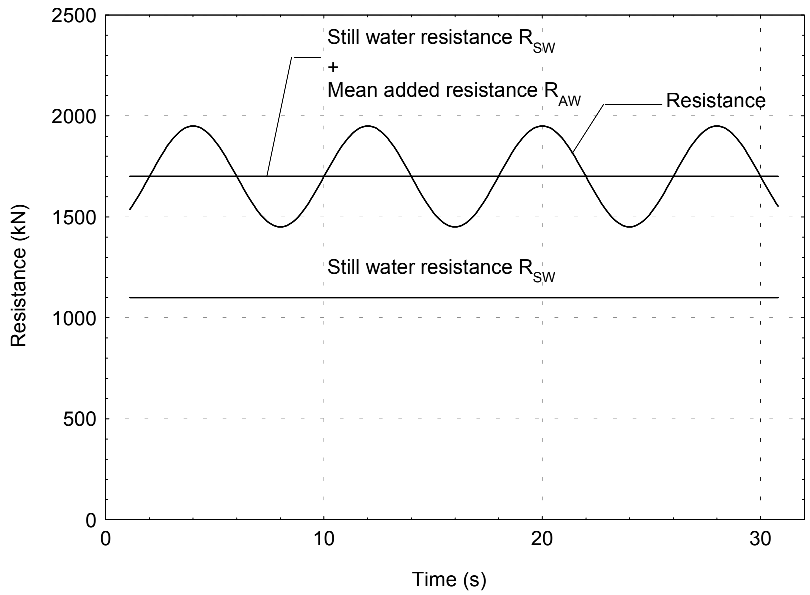

Figure 11.1: Increase of Resistance in Regular Waves

Two theoretical methods have been used for the estimation of the time-averaged added resistance of a ship due to the waves and the resulting ship motions:

[0] J.M.J. Journée, *"Theoretical Manual of SEAWAY, Release 4.19"*, Report 1216a, February 2001, Ship Hydromechanics Laboratory, Delft University of Technology, Mekelweg 2, 2628 CD Delft, The Netherlands. For updates see web site: http://dutw189.wbmt.tudelft.nl/~johan or http://www.shipmotions.nl.

- a radiated wave energy method, as introduced by [Gerritsma and Beukelman, 1972], suitable for head to beam waves.
- an integrated pressure method, as introduced by [Boese, 1970], suitable for all wave directions.

Because of the added resistance of a ship due to the waves is proportional to the relative motions squared, its inaccuracy will be gained strongly by inaccuracies in the predicted motions.
The transfer function of the mean added resistance is presented as:

$$R''_{aw} = \frac{R_{aw}}{\zeta_a^2}$$

In a non-dimensional way the transfer function of the mean added resistance is presented as:

$$R''_{aw} = \frac{R_{aw}}{\rho g \zeta_a^2 B^2 / L}$$

in which:

L = length between perpendiculars
B = maximum breadth of the waterline

Both methods will be described here.

11.1 Radiated Energy Method

The radiated wave energy during one period of oscillation of a ship in regular waves is defined by [Gerritsma and Beukelman, 1972] as:

$$P = \int_0^{T_e} \int_L b'_{33} \cdot V_z^{*2} \cdot dx_b \cdot dt$$

in which:

b'_{33} = hydrodynamic damping coefficient of the vertical motion of the cross section
V_z^* = vertical average velocity of the water particles, relative to the cross sections
T_e = period of vertical oscillation of the cross section

The speed dependent hydrodynamic damping coefficient for the vertical motion of a cross section is defined here as shown before:

$$b'_{33} = N'_{33} - V \cdot \frac{dM'_{33}}{dx_b}$$

The harmonic vertical relative velocity of a point on the ship with respect to the water particles is defined by:

$$\begin{aligned} V_z &= \dot{\zeta}'_{w_3} - \frac{D}{Dt}\left\{z - x_b \cdot \theta + y_b \cdot \phi\right\} \\ &= \dot{\zeta}'_{w_3} - \left(\dot{z} - x_b \cdot \dot{\theta} + V \cdot \theta + y_b \cdot \dot{\phi}\right) \end{aligned}$$

For a cross section of the ship, an equivalent harmonic vertical relative velocity has to be found.
This equivalent relative velocity is defined by:

$$\begin{aligned} V_z^* &= \dot{\zeta}_{w_3}^* - \left(\dot{z} - x_b \cdot \dot{\theta} + V \cdot \theta\right) \\ &= V_{z_a}^* \cdot \cos(\omega_e t + \varepsilon_{V_z^* \zeta}) \end{aligned}$$

With this the radiated energy during one period of oscillation is given by:

$$P = \frac{\pi}{\omega_e} \int\limits_L \left(N'_{33} - V \cdot \frac{dM'_{33}}{dx_b}\right) \cdot {V_{z_a}^*}^2 \cdot dx_b$$

To maintain a constant forward ship speed, this energy should be delivered by the ship's propulsion plant. A mean added resistance R_{aw} has to be gained.
The energy delivered to the surrounding water is given by:

$$\begin{aligned} P &= R_{aw} \cdot \left(v - \frac{c}{\cos\mu}\right) \cdot T_e \\ &= R_{aw} \cdot \frac{2\pi}{-k\cos\mu} \end{aligned}$$

From this the transfer function of the mean added resistance according to Gerritsma and Beukelman can be found:

$$\frac{R_{aw}}{\zeta_a^2} = \frac{-k\cos\mu}{2\omega_e} \cdot \int\limits_L \left(N'_{33} - V \cdot \frac{dM'_{33}}{dx_b}\right) \cdot \frac{{V_{z_a}^*}^2}{\zeta_a^2} \cdot dx_b$$

This method gives good results in head to beam waves. However, in following waves this method fails.
When the wave speed in following waves approaches the ship speed the frequency of encounter in the denominator tends to zero. At these low frequencies, the potential sectional mass is very high and the potential sectional damping is almost zero. The damping multiplied with the relative velocity squared in the nominator does not tend to zero, as fast as the frequency of encounter. This is caused by the presence of a natural frequency for heave and pitch at this low ω_e, so a high motion peak can be expected. This results into extreme positive and negative added resistances.

11.2 Integrated Pressure Method

[Boese, 1970] calculates the added resistance by integrating the longitudinal components of the oscillating pressures on the wetted surface of the hull. A second small contribution of the longitudinal component of the vertical hydrodynamic and wave forces has been added.
The wave elevation is given by:

$$\zeta = \zeta_a \cos(\omega_e t - kx_b \cos\mu - ky_b \sin\mu)$$

The pressure in the undisturbed waves is given by:

$$\begin{aligned} p &= -\rho g z + \rho g \cdot \frac{\cosh k(h+z_b)}{\cosh(kh)} \cdot \zeta \\ &= -\rho g z + \rho g \cdot \frac{\cosh k(h+z_b)}{\cosh(kh)} \cdot \zeta_a \cos(\omega_e t - kx_b \cos\mu - ky_b \sin\mu) \end{aligned}$$

The horizontal force on an oscillating cross section is given by:

$$\begin{aligned} f(x_b,t) &= \int\limits_{-D_s+z_x}^{\zeta} p \cdot dz_b \\ &= \rho g \cdot \left(\frac{-\zeta^2 + (-D_s+z_x)^2}{2} + \frac{\zeta}{\tanh[kh]} \cdot (\zeta + D_s - z_x) \right) \end{aligned}$$

with: $z_x = z \; - x_b\theta$.
As the mean added resistance during one period will be calculated, the constant term and the first harmonic term can be ignored. So:

$$f^*(x_b,t) = \rho g \cdot \left(\frac{-\zeta^2 + z_x^2}{2} + \frac{\zeta \cdot (\zeta - z_x)}{\tanh[kh]} \right)$$

The vertical relative motion is defined by $s = \zeta - z_x$, so:

$$f^*(x_b,t) = \rho g \cdot \left(\frac{-\zeta^2 + z_x^2}{2} + \frac{\zeta \cdot s}{\tanh[kh]} \right)$$

The average horizontal force on a cross section follows from:

$$\begin{aligned} \overline{f^*}(x_b) &= \int\limits_0^{T_e} f^*(x_b,t) \cdot dt \\ &= \frac{\rho g \zeta_a^2}{4} \cdot \left(-1 + \frac{z_{x_a}^2}{\zeta_a^2} + \frac{2s_a \cdot \cos(-kx_b \cos\mu - \varepsilon_{s\zeta})}{\zeta_a \cdot \tanh[kh]} \right) \end{aligned}$$

The added resistance due to this force is:

$$\begin{aligned} R_{aw_1} &= 2 \int\limits_L \overline{f^*}(x_b) \cdot \left(-\frac{dy_w}{dx_b} \right) \cdot dx_b \\ &= \frac{\rho g \zeta_a^2}{2} \int\limits_L \left(1 - \frac{z_{x_a}^2}{\zeta_a^2} - \frac{2s_a \cdot \cos(-kx_b \cos\mu - \varepsilon_{s\zeta})}{\zeta_a \cdot \tanh[kh]} \right) \cdot \frac{dy_w}{dx_b} \cdot dx_b \end{aligned}$$

For deep water, this part of the mean added resistance reduces to:

$$R_{aw_1} = \frac{-\rho g}{2} \int\limits_L s_a^2 \cdot \frac{dy_w}{dx_b} \cdot dx_b \qquad \text{(as given by Boese for deep water)}$$

The integrated vertical hydromechanical and wave forces in the shipborne axis system varies not only in time but also in direction with the pitch angle.

From this follows a second contribution to the mean added resistance:

$$R_{aw_2} = \frac{-1}{T_e} \int_0^{T_e} (Z_h(t) + Z_w(t)) \cdot \theta(t) \cdot dt$$
$$= \frac{-1}{T_e} \int_0^{T_e} \rho\nabla \cdot \ddot{z}(t) \cdot \theta(t) \cdot dt$$

For this second contribution can be written:

$$R_{aw_2} = \frac{1}{2}\rho\nabla \cdot \omega_e^2 \cdot z_a \cdot \theta_a \cdot \cos(\varepsilon_{z\zeta} - \varepsilon_{\theta\zeta})$$

So the transfer function of the total mean added resistance according to Boese is given by:

$$\frac{R_{aw_2}}{\zeta_a^2} = \frac{1}{2}\rho\nabla \cdot \omega_e^2 \cdot \frac{z_a}{\zeta_a} \cdot \frac{\theta_a}{\zeta_a} \cdot \cos(\varepsilon_{z\zeta} - \varepsilon_{\theta\zeta})$$
$$+\frac{1}{2}\rho g \int_L \left(1 - \frac{z_{x_a}^2}{\zeta_a^2} - \frac{2s_a \cdot \cos(-kx_b \cos\mu - \varepsilon_{s\zeta})}{\zeta_a \cdot \tanh[kh]}\right) \cdot \frac{dy_w}{dx_b} \cdot dx_b$$

11.3 Comparison of Results

Figure 11.2 shows an example of a comparison between computed and experimental data.

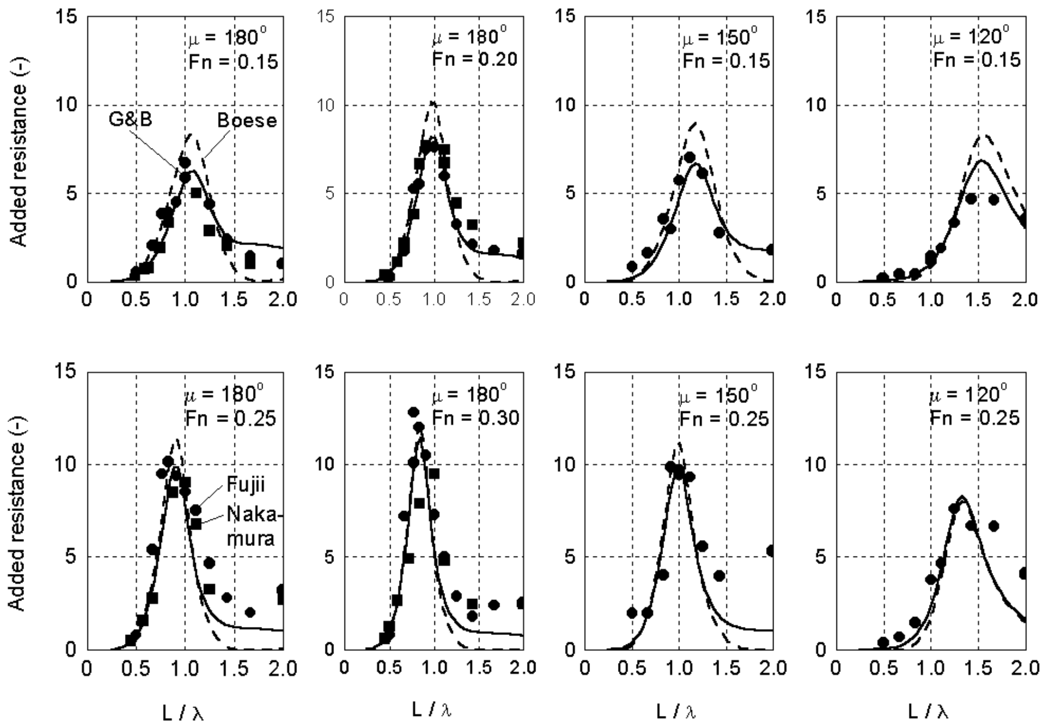

Figure 11.2: Added Resistance of the S-175 Containership Design

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Chapter 12

Bending and Torsional Moments

The axes system (of which the hydrodynamic sign convention differs from that commonly used in structural engineering) and the internal load definitions are given in figure 12.1.

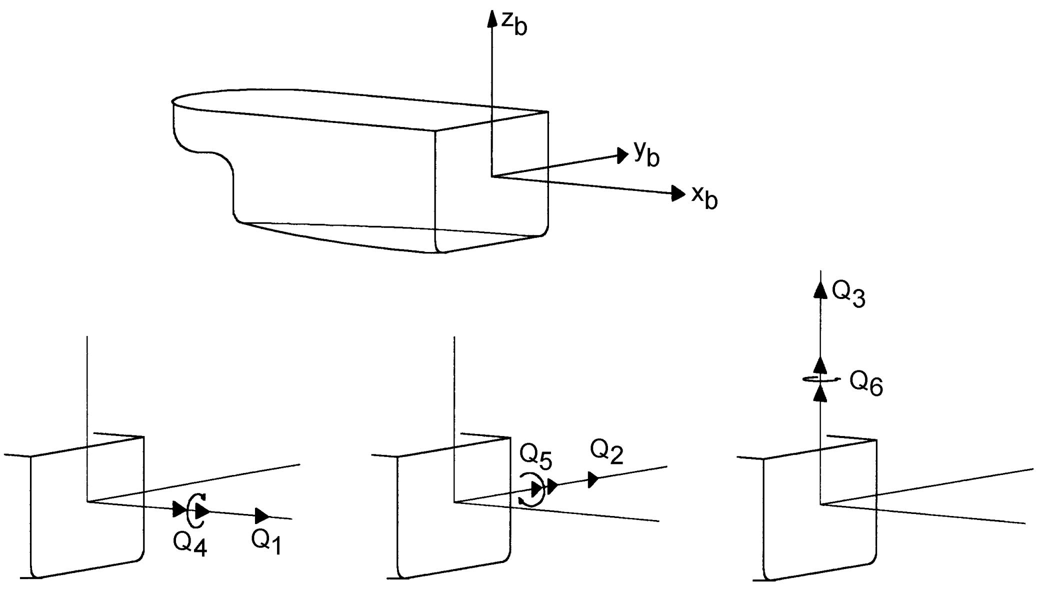

Figure 12.1: Axis System and Internal Load Definitions

To obtain the vertical and lateral shear forces and bending moments and the torsional moments the following information over a length L_m on the solid mass distribution of the ship including its cargo is required:

[0]J.M.J. Journée, *"Theoretical Manual of SEAWAY, Release 4.19"*, Report 1216a, February 2001, Ship Hydromechanics Laboratory, Delft University of Technology, Mekelweg 2, 2628 CD Delft, The Netherlands. For updates see web site: http://dutw189.wbmt.tudelft.nl/~johan or http://www.shipmotions.nl.

$m'(x_b)$: distribution over the ship length of the solid mass of the ship per unit length
$z'_m(x_b)$: distribution over the ship length of the vertical z_b values of the centre of gravity of the solid mass of the ship per unit length
$k'_{xx}(x_b)$: distribution over the ship length of the radius of inertia of the solid mass of the ship per unit length, about a horizontal longitudinal axis through the centre of gravity

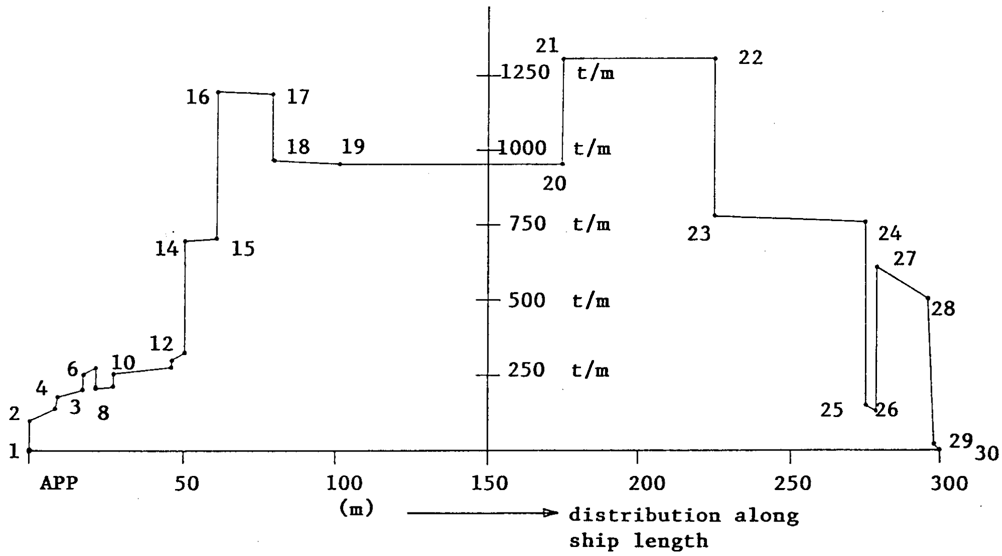

Figure 12.2: Distribution of Solid Mass

The input values for the calculation of shear forces and bending and torsional moments are often more or less inaccurate. Mostly small adaptions are necessary, for instance to avoid a remaining calculated bending moment at the forward end of the ship.
The total mass of the ship is found by an integration of the mass per unit length:

$$m = \int_{L_m} m'(x_b) \cdot dx_b$$

It is obvious that this integrated mass should be equal to the mass of displacement, calculated from the underwater hull form:

$$m = \rho \nabla$$

Both terms will be calculated from independently derived data, so small deviations are possible. A proportional correction of the masses per unit length $m'(x_b)$ can be used. Then $m'(x_b)$ will be replaced by:

$$m'(x_b) \cdot \frac{\rho \nabla}{m}$$

The longitudinal position of the centre of gravity is found from the distribution of the mass per unit length:

$$x_G = \frac{1}{m} \int_{L_m} m'(x_b) \cdot x_b \cdot dx_b$$

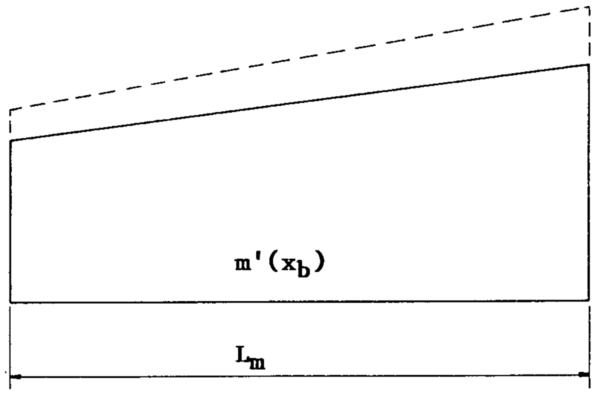

Figure 12.3: Mass Correction for Buoyancy

An equal longitudinal position of the ship's centre of buoyancy x_B is required, so:

$$x_G = x_B$$

Again, because of independently derived data, a small deviation is possible.
Then, for instance, $m'(x_b)$ can be replaced by $m'(x_b) + c(x_b)$, with:

$$\begin{array}{rcll}
c(x_b) & = & -c_1 \cdot (x_b - x_A - 0) & \text{for:} \quad 0 < x_b - x_A < \dfrac{L_m}{4} \\
c(x_b) & = & +c_1 \cdot \left(x_b - x_A - \dfrac{L_m}{2} \right) & \text{for:} \quad \dfrac{L_m}{4} < x_b - x_A < \dfrac{3L_m}{4} \\
c(x_b) & = & -c_1 \cdot (x_b - x_A - L_m) & \text{for:} \quad \dfrac{3L_m}{4} < x_b - x_A < L_m
\end{array}$$

with:

$$c_1 = \frac{32 \cdot \rho \nabla \cdot (x_B - x_G)}{L_m^3}$$

In here:

x_A = x_b-coordinate of aftmost part of mass distribution
L_m = total length of mass distribution

For relatively slender bodies, the longitudinal gyradius of the mass can be found from the distribution of the mass per unit length:

$$k_{yy}^2 = \frac{1}{m} \int\limits_{L_m} m'(x_b) \cdot x_b^2 \cdot dx_b$$

It can be desirable to change the mass distribution in such a way that a certain required longitudinal gyradius $k_{yy}(new)$ or $k_{zz}(new)$ will be achieved, without changing the total mass or the position of its centre of gravity.

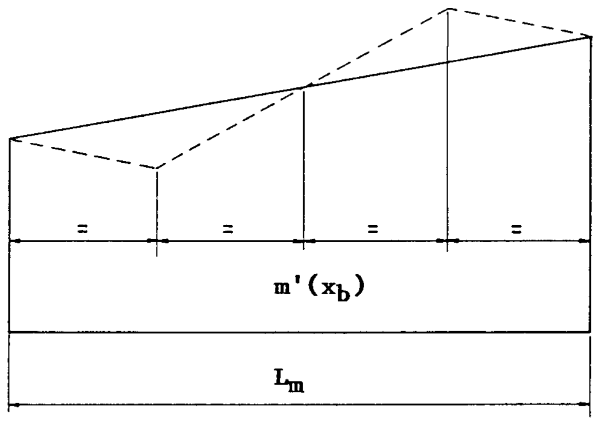

Figure 12.4: Mass Correction for Center of Buoyancy

Then, for instance, $m'(x_b)$ can be replaced by $m'(x_b) + c(x_b)$, with:

$$
\begin{array}{lllll}
c(x_b) & = & +c_2 \cdot (x_b - x_A - 0) & \text{for:} & 0 < x_b - x_A < \frac{L_m}{8} \\
c(x_b) & = & -c_2 \cdot \left(x_b - x_A - \frac{2L_m}{8} \right) & \text{for:} & \frac{L_m}{8} < x_b - x_A < \frac{3L_m}{8} \\
c(x_b) & = & +c_2 \cdot \left(x_b - x_A - \frac{4L_m}{8} \right) & \text{for:} & \frac{3L_m}{8} < x_b - x_A < \frac{4L_m}{8} \\
c(x_b) & = & -c_2 \cdot \left(x_b - x_A - \frac{4L_m}{8} \right) & \text{for:} & \frac{4L_m}{8} < x_b - x_A < \frac{5L_m}{8} \\
c(x_b) & = & +c_2 \cdot \left(x_b - x_A - \frac{6L_m}{8} \right) & \text{for:} & \frac{5L_m}{8} < x_b - x_A < \frac{7L_m}{8} \\
c(x_b) & = & -c_2 \cdot (x_b - x_A - L_m) & \text{for:} & \frac{7L_m}{8} < x_b - x_A < L_m
\end{array}
$$

with:

$$c_2 = \frac{3204 \cdot \rho\nabla \cdot \left(k_{yy}^2(new) - k_{yy}^2(old) \right)}{9L_m^3}$$

In here:

x_A = x_b-coordinate of aftmost part of mass distribution
L_m = total length of mass distribution

The position in height of the centre of gravity is found from the distribution of the heights of the centre of gravity of the masses per unit length:

$$z_G = \frac{1}{m} \int_{L_m} m'(x_b) \cdot z'_m(x_b) \cdot dx_b$$

It is obvious that this value should be zero. If not so, this value has to be subtracted from $z'_m(x_b)$.
So, $z'_m(x_b)$ will be replaced by $z'_m(x_b) - z_G$.

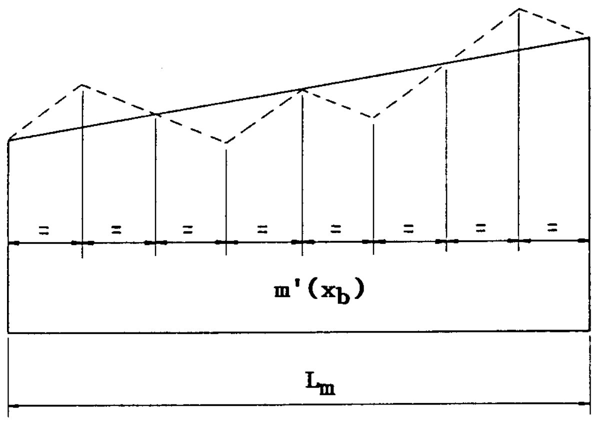

Figure 12.5: Mass Correction for Radius of Inertia

The transverse radius of inertia k_{xx} is found from the distribution of the radii of inertia of the masses per unit length:

$$k_{xx}^2 = \frac{1}{m} \int\limits_{L_m} m'(x_b) \cdot k'^2_{xx}(x_b) \cdot dx_b$$

If this value of k_{xx} differs from a required value $k_{xx}(new)$ of the gyradius, a proportional correction of the longitudinal distribution of the radii of inertia can be used:

$$k'_{xx}(x_b, new) = k'_{xx}(x_b, old) \cdot \frac{k_{xx}(new)}{k_{xx}(old)}$$

Consider a section of the ship with a length dx_b to calculate the shear forces and the bending and the torsional moments.

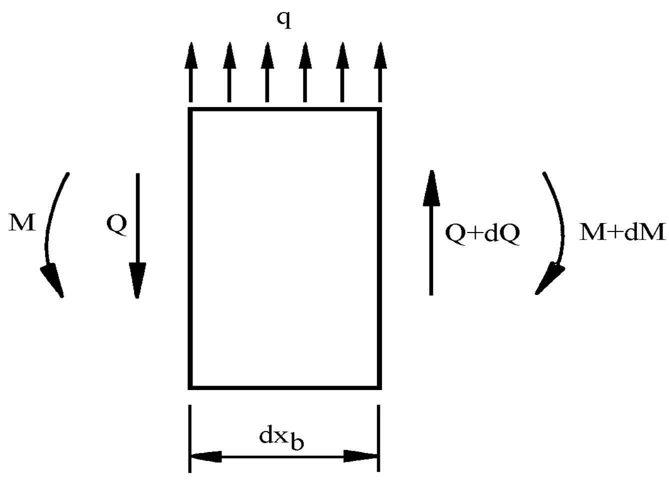

Figure 12.6: Loads on a Cross Section

When the disk is loaded by a load $q(x_b)$, this implies for the disk:

$$\begin{aligned} q(x_b) \cdot dx_b &= -dQ(x_b) \qquad \text{so:} \qquad \frac{dQ(x_b)}{dx_b} = -q(x_b) \\ Q(x_b) \cdot dx_b &= +dM(x_b) \qquad \text{so:} \qquad \frac{dM(x_b)}{dx_b} = +Q(x_b) \end{aligned}$$

in which:

$$\begin{aligned} Q(x_b) &= \text{shear force} \\ M(x_b) &= \text{bending moment} \end{aligned}$$

The shear force and the bending moment in a cross section x_1 follows from an integration of the loads from the aftmost part of the ship x_0 to this cross section x_1:

$$\begin{aligned} Q(x_1) &= -\int_{x_0}^{x_1} \frac{dQ(x_b)}{dx_b} \cdot dx_b \\ M(x_1) &= -\int_{x_0}^{x_1} Q(x_b) \cdot dx_b \\ &= +\int_{x_0}^{x_1} \left(\int_{x_0}^{x_b} \frac{dQ(x_b)}{dx_b} \cdot dx_b \right) \cdot dx_b \end{aligned}$$

So, the shear force $Q(x_1)$ and the bending moment $M(x_1)$ in a cross section can be expressed in the load $q(x_b)$ by the following integrals:

$$\begin{aligned} Q(x_1) &= -\int_{x_0}^{x_1} q(x_b) \cdot dx_b \\ M(x_1) &= +\int_{x_0}^{x_1} q(x_b) \cdot (x_1 - x_b) \cdot dx_b \\ &= +\int_{x_0}^{x_1} q(x_b) \cdot x_b \cdot dx_b - x_1 \cdot \int_{x_0}^{x_1} q(x_b) \cdot dx_b \end{aligned}$$

For the torsional moment an approach similar to the approach for the shear force can be used.
The load $q(x_b)$ consists of solid mass and hydromechanical terms. The ordinates of these terms will differ generally, so the numerical integrations of these two terms have to be carried out separately.

12.1 Still Water Loads

Consider the forces acting on a section of the ship with a length dxb.

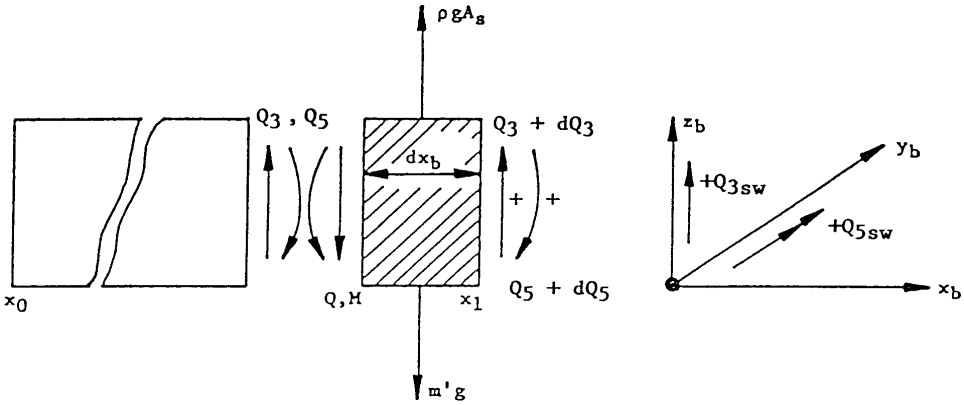

Figure 12.7: Still Water Loads on a Cross Section

According to Newton's second law of dynamics, the vertical forces on the unfastened disk of a ship in still water are given by:

$$(m' \cdot dx_b) \cdot (-g) = q_{3_{sw}}(x_b) \cdot dx_b$$

with:

$$q_{3_{sw}}(x_b) = \rho A_s g - m' g$$

So, the vertical shear force $Q_{3_{sw}}(x_1)$ and the bending moment $Q_{5_{sw}}(x_1)$ in still water in a cross section can be obtained from the vertical load $q_{3_{sw}}(x_b)$ by the following integrals:

$$\begin{aligned}
Q_{3_{sw}}(x_1) &= -\int_{x_0}^{x_1} q_{3_{sw}}(x_b) \cdot dx_b \\
Q_{5_{sw}}(x_1) &= +\int_{x_0}^{x_1} q_{3_{sw}}(x_b) \cdot x_b \cdot dx_b - x_1 \int_{x_0}^{x_1} q_{3_{sw}}(x_b) \cdot dx_b
\end{aligned}$$

For obtaining the dynamic parts of the vertical shear forces and the vertical bending moments in regular waves, reference is given to [Fukuda, 1962]. For the lateral mode and the roll mode a similar procedure can be followed. This will be shown at the following pages.

12.2 Lateral Dynamic Loads

Consider the forces acting on a section of the ship with a length dxb.
According to Newton's second law of dynamics, the harmonic lateral dynamic load per unit length on the unfastened disk is given by:

$$\begin{aligned}
q_2(x_b) = \quad & +X'_{h_2}(x_b) \ + X'_{w_2}(x_b) \\
& +\rho g A_s \phi \\
& -m'(x_b) \cdot \left(\ddot{y} \ + x_b \cdot \ddot{\psi} \ - z'_m \cdot \ddot{\phi} \ + g \cdot \phi \right)
\end{aligned}$$

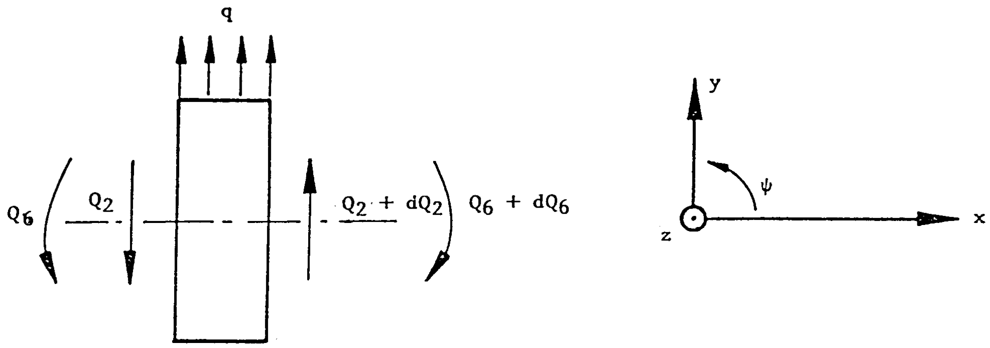

Figure 12.8: Lateral Loads on a Cross Section in Waves

The sectional hydromechanical load for sway is given by:

$$
\begin{aligned}
X'_{h_2} \quad = \quad & -a'_{22} \cdot \ddot{y} \; - b'_{22} \cdot \dot{y} \; - c'_{22} \cdot y \\
& -a'_{24} \cdot \ddot{\phi} \; - b'_{24} \cdot \dot{\phi} \; - c'_{24} \cdot \phi \\
& -a'_{26} \cdot \ddot{\psi} \; - b'_{26} \cdot \dot{\psi} \; - c'_{26} \cdot \psi
\end{aligned}
$$

with:

$$
\begin{aligned}
a'_{22} &= +M'_{22} \\
& \quad + \boxed{\frac{V}{\omega_e^2} \cdot \frac{dN'_{22}}{dx_b}} \\
b'_{22} &= +N'_{22} - V \cdot \frac{dM'_{22}}{dx_b} \\
c'_{22} &= 0 \\
a'_{24} &= +M'_{24} + \overline{OG} \cdot M'_{22} \\
& \quad + \boxed{\frac{V}{\omega_e^2} \cdot \frac{dN'_{24}}{dx_b} + \frac{V}{\omega_e^2} \cdot \overline{OG} \cdot \frac{dN'_{22}}{dx_b}} \\
b'_{24} &= +N'_{24} - V \cdot \frac{dM'_{24}}{dx_b} + \overline{OG} \cdot \left(N'_{22} - V \cdot \frac{dM'_{22}}{dx_b} \right) \\
c'_{24} &= 0 \\
a'_{26} &= +M'_{22} \cdot x_b + \frac{V}{\omega_e^2} \cdot \left(N'_{22} - V \cdot \frac{dM'_{22}}{dx_b} \right) \\
& \quad + \boxed{\frac{V}{\omega_e^2} \cdot N'_{22} + \frac{V}{\omega_e^2} \cdot \frac{dN'_{22}}{dx_b} \cdot x_b} \\
b'_{26} &= + \left(N'_{22} - V \cdot \frac{dM'_{22}}{dx_b} \right) \cdot x_b - 2V \cdot M'_{22} \\
& \quad + \boxed{\frac{V^2}{\omega_e^2} \cdot \frac{dN'_{22}}{dx_b}} \\
c'_{26} &= 0
\end{aligned}
$$

The sectional wave load for sway is given by:

$$\begin{aligned}
X'_{w_2} &= +M'_{22} \cdot \ddot{\zeta}^{*}_{w_2} \\
&\quad + \boxed{\frac{V}{\omega \cdot \omega_e} \cdot \frac{dN'_{22}}{dx_b} \cdot \ddot{\zeta}^{*}_{w_2}} \\
&\quad + \left(\boxed{\frac{\omega}{\omega_e}} \cdot N'_{22} - V \cdot \frac{dM'_{22}}{dx_b} \right) \cdot \dot{\zeta}^{*}_{w_2} \\
&\quad + X'_{FK_2} \\
&\quad + M'_{24} \cdot \ddot{\zeta}^{*}_{w_4} \\
&\quad + \boxed{\frac{V}{\omega \cdot \omega_e} \cdot \frac{dN'_{24}}{dx_b} \cdot \ddot{\zeta}^{*}_{w_4}} \\
&\quad + \left(\boxed{\frac{\omega}{\omega_e}} \cdot N'_{24} - V \cdot \frac{dM'_{24}}{dx_b} \right) \cdot \dot{\zeta}^{*}_{w_4}
\end{aligned}$$

The "Modified Strip Theory" includes the outlined terms. When ignoring the outlined terms the "Ordinary Strip Theory" is presented.
Then the harmonic lateral shear forces $Q_2(x_1)$ and the bending moments $Q_6(x_1)$ in waves in cross section x_1 can be obtained from the horizontal load $q_2(x_b)$ by the following integrals:

$$\begin{aligned}
Q_2(x_1) &= Q_{2_a} \cos(\omega_e t + \varepsilon_{Q_2\zeta}) \\
&= -\int_{x_0}^{x_1} q_2(x_b) \cdot dx_b \\
Q_6(x_1) &= Q_{6_a} \cos(\omega_e t + \varepsilon_{Q_6\zeta}) \\
&= +\int_{x_0}^{x_1} q_2(x_b) \cdot x_b \cdot dx_b \; - x_1 \int_{x_0}^{x_1} q_2(x_b) \cdot dx_b
\end{aligned}$$

12.3 Vertical Dynamic Loads

Consider the forces acting on a section of the ship with a length dxb.
According to Newton's second law of dynamics, the harmonic longitudinal and vertical dynamic loads per unit length on the unfastened disk are given by:

$$\begin{aligned}
q_1(x_b) &= +X'_{h_1}(x_b) \; + X'_{w_1}(x_b) \\
&\quad -m'(x_b) \cdot \left(\ddot{x} \; - \overline{bG} \cdot \ddot{\theta} \right) \\
q_3(x_b) &= +X'_{h_3}(x_b) \; + X'_{w_3}(x_b) \\
&\quad -m'(x_b) \cdot \left(\ddot{z} \; - x_b \cdot \ddot{\theta} \right)
\end{aligned}$$

The sectional hydromechanical load for surge is given by:

$$\begin{aligned}
X'_{h_1}(x_b) &= -a'_{11} \cdot \ddot{x} \; - b'_{11} \cdot \dot{x} \; - c'_{11} \cdot x \\
&\quad -a'_{13} \cdot \ddot{z} \; - b'_{13} \cdot \dot{z} \; - c'_{13} \cdot z \\
&\quad -a'_{15} \cdot \ddot{\theta} \; - b'_{15} \cdot \dot{\theta} \; - c'_{15} \cdot \theta
\end{aligned}$$

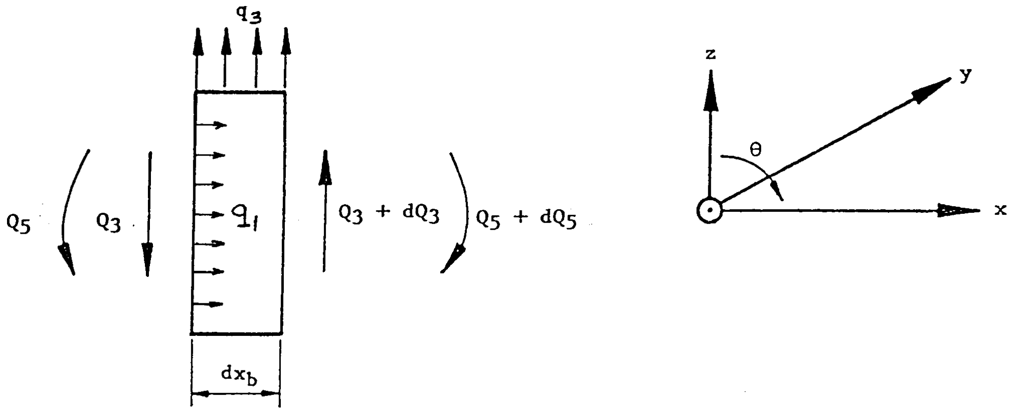

Figure 12.9: Vertical Loads on a Cross Section in Waves

with:

$$
\begin{aligned}
a'_{11} &= +M'_{11} \\
&\quad +\left|\frac{V}{\omega_e^2}\cdot\frac{dN'_{11}}{dx_b}\right| \\
b'_{11} &= +N'_{11} - V\cdot\frac{dM'_{11}}{dx_b} + b'_{11_V} \\
c'_{11} &= 0 \\
a'_{13} &= 0 \\
b'_{13} &= 0 \\
c'_{13} &= 0 \\
a'_{15} &= -M'_{11}\cdot\overline{bG} \\
&\quad +\left|\frac{-V}{\omega_e^2}\cdot\frac{dN'_{11}}{dx_b}\cdot\overline{bG}\right| \\
b'_{15} &= -\left(N'_{11} - V\cdot\frac{dM'_{11}}{dx_b}\right)\cdot\overline{bG} - b'_{11_V}\cdot\overline{bG} \\
c'_{15} &= 0
\end{aligned}
$$

The sectional hydromechanical load for heave is given by:

$$
\begin{aligned}
X'_{h_3}(x_b) \;=\; & -a'_{31}\cdot\ddot{x} - b'_{31}\cdot\dot{x} - c'_{31}\cdot x \\
& -a'_{33}\cdot\ddot{z} - b'_{33}\cdot\dot{z} - c'_{33}\cdot z \\
& -a'_{35}\cdot\ddot{\theta} - b'_{35}\cdot\dot{\theta} - c'_{35}\cdot\theta
\end{aligned}
$$

with:

$$
\begin{aligned}
a'_{31} &= 0 \\
b'_{31} &= 0 \\
c'_{31} &= 0 \\
a'_{33} &= +M'_{33}
\end{aligned}
$$

$$
\begin{array}{rcl}
 & & +\boxed{\dfrac{V}{\omega_e^2} \cdot \dfrac{dN'_{33}}{dx_b}} \\
b'_{33} & = & +N'_{33} - V \cdot \dfrac{dM'_{33}}{dx_b} \\
c'_{33} & = & +2\rho g \cdot y_w \\
a'_{35} & = & -M'_{33} \cdot x_b - \dfrac{V}{\omega_e^2} \cdot \left(N'_{33} - V \cdot \dfrac{dM'_{33}}{dx_b} \right) \\
 & & +\boxed{\dfrac{-V}{\omega_e^2} \cdot N'_{33} - \dfrac{V}{\omega_e^2} \cdot \dfrac{dN'_{33}}{dx_b} \cdot x_b} \\
b'_{35} & = & -\left(N'_{33} - V \cdot \dfrac{dM'_{33}}{dx_b} \right) \cdot x_b + 2V \cdot M'_{33} \\
 & & +\boxed{\dfrac{V^2}{\omega_e^2} \cdot \dfrac{dN'_{33}}{dx_b}} \\
c'_{35} & = & -2\rho g \cdot y_w \cdot x_b
\end{array}
$$

The sectional wave loads for surge and heave are given by:

$$
\begin{array}{rcl}
X'_{w_1}(x_b) & = & +M'_{11} \cdot \ddot{\zeta}^*_{w_1} \\
 & & +\boxed{\dfrac{V}{\omega \cdot \omega_e} \cdot \dfrac{dN'_{11}}{dx_b} \cdot \ddot{\zeta}^*_{w_1}} \\
 & & +\left(\boxed{\dfrac{\omega}{\omega_e}} \cdot N'_{11} - V \cdot \dfrac{dM'_{11}}{dx_b} \right) \cdot \dot{\zeta}^*_{w_1} \\
 & & +X'_{FK_1} \\
X'_{w_3}(x_b) & = & +M'_{33} \cdot \ddot{\zeta}^*_{w_3} \\
 & & +\boxed{\dfrac{V}{\omega \cdot \omega_e} \cdot \dfrac{dN'_{33}}{dx_b} \cdot \ddot{\zeta}^*_{w_3}} \\
 & & +\left(\boxed{\dfrac{\omega}{\omega_e}} \cdot N'_{33} - V \cdot \dfrac{dM'_{33}}{dx_b} \right) \cdot \dot{\zeta}^*_{w_3} \\
 & & +X'_{FK_3}
\end{array}
$$

The "Modified Strip Theory" includes the outlined terms. When ignoring the outlined terms the "Ordinary Strip Theory" is presented.
Then the harmonic vertical shear forces $Q_3(x_1)$ and the bending moments $Q_5(x_1)$ in waves in cross section x_1 can be obtained from the longitudinal and vertical load $q_1(x_b)$ and $q_3(x_b)$ by the following integrals:

$$
\begin{array}{rcl}
Q_3(x_1) & = & Q_{3_a} \cos(\omega_e t + \varepsilon_{Q_3\zeta}) \\
 & = & -\displaystyle\int_{x_0}^{x_1} q_3(x_b) \cdot dx_b \\
Q_5(x_1) & = & Q_{5_a} \cos(\omega_e t + \varepsilon_{Q_5\zeta})
\end{array}
$$

$$= +\int_{x_0}^{x_1} q_1(x_b)\cdot \overline{bG}(x_b)\cdot dx_b + \int_{x_0}^{x_1} q_3(x_b)\cdot x_b\cdot dx_b \; - x_1 \int_{x_0}^{x_1} q_3(x_b)\cdot dx_b$$

Figure 12.10 shows a comparison between measured and calculated distributions of the vertical wave bending moment amplitudes over the length of the ship.

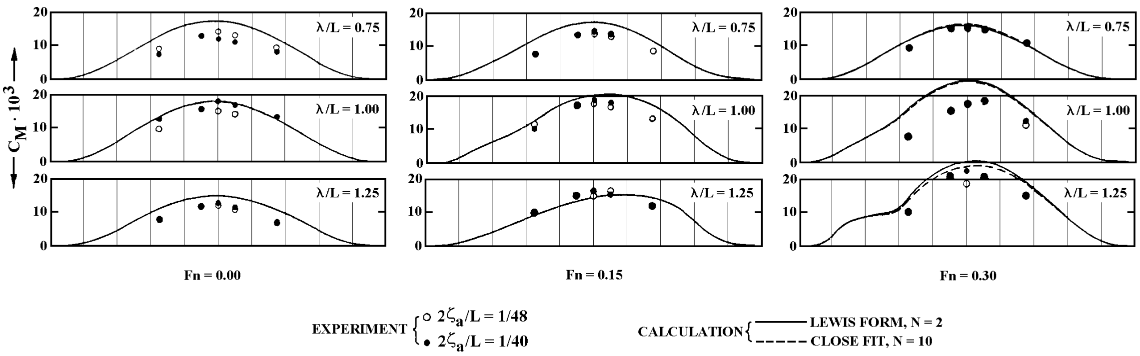

Figure 12.10: Distribution of Vertical Bending Moment Amplitudes

12.4 Torsional Dynamic Loads

Consider the forces acting on a section of the ship with a length dxb.

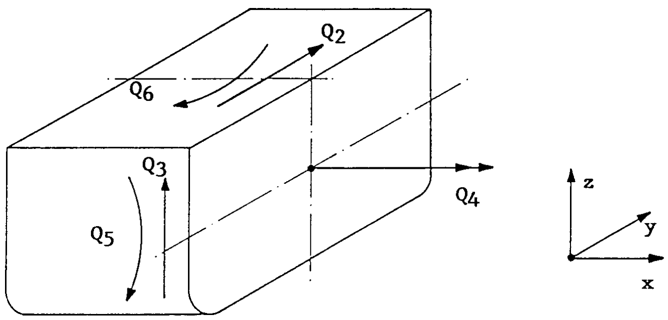

Figure 12.11: Torsional Loads on a Cross Section in Waves

According to Newton's second law of dynamics, the harmonic torsional dynamic load per unit length on the unfastened disk about an longitudinal axis at a distance z_1 above the ship's center of gravity is given by:

$$\begin{aligned} q_4(x_b, z_1) \;=\; & +X'_{h_4}(x_b) \; + X'_{w_4}(x_b) \\ & -m'(x_b)\cdot\left(k'^2_{xx}\cdot\ddot{\phi} \; - z'_m\cdot(\ddot{y} \; + x_b\cdot\ddot{\psi} + g\cdot\phi)\right) \\ & +z_1\cdot q_2(x_b) \end{aligned}$$

The sectional hydromechanical load for roll is given by:

$$\begin{aligned} X'_{h_4}(x_b) \;=\; & -a'_{42} \cdot \ddot{y} - b'_{42} \cdot \dot{y} - c'_{42} \cdot y \\ & -a'_{44} \cdot \ddot{\phi} - b'_{44} \cdot \dot{\phi} - c'_{44} \cdot \phi \\ & -a'_{46} \cdot \ddot{\psi} - b'_{46} \cdot \dot{\psi} - c'_{46} \cdot \psi \end{aligned}$$

with:

$$\begin{aligned}
a'_{42} &= +M'_{42} + \overline{OG} \cdot a'_{22} \\
&\quad + \boxed{\frac{V}{\omega_e^2} \cdot \frac{dN'_{42}}{dx_b}} \\
b'_{42} &= +N'_{42} - V \cdot \frac{dM'_{42}}{dx_b} + \overline{OG} \cdot b'_{22} \\
c'_{42} &= 0 + \overline{OG} \cdot c'_{22} \\
a'_{44} &= +M'_{44} + \overline{OG} \cdot M'_{42} + \overline{OG} \cdot a'_{24} \\
&\quad + \boxed{\frac{V}{\omega_e^2} \cdot \frac{dN'_{44}}{dx_b} + \frac{V}{\omega_e^2} \cdot \overline{OG} \cdot \frac{dN'_{42}}{dx_b}} \\
b'_{44} &= +N'_{44} - V \cdot \frac{dM'_{44}}{dx_b} + \overline{OG} \cdot \left(N'_{42} - V \cdot \frac{dM'_{42}}{dx_b} \right) + b'_{44_V} + \overline{OG} \cdot b'_{24} \\
c'_{44} &= +2\rho g \cdot \left(\frac{y_w^3}{3} - \frac{A_s}{2} \cdot \overline{bG} \right) \\
a'_{46} &= +M'_{42} \cdot x_b + \frac{V}{\omega_e^2} \left(N'_{42} - V \cdot \frac{dM'_{42}}{dx_b} \right) + \overline{OG} \cdot a'_{26} \\
&\quad + \boxed{\frac{V}{\omega_e^2} \cdot N'_{42} + \frac{V}{\omega_e^2} \cdot \frac{dN'_{42}}{dx_b} \cdot x_b} \\
b'_{46} &= + \left(N'_{42} - V \cdot \frac{dM'_{42}}{dx_b} \right) \cdot x_b - 2V \cdot M'_{42} + \overline{OG} \cdot b'_{26} \\
&\quad + \boxed{\frac{V^2}{\omega_e^2} \cdot \frac{dN'_{42}}{dx_b}} \\
c'_{46} &= 0 + \overline{OG} \cdot c'_{26}
\end{aligned}$$

In here, $\overline{bG}$ is the vertical distance of the centre of gravity of the ship G above the centroid b of the local submerged sectional area.
The sectional wave load for roll is given by:

$$\begin{aligned} X'_{w_4}(x_b) \;=\; & +M'_{44} \cdot \ddot{\zeta}^*_{w_4} \\ & + \boxed{\frac{V}{\omega \cdot \omega_e} \cdot \frac{dN'_{44}}{dx_b} \cdot \dot{\zeta}^*_{w_4}} \\ & + \left(\boxed{\frac{\omega}{\omega_e}} \cdot N'_{44} - V \cdot \frac{dM'_{44}}{dx_b} \right) \cdot \dot{\zeta}^*_{w_4} \\ & + X'_{FK_4} \\ & + M'_{42} \cdot \ddot{\zeta}^*_{w_2} \end{aligned}$$

$$
\begin{aligned}
&+\boxed{\frac{V}{\omega \cdot \omega_e} \frac{dN'_{42}}{dx_b} \cdot \ddot{\zeta}^*_{w_2}} \\
&+\left(\boxed{\frac{\omega}{\omega_e}} \cdot N'_{42} - V \cdot \frac{dM'_{42}}{dx_b}\right) \cdot \dot{\zeta}^*_{w_2} \\
&+\overline{OG} \cdot X'_{w_2}
\end{aligned}
$$

The "Modified Strip Theory" includes the outlined terms. When ignoring the outlined terms the "Ordinary Strip Theory" is presented.
Then the harmonic torsional moments $Q_4(x_1, z_1)$ in waves in cross section x_1 at a distance z1 above the ship's centre of gravity can be obtained from the torsional load $q_4(x_b, z_1)$ by the following integral:

$$
\begin{aligned}
Q_4(x_1, z_1) &= Q_{4_a} \cos(\omega_e t + \varepsilon_{Q_4\zeta}) \\
&= -\int_{x_0}^{x_1} q_4(x_b, z_1) \cdot dx_b
\end{aligned}
$$

Chapter 13

Statistics in Irregular Waves

To compare the calculated behaviour of different ship designs or to get an impression of the behaviour of a specific ship design in a seaway, standard representations of the wave energy distributions are necessary.
Three well known types of normalised wave energy spectra are described here:

- the Neumann wave spectrum, a somewhat wide wave spectrum, which is sometimes used for open sea areas
- the Bretschneider wave spectrum, an average wave spectrum, frequently used in open sea areas
- the Mean JONSWAP wave spectrum, a narrow wave spectrum, frequently used in North Sea areas.

The mathematical formulations of these normalised uni-directional wave energy spectra are based on two parameters:

- the significant wave height $H_{1/3}$
- the average wave period T_1, based on the centroid of the spectral area curve.

To obtain the average zero-crossing period T_2 or the spectral peak period T_p, a fixed relation with T_1 can be used not-truncated spectra.
From these wave energy spectra and the transfer functions of the responses, the response energy spectra can be obtained.
Generally the frequency ranges of the energy spectra of the waves and the responses of the ship on these waves are not very wide. Then the Rayleigh distribution can be used to obtain a probability density function of the maximum and minimum values of the waves and the responses. With this function, the probabilities on exceeding threshold values by the ship motions can be calculated.
Bow slamming phenomena are defined by a relative bow velocity criterium and a peak bottom impact pressure criterium.

[0] J.M.J. Journée, *"Theoretical Manual of SEAWAY, Release 4.19"*, Report 1216a, February 2001, Ship Hydromechanics Laboratory, Delft University of Technology, Mekelweg 2, 2628 CD Delft, The Netherlands. For updates see web site: http://dutw189.wbmt.tudelft.nl/~johan or http://www.shipmotions.nl.

13.1 Normalized Wave Energy Spectra

Three mathematical definitions with two parameters of normalized spectra of irregular uni-directional waves have been described:

- the Neumann wave spectrum, a somewhat wide spectrum
- the Bretschneider wave spectrum, an average spectrum
- the mean JONSWAP wave spectrum, a narrow spectrum

A comparison of the Neumann, the Bretschneider and the mean JONSWAP wave spectra is given here for a sea state with a significant wave height of 4 meters and an average wave period of 8 seconds.

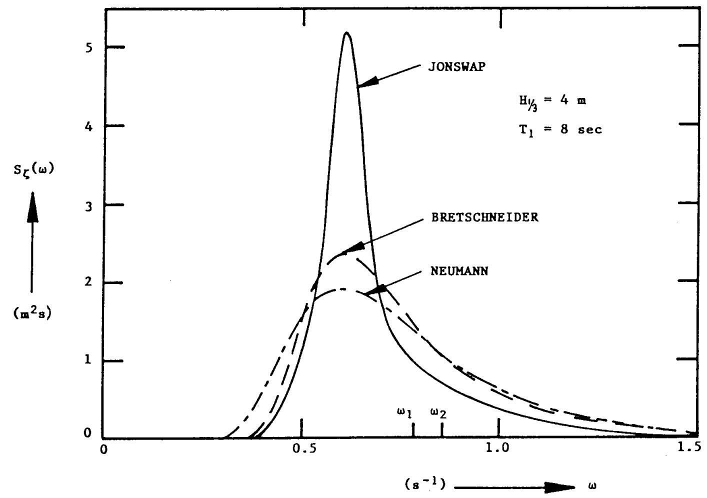

Figure 13.1: Comparison of Three Spectral Formulations

13.1.1 Neumann Wave Spectrum

In some cases in literature the Neumann definition of a wave spectrum for open sea areas is used:

$$S_\zeta(\omega) = \frac{3832 \cdot H_{1/3}^2}{T_1^5} \cdot \omega^{-6} \cdot \exp\left\{\frac{-69.8}{T_1^2} \cdot \omega^{-2}\right\}$$

13.1.2 Bretschneider Wave Spectrum

A very well known two-parameter wave spectrum of open seas is defined by Bretschneider as:

$$S_\zeta(\omega) = \frac{172.8 \cdot H_{1/3}^2}{T_1^4} \cdot \omega^{-5} \cdot \exp\left\{\frac{-691.2}{T_1^4} \cdot \omega^{-4}\right\}$$

Another name of this wave spectrum is the Modified Two-Parameter Pierson-Moskowitz Wave Spectrum.

This formulation is accepted by the 2nd International Ship Structures Congress in 1967 and the 12th International Towing Tank Conference in 1969 as a standard for seakeeping calculations and model experiments. This is reason why this spectrum is also called I.S.S.C. or I.T.T.C. Wave Spectrum.
The original One-Parameter Pierson-Moskowitz Wave Spectrum for fully developed seas can be obtained from this definition by using a fixed relation between the significant wave height and the average wave period in this Bretschneider definition: $T_1 = 3.861 \cdot \sqrt{H_{1/3}}$.

13.1.3 Mean JONSWAP Wave Spectrum

In 1968 and 1969 an extensive wave measurement program, known as the Joint North Sea Wave Project (JONSWAP) was carried out along a line extending over 100 miles into the North Sea from Sylt Island. From analysis of the measured spectra, a spectral formulation of wind generated seas with a fetch limitation was found.
The following definition of a Mean JONSWAP wave spectrum is advised by the 15th ITTC in 1978 for fetch limited situations:

$$S_\zeta(\omega) = \frac{172.8 \cdot H_{1/3}^2}{T_1^4} \cdot \omega^{-5} \cdot \exp\left\{ \frac{-691.2}{T_1^4} \cdot \omega^{-4} \right\} \cdot A \cdot \gamma^B$$

with:

$$\begin{aligned}
A &= 0.658 \\
B &= exp\left\{ -\left(\frac{\frac{\omega}{\omega_p} - 1.0}{\sigma\sqrt{2}} \right)^2 \right\} \\
\gamma &= 3.3 \quad \text{(peakedness factor)} \\
\omega_p &= \frac{2\pi}{T_p} \quad \text{(circular frequency at spectral peak)} \\
\sigma &= \text{ a step function of } \omega\text{: if } \omega < \omega_p \text{ then } \sigma = 0.07 \\
& \qquad \text{if } \omega > \omega_p \text{ then } \sigma = 0.09
\end{aligned}$$

The JONSWAP expression is equal to the Bretschneider definition multiplied by the frequency function $A \cdot \gamma^B$.
Sometimes, a third free parameter is introduced in the JONSWAP wave spectrum by varying the peakedness factor γ.

13.1.4 Definition of Parameters

The n-th order spectral moments of the wave spectrum, defined as a function of the circular wave frequency ω, are:

$$m_{n\zeta} = \int_0^\infty S_\zeta(\omega) \cdot \omega^n \cdot d\omega$$

The breadth of a wave spectrum is defined by:

$$\varepsilon = \sqrt{1 - \frac{m_{2\zeta}^2}{m_{0\zeta} \cdot m_{4\zeta}}}$$

The significant wave height is defined by:

$$H_{1/3} = 4.0 \cdot \sqrt{m_{0\zeta}}$$

The several definitions of the average wave period are:

$$\begin{aligned} T_p &= && \text{peak or modal wave period,} \\ & && \text{corresponding to peak of spectral curve} \\ T_1 = 2\pi \cdot \frac{m_{0\zeta}}{m_{1\zeta}} &= && \text{average wave period,} \\ & && \text{corresponding to centroid of spectral curve} \\ T_2 = 2\pi \cdot \sqrt{\frac{m_{0\zeta}}{m_{2\zeta}}} &= && \text{average zero-crossing wave period,} \\ & && \text{corresponding to radius of inertia of spectral curve} \end{aligned}$$

For not-truncated mathematically defined spectra, the theoretical relations between the periods are tabled below:

$$\left.\begin{array}{rcccl} T_1 &=& 1.086 \cdot T_2 &=& 0.772 \cdot T_p \\ 0.921 \cdot T_1 &=& T_2 &=& 0.711 \cdot T_p \\ 1.296 \cdot T_1 &=& 1.407 \cdot T_2 &=& T_p \end{array}\right\} \text{for Bretschneider Wave Spectra}$$

$$\left.\begin{array}{rcccl} T_1 &=& 1.073 \cdot T_2 &=& 0.834 \cdot T_p \\ 0.932 \cdot T_1 &=& T_2 &=& 0.777 \cdot T_p \\ 1.199 \cdot T_1 &=& 1.287 \cdot T_2 &=& T_p \end{array}\right\} \text{for JONSWAP Wave Spectra}$$

Truncation of wave spectra during numerical calculations, can cause differences between input and calculated wave periods. Generally, the wave heights will not differ much.

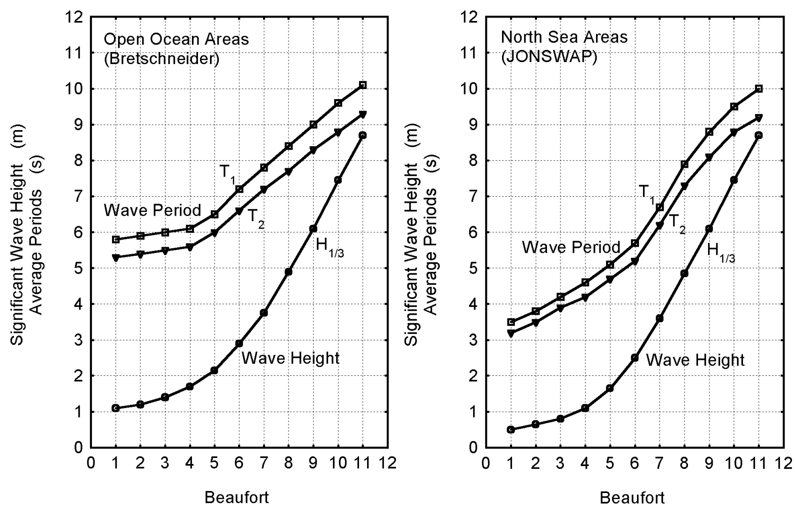

Figure 13.2: Wave Spectra Parameter Estimates

In figure 13.2 and the table below, for "Open Ocean Areas" and "North Sea Areas" an indication is given of a possible average relation between the scale of Beaufort or the wind velocity at 19.5 meters above the sea level and the significant wave height $H_{1/3}$ and the average wave periods T_1 or T_2.

Indication of Wave Spectra Parameters

Scale of Beaufort	Wind Speed at 19.5 m above sea	Open Ocean Areas (Bretschneider)			North Sea Areas (JONSWAP)			
	(kn)	$H_{1/3}$ (m)	T_1 (s)	T_2 (s)	$H_{1/3}$ (m)	T_1 (s)	T_2 (s)	γ (-)
1	2.0	1.10	5.8	5.35	0.50	3.5	3.25	3.3
2	5.0	1.20	5.9	5.45	0.65	3.8	3.55	3.3
3	8.5	1.40	6.0	5.55	0.80	4.2	3.90	3.3
4	13.5	1.70	6.1	5.60	1.10	4.6	4.30	3.3
5	19.0	2.15	6.5	6.00	1.65	5.1	4.75	3.3
6	24.5	2.90	7.2	6.65	2.50	5.7	5.30	3.3
7	30.5	3.75	7.8	7.20	3.60	6.7	6.25	3.3
8	37.0	4.90	8.4	7.75	4.85	7.9	7.35	3.3
9	44.0	6.10	9.0	8.30	6.10	8.8	8.20	3.3
10	51.5	7.45	9.6	8.85	7.45	9.5	8.85	3.3
11	59.5	8.70	10.1	9.30	8.70	10.0	9.30	3.3
12	>64.0	10.25	10.5	9.65	10.25	10.5	9.80	3.3

These data are an indication only, a generally applicable fixed relation between wave heights and wave periods does not exist.

Other open ocean definitions for the North Atlantic and the North Pacific, obtained from [Bales, 1983] and adopted by the 17th ITTC (1984), are given in the tables below.

The modal or central periods in these tables correspond with the peak period T_p.

For not-truncated spectra, the relations with T_1 and T_2 are defined before.

Open Ocean Annual Sea State Occurrences of Bales (1983) of the North Atlantic and the North Pacific

Sea State Number (−)	Significant Wave Height $H_{1/3}$ (m)		Sustained Wind Speed 1) (kn)		Probability of Sea State (%)	Modal Wave Period T_p (s)	
	Range	Mean	Range	Mean		Range 2)	Most 3) Probable
North Atlantic							
0 - 1	0.0 - 0.1	0.05	0 - 6	3	0	-	-
2	0.1 - 0.5	0.3	7 - 10	8.5	7.2	3.3 - 12.8	7.5
3	0.50 - 1.25	0.88	11 - 16	13.5	22.4	5.0 - 14.8	7.5
4	1.25 - 2.50	1.88	17 - 21	19	28.7	6.1 - 15.2	8.8
5	2.5 - 4.0	3.25	22 - 27	24.5	15.5	8.3 - 15.5	9.7
6	4 - 6	5.0	28 - 47	37.5	18.7	9.8 - 16.2	12.4
7	6 - 9	7.5	48 - 55	51.5	6.1	11.8 - 18.5	15.0
8	9 - 14	11.5	56 - 63	59.5	1.2	14.2 - 18.6	16.4
>8	>14	>14	>63	>63	<0.05	18.0 - 23.7	20.0
North Pacific							
0 - 1	0.0 - 0.1	0.05	0 - 6	3	0	-	-
2	0.1 - 0.5	0.3	7 - 10	8.5	4.1	3.0 - 15.0	7.5
3	0.50 - 1.25	0.88	11 - 16	13.5	16.9	5.2 - 15.5	7.5
4	1.25 - 2.50	1.88	17 - 21	19	27.8	5.9 - 15.5	8.8
5	2.5 - 4.0	3.25	22 - 27	24.5	23.5	7.2 - 16.5	9.7
6	4 - 6	5.0	28 - 47	37.5	16.3	9.3 - 16.5	13.8
7	6 - 9	7.5	48 - 55	51.5	9.1	10.0 - 17.2	15.8
8	9 - 14	11.5	56 - 63	59.5	2.2	13.0 - 18.4	18.0
>8	>14	>14	>63	>63	0.1	20.0	20.0

Note:
1) Ambient wind sustained at 19.5 m above surface to generate fully-developed seas. To convert to another altitude h_2, apply $V_2 = V_1 \cdot (h_2/19.5)^{1/7}$.
2) Minimum is 5 percentile and maximum is 95 percentile for periods given wave height range.
3) Based on periods associated with central frequencies included in Hindcast Climatology.

13.2 Response Spectra and Statistics

The energy spectrum of the responses $r(t)$ of a sailing ship on the irregular waves follows from the transfer function of the response and the wave energy spectrum by:

$$S_r(\omega) = \left(\frac{r_a}{\zeta_a}\right)^2 \cdot S_\zeta(\omega)$$

or:

$$S_r(\omega_e) = \left(\frac{r_a}{\zeta_a}\right)^2 \cdot S_\zeta(\omega_e)$$

This has been visualized for a heave motion in figures 13.3 and 13.4.

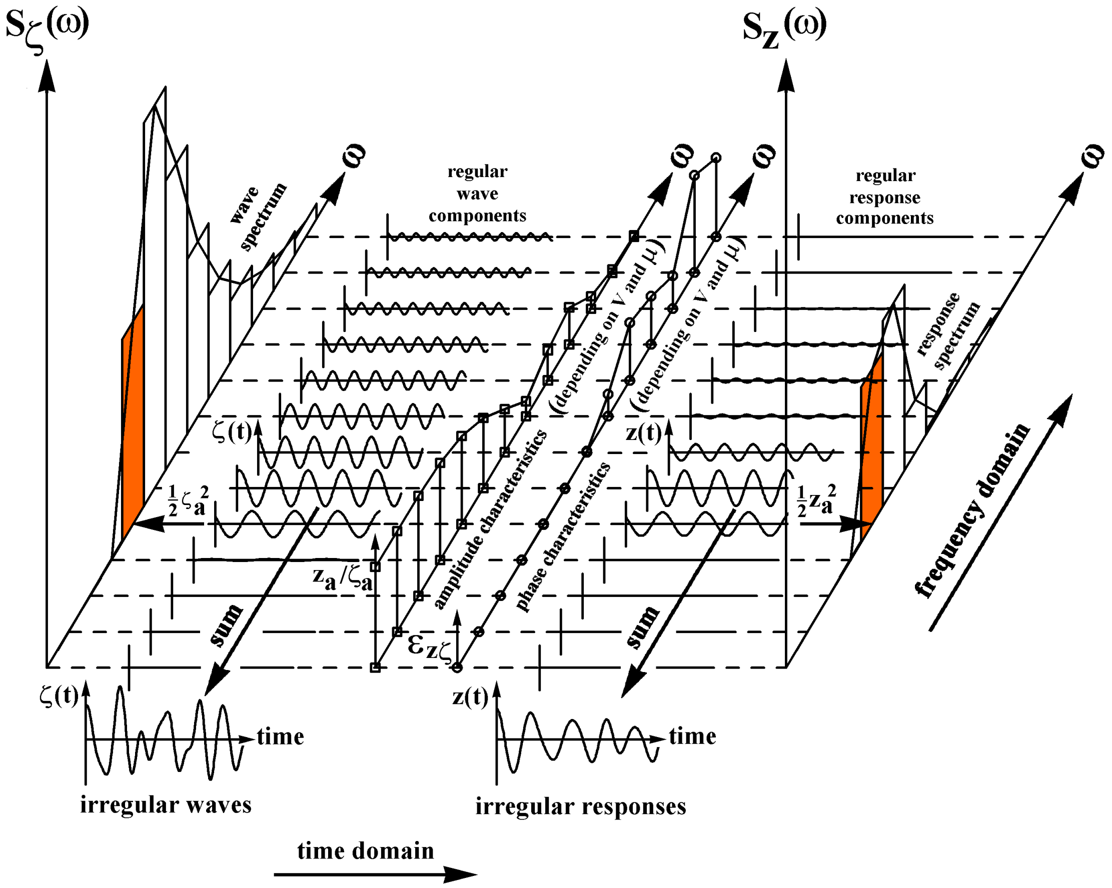

Figure 13.3: Principle of Transfer of Waves into Responses

The moments of the response spectrum are given by:

$$m_{nr} = \int_0^\infty S_r(\omega_e) \cdot \omega_e^n \cdot d\omega_e \qquad \text{with: } n = 0, 1, 2, ...$$

From the spectral density function of a response the significant amplitude can be calculated.

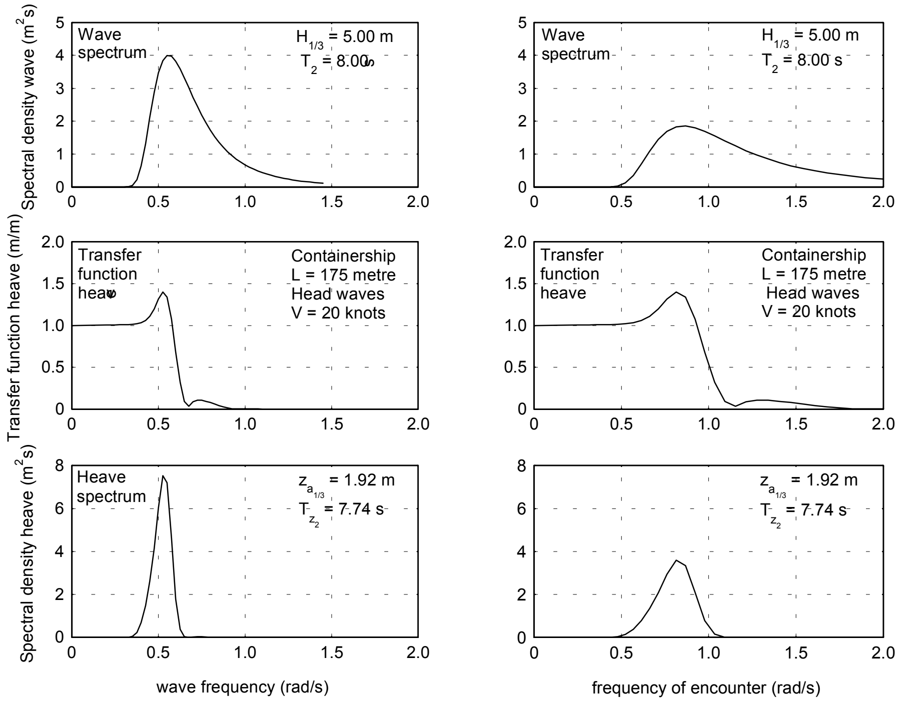

Figure 13.4: Heave Spectra in the Wave and Encounter Frequency Domain

The significant amplitude is defined to be the mean value of the highest one-third part of the highest wave heights, so:

$$r_{a_{1/3}} = 2\sqrt{m_{0r}}$$

A mean period can be found from the centroid of the spectrum by:

$$T_{1r} = 2\pi \cdot \frac{m_{0r}}{m_{1r}}$$

An other definition, which is equivalent to the average zero-crossing period, is found from the spectral radius of inertia by:

$$T_{2r} = 2\pi \cdot \sqrt{\frac{m_{0r}}{m_{2r}}}$$

The probability density function of the maximum and minimum values, in case of a spectrum with a frequency range which is not too wide, is given by the Rayleigh distribution:

$$f(r_a) = \frac{r_a}{m_{0r}} \cdot \exp\left\{\frac{-r_a^2}{2m_{0r}}\right\}$$

This implies that the probability of exceeding a threshold value a by the response amplitude r_a becomes:

$$\begin{aligned} P\{r_a > a\} &= \int_a^\infty \frac{r_a}{m_{0r}} \cdot \exp\left\{\frac{-r_a^2}{2m_{0r}}\right\} \cdot dr_a \\ &= \exp\left\{\frac{-a^2}{2m_{0r}}\right\} \end{aligned}$$

The number of times per hour that this happens follows from:

$$N_{hour} = \frac{3600}{T_{2r}} \cdot P\left\{r_a > a\right\}$$

The spectral value of the waves $S_\zeta(\omega_e)$, based on ω_e, is not equal to the spectral value $S_\zeta(\omega)$, based on ω.
Because of the requirement of an equal amount of energy in the frequency bands $\Delta\omega$ and $\Delta\omega_e$, it follows:

$$S_\zeta(\omega_e) \cdot d\omega_e = S_\zeta(\omega) \cdot d\omega$$

From this the following relation is found:

$$S_\zeta(\omega_e) = \frac{S_\zeta(\omega)}{\frac{d\omega_e}{d\omega}}$$

The relation between the frequency of encounter and the wave frequency, of which an example is illustrated in figure 13.5, is given by:

$$\omega_e = \omega - kV \cdot \cos\mu$$

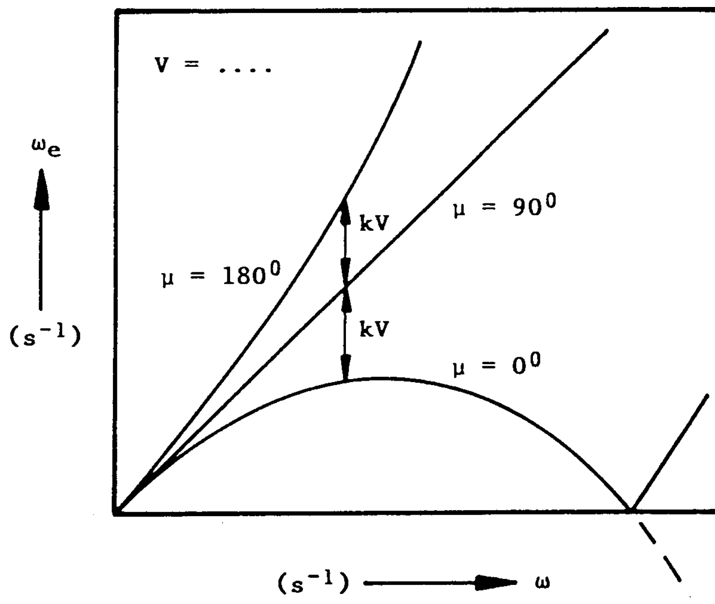

Figure 13.5: Example of Relation Between ω_e and ω

From the relation between ω_e and ω follows:

$$\frac{d\omega_e}{d\omega} = 1.0 - \frac{V \cdot \cos\mu}{\frac{d\omega}{dk}}$$

The derivative $d\omega/dk$ follows from the relation between ω and k:

$$\omega = \sqrt{kg \cdot \tanh(kh)}$$

So:

$$\frac{d\omega}{dk} = \frac{g \cdot \tanh(kh) + \frac{kg}{h \cdot \cosh^2(kh)}}{2\sqrt{kg \cdot \tanh(kh)}}$$

As can be seen in figure 13.5, in following waves the derivative $d\omega_e/d\omega$ can approach from both sides, a positive or a negative side, to zero. As a result of this, around a wave speed equal to twice the forward ship speed component in the direction of the wave propagation, the transformed spectral values will range from plus infinite to minus infinite. This implies that numerical problems will arise in the numerical integration routine.
This is the reason why the spectral moments have to be written in the following format:

$$\begin{aligned}
m_{0r} &= \int_0^\infty S_r(\omega_e) \cdot d\omega_e &&= \int_0^\infty S_r(\omega) \cdot d\omega \\
m_{1r} &= \int_0^\infty S_r(\omega_e) \cdot \omega_e \cdot d\omega_e &&= \int_0^\infty S_r(\omega) \cdot \omega_e \cdot d\omega \\
m_{2r} &= \int_0^\infty S_r(\omega_e) \cdot \omega_e^2 \cdot d\omega_e &&= \int_0^\infty S_r(\omega) \cdot \omega_e^2 \cdot d\omega
\end{aligned}$$

with:

$$S_r(\omega) = \left(\frac{r_a}{\zeta_a}\right)^2 \cdot S_\zeta(\omega)$$

If $S_r(\omega_e)$ has to be known, for instance for a comparison of the calculated response spectra with measured response spectra, these values can be obtained from this $S_r(\omega)$ and the derivative $d\omega_e/d\omega$. So an integration of $S_r(\omega_e)$ over ω_e has to be avoided.
Because of the linearities, the calculated significant values can be presented by:

$$\frac{r_{a_{1/3}}}{H_{1/3}} \quad \text{versus } T_1 \text{ or } T_2$$

$$\begin{aligned}
H_{1/3} &= \text{significant wave height} \\
T_1, T_2 &= \text{average wave periods}
\end{aligned}$$

The mean added resistance in a seaway follows from:

$$R_{AW} = 2\int_0^\infty \frac{R_{AW}}{\zeta_a^2} \cdot S_\zeta(\omega) \cdot d\omega$$

Because of the linearities in the motions, the calculated mean added resistance values can be presented by:

$$\frac{R_{AW}}{H_{1/3}^2} \quad \text{versus } T_1 \text{ or } T_2$$

13.3 Shipping Green Water

The effective dynamic freeboard will differ from the results obtained from the geometric freeboard at zero forward speed in still water and the calculated vertical relative motions of a sailing ship in waves.
When sailing in still water, sinkage, trim and the ship's wave system will effect the local geometric freeboard. A static swell up should be taken into account.
An empirical formula, based on model experiments, for the static swell up at the forward perpendicular is given by [Tasaki, 1963]:

$$f_e = f - 0.75 \cdot B \cdot \frac{L}{L_E} \cdot F_n^2$$

with:

f_e	= effective freeboard at the forward perpendicular
f	= geometric freeboard at the forward perpendicular
L	= length of the ship
B	= breadth of the ship
L_E	= length of entrance of the waterline
F_n	= Froude number

An oscillating ship will produce waves and these dynamic phenomena will influence the amplitude of the relative motion. A dynamic swell up should be taken into account.
[Tasaki, 1963] carried out forced oscillation tests with ship models in still water and obtained an empirical formula for the dynamic swell-up at the forward perpendicular in head waves:

$$\frac{\Delta s_a}{s_a} = \frac{C_B - 0.45}{3} \cdot \sqrt{\frac{\omega_e^2 L}{g}}$$

with the restrictions:

block coefficient:	$0.60 < C_B < 0.80$
Froude number:	$0.16 < F_n < 0.29$

In this formula s_a is the amplitude of the relative motion at the forward perpendicular as obtained in head waves, calculated from the heave, the pitch and the wave motions.
Then the actual amplitude of the relative motions becomes:

$$s_a^* = s_a + \Delta s_a$$

Then, shipping green water is defined by:

$$s_a^* > f_e \qquad \text{at the forwarde perpendicular}$$

The spectral density of the vertical relative motion at the forward perpendicular is given by:

$$S_{s^*}(\omega) = \left(\frac{s_a^*}{\zeta_a}\right)^2 \cdot S_\zeta(\omega)$$

The spectral moments are given by:

$$m_{ns^*} = \int_0^\infty S_{s^*}(\omega) \cdot \omega_e^n \cdot d\omega \qquad \text{with:} \;\; n = 0, 1, 2, ...$$

When using the Rayleigh distribution the probability of shipping green water is given by:

$$P\left\{s_a^* > f_e\right\} = \exp\left\{\frac{-f_e^2}{2m_{0s^*}}\right\}$$

The average zero-crossing period of the relative motion is found from the spectral radius of inertia by:

$$T_{2s^*} = 2\pi\sqrt{\frac{m_{0s^*}}{m_{2s^*}}}$$

The number of times per hour that green water will be shipped follows from:

$$N_{hour} = \frac{3600}{T_{2s^*}} \cdot P\left\{s_a^* > f_e\right\}$$

13.4 Bow Slamming

Slamming is a two-node vibration of the ship caused by suddenly pushing the ship by the waves. A complete prediction of slamming phenomena is a complex task, which is beyond the scope of any existing theory.
Slamming impact pressures are affected by the local hull section shape, the relative velocity between ship and waves at impact, the relative angle between the keel and the water surface, the local flexibility of the ship's bottom plating and the overall flexibility of the ship's structure.

13.4.1 Criterium of Ochi

[Ochi, 1964] translated slamming phenomena into requirements for the vertical relative motions of the ship.
He defined slamming by:

- an emergence of the bow of the ship at 10 percentile of the length aft of the forward perpendiculars
- an exceeding of a certain critical value at the instance of impact by the vertical relative velocity, without forward speed effect, between the wave surface and the bow of the ship

Ochi defines the vertical relative displacement and velocity of the water particles with respect to the keel point of the ship by:

$$\begin{aligned} s &= \zeta_{x_b} - z + x_b \cdot \theta \\ \dot{s} &= \dot{\zeta}_{x_b} - \dot{z} + x_b \cdot \dot{\theta} \end{aligned}$$

with:

$$\begin{aligned}\zeta_{x_b} &= \zeta_a \cos(\omega_e t - k x_b \cos\mu) \\ \dot{\zeta}_{x_b} &= -\omega_e \zeta_a \sin(\omega_e t - k x_b \cos\mu)\end{aligned}$$

So a forward speed effect is not included in the vertical relative velocity.
The spectral moments of the vertical relative displacements and velocities are defined by m_{0s} and $m_{0\dot{s}}$.
Emergence of the bow of the ship happens when the vertical relative displacement amplitude s_a at $0.90 \cdot L$ is larger than the ship's draft D_s at this location.
The probability of emergence of the bow follows from:

$$P\{s_a > D_s\} = \exp\left\{\frac{-D_s^2}{2m_{0s}}\right\}$$

The second requirement states that the vertical relative velocity exceeds a threshold value. According to Ochi, 12 feet per second can be taken as a threshold value for a ship with a length of 520 feet.
Scaling results into:

$$\dot{s}_{cr} = 0.0928 \cdot \sqrt{gL}$$

The probability of exceeding this threshold value is:

$$P\{\dot{s}_a > \dot{s}_{cr}\} = \exp\left\{\frac{-\dot{s}_{cr}^2}{2m_{0\dot{s}}}\right\}$$

Both occurrences, emergence of the bow and exceeding the threshold velocity, are statistically independent. In case of slamming both occurrences have to appear at the same time.
So the probability on a slam is the product of the both independent probabilities:

$$\begin{aligned}P\{slam\} &= P\{s_a > D_s\} \cdot P\{\dot{s}_a > \dot{s}_{cr}\} \\ &= \exp\left\{\frac{-D_s^2}{2m_{0s}} + \frac{-\dot{s}_{cr}^2}{2m_{0\dot{s}}}\right\}\end{aligned}$$

13.4.2 Criterium of Conolly

[Conolly, 1974] translated slamming phenomena into requirements for the peak impact pressure of the ship.
He defined slamming by:

- an emergence of the bow of the ship
- an exceeding of a certain critical value by the peak impact pressure at this location.

The peak impact pressure is defined by:

$$p = C_p \cdot \frac{1}{2}\rho \dot{s}_{cr}^2$$

The coefficient C_p has been taken from experimental data of slamming drop tests with wedges and cones, as given in literature.

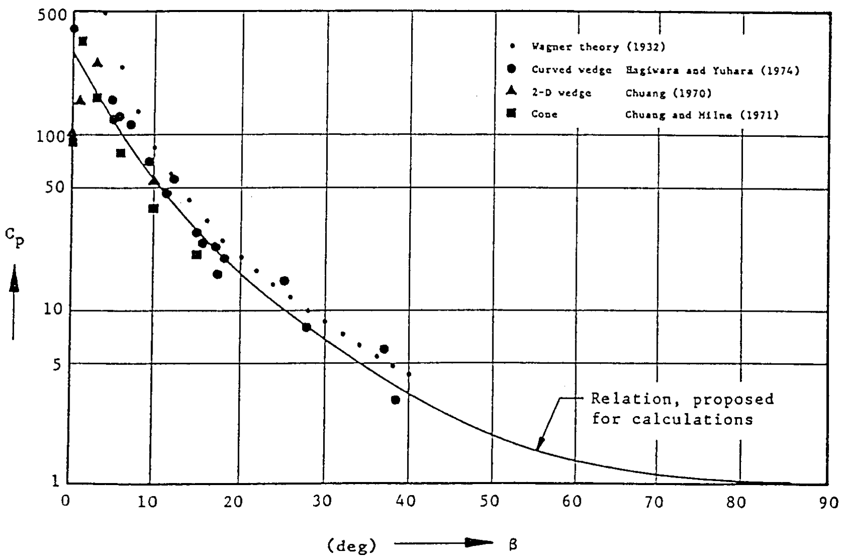

Figure 13.6: Peak Impact Pressure Coefficients

Some of these data, as for instance presented by [Lloyd, 1989] as a function of the deadrise angle β, are illustrated in figure 13.6.
An equivalent deadrise angle β is defined here by the determination of an equivalent wedge. The contour of the cross section inside 10 percentile of the half breadth $B/2$ of the ship has been used to define an equivalent wedge with a half breadth: $b = 0.10 \cdot B/2$.
The accessory draught t of the wedge follows from the section contour.
In the forebody of the ship, this draught can be larger than 10 percentile of the amidships draught T. If so, the section contour below $0.10 \cdot T$ has been used to define an equivalent wedge: $t = 0.10 \cdot T$.
If this draught is larger than the local draught, the local draught has been used.
The accessory half breadth b of the wedge follows from the section contour.

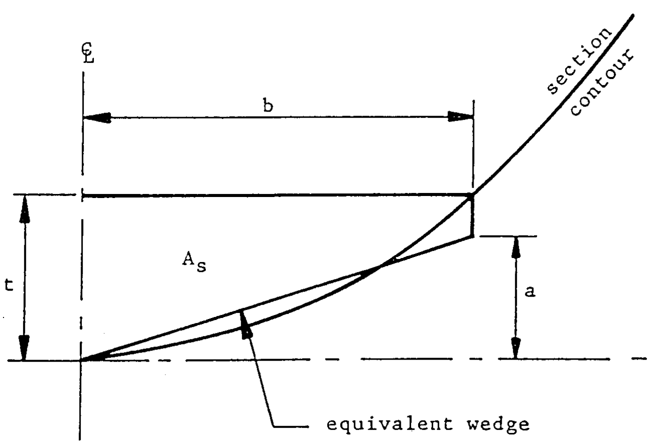

Figure 13.7: Definition of an Equivalent Wedge

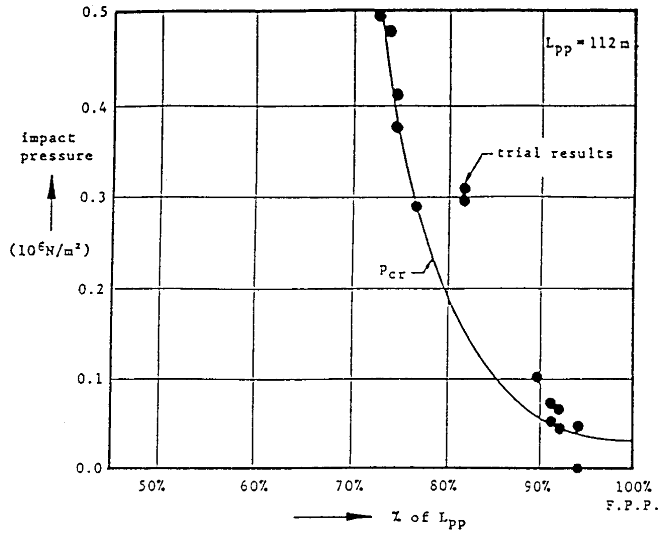

Figure 13.8: Measured Impact Pressures of a 112 Meter Ship

Then the sectional area A_s below local draught t has to be calculated.
Now the equivalent deadrise angle β follows from:

$$\begin{aligned} \beta &= \arctan\left(\frac{a}{b}\right) \qquad 0 \leq \beta \leq \pi/2 \\ a &= \frac{2(b \cdot t - A_s)}{b} \end{aligned}$$

Critical peak impact pressures p_{cr} have been taken from [Conolly, 1974]. He gives measured impact pressures at a ship with a length of 112 meter over 30 per cent of the ship length from forward. From this, a lower limit of p_{cr} has been assumed. This lower limit is presented in figure 13.8.
These values have to be scaled to the actual ship size. Bow emergence and exceeding of this limit is supposed to cause slamming.
This approach can be translated into local hull shape-depending threshold values of the vertical relative velocity too:

$$\dot{s}_{cr} = \sqrt{\frac{2p_{cr}}{\rho C_p}}$$

The vertical relative velocity, including a forward speed effect, of the water particles with respect to the keel point of the ship is defined by:

$$\begin{aligned} \dot{s} &= \frac{D}{Dt}\left\{\zeta_{x_b} - z + x_b \cdot \theta\right\} \\ &= \dot{\zeta}_{x_b} - \dot{z} + x_b \cdot \dot{\theta} - V \cdot \theta \end{aligned}$$

with:

$$\begin{aligned} \zeta_{x_b} &= \zeta_a \cos(\omega_e t - k x_b \cos\mu) \\ \dot{\zeta}_{x_b} &= -\omega\zeta_a \sin(\omega_e t - k x_b \cos\mu) \end{aligned}$$

Then:

$$P\left\{slam\right\} = \exp\left\{\frac{-D_s^2}{2m_{0s}} + \frac{-\dot{s}_{cr}^2}{2m_{0\dot{s}}}\right\}$$

Note that, because of including the forward speed effect, the spectral moment of the velocities does not follow from the spectral density of the relative displacement as showed in the definition of Ochi.

The average period of the relative displacement is found by:

$$T_{2s} = 2\pi\sqrt{\frac{m_{0s}}{m_{2s}}} = 2\pi\sqrt{\frac{m_{0s}}{m_{0\dot{s}}}}$$

Then the number of times per hour that a slam will occur follows from:

$$N_{hour} = \frac{3600}{T_{2s}} \cdot P\left\{slam\right\}$$

Chapter 14

Twin-Hull Ships

When not taking into account the interaction effects between the two individual hulls, the wave loads and motions of twin-hull ships can be calculated easily. Each individual hull has to be symmetric with respect to its centre plane. The distance between the two centre planes of the single hulls should be constant. The coordinate system for the equations of motion of a twin-hull ship is given in figure 14.1.

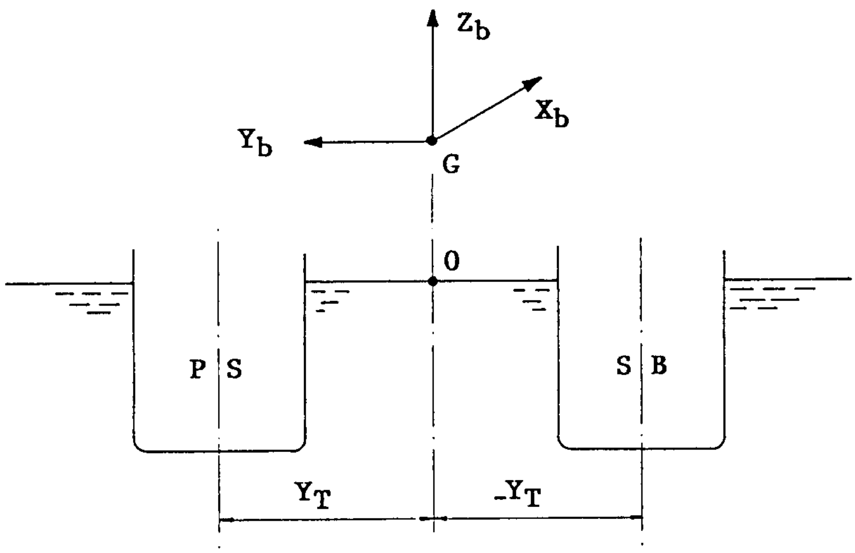

Figure 14.1: Coordinate System of Twin-Hull Ships

14.1 Hydromechanical Coefficients

The hydromechanical coefficients a_{ij}, b_{ij} and c_{ij} in this chapter are those of one individual hull, defined in the coordinate system of the single hull, as given and discussed before.

[0] J.M.J. Journée, *"Theoretical Manual of SEAWAY, Release 4.19"*, Report 1216a, February 2001, Ship Hydromechanics Laboratory, Delft University of Technology, Mekelweg 2, 2628 CD Delft, The Netherlands. For updates see web site: http://dutw189.wbmt.tudelft.nl/~johan or http://www.shipmotions.nl.

14.2 Equations of Motion

The equations of motion for six degrees of freedom of a twin-hull ship are defined by:

$$
\begin{array}{rll}
\text{Surge:} & \rho\nabla_T \cdot \ddot{x} & -X_{Th_1} = X_{Tw_1} \\
\text{Sway:} & \rho\nabla_T \cdot \ddot{y} & -X_{Th_2} = X_{Tw_2} \\
\text{Heave:} & \rho\nabla_T \cdot \ddot{z} & -X_{Th_3} = X_{Tw_3} \\
\text{Roll:} & I_{Txx} \cdot \ddot{\phi} - I_{Txz} \cdot \ddot{\psi} & -X_{Th_4} = X_{Tw_4} \\
\text{Pitch:} & I_{Txx} \cdot \ddot{\theta} & -X_{Th_5} = X_{Tw_5} \\
\text{Yaw:} & I_{Tzz} \cdot \ddot{\psi} - I_{Tzx} \cdot \ddot{\phi} & -X_{Th_6} = X_{Tw_6}
\end{array}
$$

in which:

∇_T	=	volume of displacement of the twin-hull ship
I_{Tij}	=	solid mass moment of inertia of the twin-hull ship
X_{Th_1}, X_{Th_2}, X_{Th_3}	=	hydromechanical forces in the x-, y- and z-direction
X_{Th_4}, X_{Th_5}, X_{Th_6}	=	hydromechanical moments about the x-, y- and z-axis
X_{Tw_1}, X_{Tw_2}, X_{Tw_3}	=	exciting wave forces in the x-, y- and z-direction
X_{Tw_4}, X_{Tw_5}, X_{Tw_6}	=	exciting wave moments about the x-, y- and z-axis

14.3 Hydromechanical Forces and Moments

The equations of motion for six degrees of freedom and the hydro- mechanic forces and moments in here, are defined by:

$$
\begin{array}{rcl}
-X_{Th_1} & = & +2a_{11} \cdot \ddot{x} + 2b_{11} \cdot \dot{x} + 2c_{11} \cdot x \\
& & +2a_{13} \cdot \ddot{z} + 2b_{13} \cdot \dot{z} + 2c_{13} \cdot z \\
& & +2a_{15} \cdot \ddot{\theta} + 2b_{15} \cdot \dot{\theta} + 2c_{15} \cdot \theta \\
-X_{Th_2} & = & +2a_{22} \cdot \ddot{y} + 2b_{22} \cdot \dot{y} + 2c_{22} \cdot y \\
& & +2a_{24} \cdot \ddot{\phi} + 2b_{24} \cdot \dot{\phi} + 2c_{24} \cdot \phi \\
& & +2a_{26} \cdot \ddot{\psi} + 2b_{26} \cdot \dot{\psi} + 2c_{26} \cdot \psi \\
-X_{Th_3} & = & +2a_{31} \cdot \ddot{x} + 2b_{31} \cdot \dot{x} + 2c_{31} \cdot x \\
& & +2a_{33} \cdot \ddot{z} + 2b_{33} \cdot \dot{z} + 2c_{33} \cdot z \\
& & +2a_{35} \cdot \ddot{\theta} + 2b_{35} \cdot \dot{\theta} + 2c_{35} \cdot \theta \\
-X_{Th_4} & = & +2a_{42} \cdot \ddot{y} + 2b_{42} \cdot \dot{y} + 2c_{42} \cdot y \\
& & +2a_{44} \cdot \ddot{\phi} + 2b_{44} \cdot \dot{\phi} + 2c_{44} \cdot \phi \\
& & +2y_T^2 \cdot a_{33} \cdot \ddot{\phi} + 2y_T^2 \cdot b_{33} \cdot \dot{\phi} + 2y_T^2 \cdot c_{33} \cdot \phi \\
& & +2a_{46} \cdot \ddot{\psi} + 2b_{46} \cdot \dot{\psi} + 2c_{46} \cdot \psi \\
-X_{Th_5} & = & +2a_{51} \cdot \ddot{x} + 2b_{51} \cdot \dot{x} + 2c_{51} \cdot x \\
& & +2a_{53} \cdot \ddot{z} + 2b_{53} \cdot \dot{z} + 2c_{53} \cdot z \\
& & +2a_{55} \cdot \ddot{\theta} + 2b_{55} \cdot \dot{\theta} + 2c_{55} \cdot \theta \\
-X_{Th_6} & = & +2a_{62} \cdot \ddot{y} + 2b_{62} \cdot \dot{y} + 2c_{62} \cdot y \\
& & +2a_{64} \cdot \ddot{\phi} + 2b_{64} \cdot \dot{\phi} + 2c_{64} \cdot \phi \\
& & +2a_{66} \cdot \ddot{\psi} + 2b_{66} \cdot \dot{\psi} + 2c_{66} \cdot \psi \\
& & +2y_T^2 \cdot a_{11} \cdot \ddot{\psi} + 2y_T^2 \cdot b_{11} \cdot \dot{\psi} + 2y_T^2 \cdot c_{11} \cdot \psi
\end{array}
$$

In here, y_T is half the distance between the centre planes.

14.4 Exciting Wave Forces and Moments

The first order wave potential for an arbitrary water depth h is defined in the new coordinate system by:

$$\Phi_w = \frac{-g}{\omega} \cdot \frac{\cosh k(h+z_b)}{\cosh(kh)} \cdot \zeta_a \sin(\omega_e t - kx_b \cos\mu - ky_b \sin\mu)$$

This holds that for the port side (ps) and starboard (sb) hulls the equivalent components of the orbital accelerations and velocities in the surge, sway, heave and roll directions are equal to:

$$\begin{aligned}
\ddot{\zeta}^*_{w_1}(ps) &= -kg\cos\mu \cdot \zeta^*_{a_1} \sin(\omega_e t - kx_b\cos\mu - ky_T\sin\mu) \\
\ddot{\zeta}^*_{w_1}(sb) &= -kg\cos\mu \cdot \zeta^*_{a_1} \sin(\omega_e t - kx_b\cos\mu + ky_T\sin\mu) \\
\dot{\zeta}^*_{w_1}(ps) &= \frac{+kg\cos\mu}{\omega} \cdot \zeta^*_{a_1} \cos(\omega_e t - kx_b\cos\mu - ky_T\sin\mu) \\
\dot{\zeta}^*_{w_1}(sb) &= \frac{+kg\cos\mu}{\omega} \cdot \zeta^*_{a_1} \cos(\omega_e t - kx_b\cos\mu + ky_T\sin\mu) \\
\ddot{\zeta}^*_{w_2}(ps) &= -kg\sin\mu \cdot \zeta^*_{a_2} \sin(\omega_e t - kx_b\cos\mu - ky_T\sin\mu) \\
\ddot{\zeta}^*_{w_2}(sb) &= -kg\sin\mu \cdot \zeta^*_{a_2} \sin(\omega_e t - kx_b\cos\mu + ky_T\sin\mu) \\
\dot{\zeta}^*_{w_2}(ps) &= \frac{+kg\sin\mu}{\omega} \cdot \zeta^*_{a_2} \cos(\omega_e t - kx_b\cos\mu - ky_T\sin\mu) \\
\dot{\zeta}^*_{w_2}(sb) &= \frac{+kg\sin\mu}{\omega} \cdot \zeta^*_{a_2} \cos(\omega_e t - kx_b\cos\mu + ky_T\sin\mu) \\
\ddot{\zeta}^*_{w_3}(ps) &= -kg \cdot \zeta^*_{a_3} \cos(\omega_e t - kx_b\cos\mu - ky_T\sin\mu) \\
\ddot{\zeta}^*_{w_3}(sb) &= -kg \cdot \zeta^*_{a_3} \cos(\omega_e t - kx_b\cos\mu + ky_T\sin\mu) \\
\dot{\zeta}^*_{w_3}(ps) &= \frac{+kg}{\omega} \cdot \zeta^*_{a_3} \sin(\omega_e t - kx_b\cos\mu - ky_T\sin\mu) \\
\dot{\zeta}^*_{w_3}(sb) &= \frac{+kg}{\omega} \cdot \zeta^*_{a_3} \sin(\omega_e t - kx_b\cos\mu + ky_T\sin\mu)
\end{aligned}$$

From this follows the total wave loads for the degrees of freedom. In these loads on the following pages, the "Modified Strip Theory" includes the outlined terms. When ignoring these outlined terms the "Ordinary Strip Theory" is presented.
The exciting wave forces for surge are:

$$\begin{aligned}
X_{T w_1} = &+\int_L M'_{11} \cdot \left(\ddot{\zeta}^*_{w_1}(ps) + \ddot{\zeta}^*_{w_1}(sb)\right) \cdot dx_b \\
&+\boxed{\frac{V}{\omega \cdot \omega_e} \int_L \frac{dN'_{11}}{dx_b} \cdot \left(\ddot{\zeta}^*_{w_1}(ps) + \ddot{\zeta}^*_{w_1}(sb)\right) \cdot dx_b} \\
&+\int_L \left(\boxed{\frac{\omega}{\omega_e}} \cdot N'_{11} - V \cdot \frac{dM'_{11}}{dx_b}\right) \cdot \left(\dot{\zeta}^*_{w_1}(ps) + \dot{\zeta}^*_{w_1}(sb)\right) \cdot dx_b
\end{aligned}$$

$$+\int\limits_L \big(X'_{FK_1}(ps)+X'_{FK_1}(sb)\big)\cdot dx_b$$

The exciting wave forces for sway are:

$$\begin{aligned}
X_{Tw_2} \;=\;& +\int\limits_L M'_{22}\cdot\left(\ddot{\zeta}^*_{w_2}(ps)+\ddot{\zeta}^*_{w_2}(ps)\right)\cdot dx_b\\
&+\boxed{\frac{V}{\omega\cdot\omega_e}\int\limits_L \frac{dN'_{22}}{dx_b}\cdot\left(\ddot{\zeta}^*_{w_2}(ps)+\ddot{\zeta}^*_{w_2}(sb)\right)\cdot dx_b}\\
&+\int\limits_L \left(\boxed{\frac{\omega}{\omega_e}}\cdot N'_{22}-V\cdot\frac{dM'_{22}}{dx_b}\right)\cdot\left(\dot{\zeta}^*_{w_2}(ps)+\dot{\zeta}^*_{w_2}(sb)\right)\cdot dx_b\\
&+\int\limits_L \big(X'_{FK_2}(ps)+X'_{FK_2}(sb)\big)\cdot dx_b
\end{aligned}$$

The exciting wave forces for heave are:

$$\begin{aligned}
X_{Tw_3} \;=\;& +\int\limits_L M'_{33}\cdot\left(\ddot{\zeta}^*_{w_3}(ps)+\ddot{\zeta}^*_{w_3}(sb)\right)\cdot dx_b\\
&+\boxed{\frac{V}{\omega\cdot\omega_e}\int\limits_L \frac{dN'_{33}}{dx_b}\cdot\left(\ddot{\zeta}^*_{w_3}(ps)+\ddot{\zeta}^*_{w_3}(sb)\right)\cdot dx_b}\\
&+\int\limits_L \left(\boxed{\frac{\omega}{\omega_e}}\cdot N'_{33}-V\cdot\frac{dM'_{33}}{dx_b}\right)\cdot\left(\dot{\zeta}^*_{w_3}(ps)+\dot{\zeta}^*_{w_3}(sb)\right)\cdot dx_b\\
&+\int\limits_L \big(X'_{FK_3}(ps)+X'_{FK_3}(sb)\big)\cdot dx_b
\end{aligned}$$

The exciting wave moments for roll are:

$$\begin{aligned}
X_{Tw_4} \;=\;& +\int\limits_L \big(X'_{FK_4}(ps)+X'_{FK_4}(sb)\big)\cdot dx_b\\
&+\int\limits_L M'_{42}\cdot\left(\ddot{\zeta}^*_{w_2}(ps)+\ddot{\zeta}^*_{w_2}(sb)\right)\cdot dx_b\\
&+\boxed{\frac{V}{\omega\cdot\omega_e}\int\limits_L \frac{dN'_{42}}{dx_b}\cdot\left(\ddot{\zeta}^*_{w_2}(ps)+\ddot{\zeta}^*_{w_2}(sb)\right)\cdot dx_b}\\
&+\int\limits_L \left(\boxed{\frac{\omega}{\omega_e}}\cdot N'_{42}-V\cdot\frac{dM'_{42}}{dx_b}\right)\cdot\left(\dot{\zeta}^*_{w_2}(ps)+\dot{\zeta}^*_{w_2}(sb)\right)\cdot dx_b\\
&+\overline{OG}\cdot X_{Tw_2}\\
&+y_T\cdot X_{Tw_3}
\end{aligned}$$

The exciting wave moments for pitch are:

$$
\begin{aligned}
X_{Tw_5} \;=\; & -\int\limits_L M'_{11} \cdot \overline{bG} \cdot \left(\ddot{\zeta}^*_{w_1}(ps) + \ddot{\zeta}^*_{w_1}(sb)\right) \cdot dx_b \\
& + \boxed{\frac{V}{\omega \cdot \omega_e} \int\limits_L \frac{dN'_{11}}{dx_b} \cdot \overline{bG} \cdot \left(\ddot{\zeta}^*_{w_1}(ps) + \ddot{\zeta}^*_{w_1}(sb)\right) \cdot dx_b} \\
& - \int\limits_L (\left(\boxed{\frac{\omega}{\omega_e}} \cdot N'_{11} - V \cdot \frac{dM'_{11}}{dx_b} \right)) \cdot \overline{bG} \cdot \left(\dot{\zeta}^*_{w_1}(ps) + \dot{\zeta}^*_{w_1}(sb)\right) \cdot dx_b \\
& - \int\limits_L \left(X'_{FK_1}(ps) + X'_{FK_1}(sb)\right) \cdot \overline{bG} \cdot dx_b \\
& - \int\limits_L M'_{33} \cdot x_b \cdot \left(\ddot{\zeta}^*_{w_3}(ps) + \ddot{\zeta}^*_{w_3}(sb)\right) \cdot dx_b \\
& + \boxed{\frac{V}{\omega \cdot \omega_e} \int\limits_L \frac{dN'_{33}}{dx_b} \cdot x_b \cdot \left(\ddot{\zeta}^*_{w_3}(ps) + \ddot{\zeta}^*_{w_3}(sb)\right) \cdot dx_b} \\
& - \int\limits_L \left(\boxed{\frac{\omega}{\omega_e}} \cdot N'_{33} - V \cdot \frac{dM'_{33}}{dx_b} \right) \cdot x_b \cdot \left(\dot{\zeta}^*_{w_3}(ps) + \dot{\zeta}^*_{w_3}(sb)\right) \cdot dx_b \\
& - \int\limits_L \left(X'_{FK_3}(ps) + X'_{FK_3}(sb) \cdot x_b\right) \cdot dx_b
\end{aligned}
$$

The exciting wave moments for yaw are:

$$
\begin{aligned}
X_{Tw_6} \;=\; & +\int\limits_L M'_{22} \cdot x_b \cdot \left(\ddot{\zeta}^*_{w_2}(ps) + \ddot{\zeta}^*_{w_2}(sb)\right) \cdot dx_b \\
& + \boxed{\frac{V}{\omega \cdot \omega_e} \int\limits_L \frac{dN'_{22}}{dx_b} \cdot x_b \cdot \left(\ddot{\zeta}^*_{w_2}(ps) + \ddot{\zeta}^*_{w_2}(sb)\right) \cdot dx_b} \\
& + \int\limits_L \left(\boxed{\frac{\omega}{\omega_e}} \cdot N'_{22} - V \cdot \frac{dM'_{22}}{dx_b} \right) \cdot x_b \cdot \left(\dot{\zeta}^*_{w_2}(ps) + \dot{\zeta}^*_{w_2}(sb)\right) \cdot dx_b \\
& + \int\limits_L \left(X'_{FK_2}(ps) + X'_{FK_2}(sb)\right) \cdot x_b \cdot dx_b \\
& + y_T \cdot X_{Tw_1}
\end{aligned}
$$

14.5 Added Resistance due to Waves

The added resistances can be found easily from the definitions of the mono-hull ship by using the wave elevation at each individual centre line and replacing the heave motion z by:

$$z(ps) = z + y_T \cdot \phi \qquad \text{and} \qquad z(sb) = z - y_T \cdot \phi$$

14.5.1 Radiated Energy Method

The transfer function of the mean added resistance of twin-hull ships according to the method of [Gerritsma and Beukelman, 1972] becomes:

$$\frac{R_{aw}}{\zeta_a^2} = \frac{-k\cos\mu}{2\omega_e} \cdot \int\limits_L \left(N'_{33} - V \cdot \frac{dM'_{33}}{dx_b} \right) \cdot \frac{V_{z_a}^{*\,2}(ps) + V_{z_a}^{*\,2}(sb)}{\zeta_a^2} \cdot dx_b$$

with:

$$\begin{aligned}
V_z^*(ps) &= \dot{\zeta}_{w_3}^*(ps) - \left(\dot{z} - x_b \cdot \dot{\theta} + V \cdot \theta - y_T \cdot \dot{\phi} \right) \\
V_z^*(sb) &= \dot{\zeta}_{w_3}^*(sb) - \left(\dot{z} - x_b \cdot \dot{\theta} + V \cdot \theta + y_T \cdot \dot{\phi} \right) \\
\dot{\zeta}_{w_3}^*(ps) &= \frac{+kg}{\omega} \cdot \zeta_{a_3}^* \sin(\omega_e t - kx_b \cos\mu - ky_T \sin\mu) \\
\dot{\zeta}_{w_3}^*(sb) &= \frac{+kg}{\omega} \cdot \zeta_{a_3}^* \sin(\omega_e t - kx_b \cos\mu + ky_T \sin\mu)
\end{aligned}$$

14.5.2 Integrated Pressure Method

The transfer function of the mean added resistance of twin-hull ships according to the method of [Boese, 1970] becomes:

$$\begin{aligned}
\frac{R_{aw_2}}{\zeta_a^2} =& \\
&+\frac{1}{2}\rho\nabla\omega_e^2 \frac{z_a(ps)}{\zeta_a} \cdot \frac{\theta_a}{\zeta_a} \cdot \cos(\varepsilon_{z(ps)\zeta(ps)} - \varepsilon_{\theta\zeta(ps)}) \\
&+\frac{1}{2}\rho\nabla\omega_e^2 \frac{z_a(sb)}{\zeta_a} \cdot \frac{\theta_a}{\zeta_a} \cdot \cos(\varepsilon_{z(sb)\zeta(sb)} - \varepsilon_{\theta\zeta(sb)}) \\
&+\frac{1}{2}\rho g \int\limits_L \left(1 - \frac{z_{x_a}^2(ps)}{\zeta_a^2} - \frac{2s_a(ps) \cdot \cos(-kx_b\cos\mu - ky_T\sin\mu - \varepsilon_{s(ps)\zeta(ps)})}{\zeta_a \cdot \tanh(kh)} \right) \frac{dy_w}{dx_b}\, dx_b \\
&+\frac{1}{2}\rho g \int\limits_L \left(1 - \frac{z_{x_a}^2(sb)}{\zeta_a^2} - \frac{2s_a(sb) \cdot \cos(-kx_b\cos\mu - ky_T\sin\mu - \varepsilon_{s(sb)\zeta(sb)})}{\zeta_a \cdot \tanh(kh)} \right) \frac{dy_w}{dx_b}\, dx_b
\end{aligned}$$

with:

$$\begin{aligned}
\zeta(ps) &= \zeta_a \cos(\omega_e t - kx_b\cos\mu - ky_T\sin\mu) \\
\zeta(sb) &= \zeta_a \cos(\omega_e t - kx_b\cos\mu + ky_T\sin\mu) \\
z_x(ps) &= z - x_b \cdot \theta + y_T \cdot \phi \\
z_x(sb) &= z - x_b \cdot \theta - y_T \cdot \phi \\
s(ps) &= \zeta(ps) - z + x_b \cdot \theta - y_T \cdot \phi \\
s(sb) &= \zeta(sb) - z + x_b \cdot \theta + y_T \cdot \phi
\end{aligned}$$

14.6 Bending and Torsional Moments

According to Newton's second law of dynamics, the harmonic lateral, vertical and torsional dynamic loads per unit length on the unfastened disk of a twin-hull ship are given by:

$$q_{T1}(x_b) = +X'_{Th_2}(x_b) + X'_{Tw_2}(x_b)$$

$$
\begin{aligned}
& -m'_T(x_b) \cdot \left(\ddot{x} - \overline{bG} \cdot \ddot{\theta} \right) \\
q_{T2}(x_b) \;\; = \;\; & +X'_{Th_2}(x_b) + X'_{Tw_2}(x_b) + 2\rho g A_s \cdot \phi \\
& -m'_T(x_b) \cdot \left(\ddot{y} + x_b \cdot \ddot{\psi} - z'_m \cdot \ddot{\phi} + g \cdot \phi \right) \\
q_{T3}(x_b) \;\; = \;\; & +X'_{Th_3}(x_b) + X'_{Tw_3}(x_b) \\
& -m'_T(x_b) \cdot \left(\ddot{z} - x_b \cdot \ddot{\theta} \right) \\
q_{T4}(x_b, z_1) \;\; = \;\; & +X'_{Th_4}(x_b) + X'_{Tw_4}(x_b) \\
& -m'_T(x_b) \cdot \left(k'^2_{Txx} \cdot \ddot{\phi} - z'_m \cdot (\ddot{y} + x_b \cdot \ddot{\psi} + g \cdot \phi) \right) \\
& +z_1 \cdot q_{T2}(x_b)
\end{aligned}
$$

In here:

m'_T	=	the mass per unit length of the twin-hull ship
k'_{Txx}	=	the local sold mass gyradius for roll
A_s	=	sectional area of one hull

The calculation procedure of the forces and moments is similar to the procedure given before for mono-hull ships.

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Chapter 15

Numerical Recipes

Some typical numerical recipes, as used in the strip theory program SEAWAY, are described in more detail here.

15.1 Polynomials

Discrete points can be connected by a first degree or a second degree polynomial, see figure 15.1-a,b.

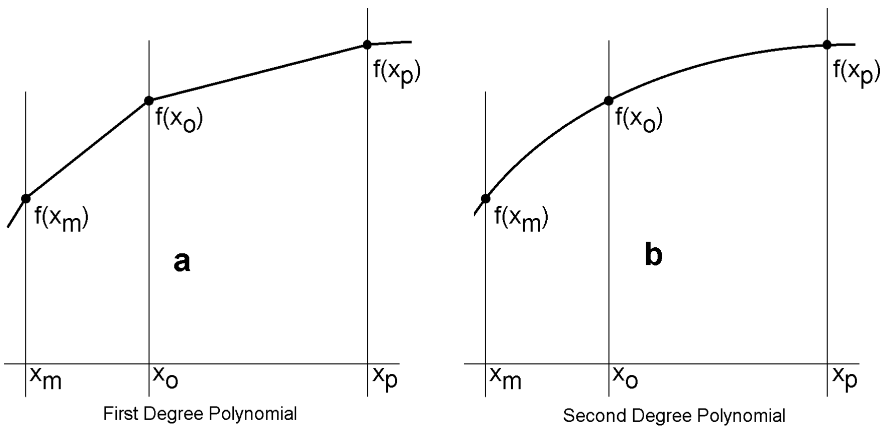

Figure 15.1: First and Second Order Polynomials Through Discrete Points

15.1.1 First Degree Polynomials

A first degree - or linear - polynomial, as given in figure 15.1-a, is defined by:

$$f(x) = ax + b$$

with in the interval $x_m < x < x_0$ the following coefficients:

[0] J.M.J. Journée, *"Theoretical Manual of SEAWAY, Release 4.19"*, Report 1216a, February 2001, Ship Hydromechanics Laboratory, Delft University of Technology, Mekelweg 2, 2628 CD Delft, The Netherlands. For updates see web site: http://dutw189.wbmt.tudelft.nl/~johan or http://www.shipmotions.nl.

$$
\begin{aligned}
a &= \frac{f(x_0) - f(x_m)}{x_0 - x_m} \\
b &= f(x_0) - a x_0
\end{aligned}
$$

and in the interval $x_0 < x < x_p$ the following coefficients:

$$
\begin{aligned}
a &= \frac{f(x_p) - f(x_0)}{x_p - x_0} \\
b &= f(x_0) - a x_0
\end{aligned}
$$

Notify that only one interval is required for obtaining the coefficients in that interval.

15.1.2 Second Degree Polynomials

A second degree polynomial, as given in figure 15.1-b, is defined by:

$$f(x) = ax^2 + bx + c$$

with in the interval $x_m < x < x_p$ the following coefficients:

$$
\begin{aligned}
a &= \frac{\frac{f(x_p)-f(x_0)}{x_p-x_0} - \frac{f(x_0)-f(x_m)}{x_0-x_m}}{x_p - x_m} \\
b &= \frac{f(x_p) - f(x_0)}{x_p - x_0} - a(x_p + x_0) \\
c &= f(x_0) - a x_0^2 - b x_0
\end{aligned}
$$

Notify that two intervals are required for obtaining these coefficients, valid in both intervals.

15.2 Integrations

Numerical integrations can be carried out by either the trapezoid rule or Simpson's general rule.
SEAWAY uses Simpson's general rule as a standard. Then, the integrations have to be carried out over a number of sets of two intervals, see figure 15.1-b. Numerical inaccuracies can be expected when $x_0 - x_m \ll x_p - x_0$ or $x_p - x_0 \ll x_0 - x_m$. In those cases the trapezoid rule has to be preferred, see figure 15.1-a.
SEAWAY makes the choice bewteen the use of the trapezoid rule andr Simpson's rule automatically, based on the following requirements:

$$
\begin{aligned}
&\text{Trapezoid rule if:} \qquad \frac{x_0 - x_m}{x_p - x_m} < 0.2 \quad \text{or:} \quad \frac{x_0 - x_m}{x_p - x_m} > 5.0 \\
&\text{Simpson's rule if:} \qquad 0.2 < \frac{x_0 - x_m}{x_p - x_m} < 5.0
\end{aligned}
$$

15.2.1 First Degree Integrations

First degree integrations - integrations carried out by the trapezoid rule, see figure 15.1-a - means the use of a linear function:

$$f(x) = ax + b$$

The integral over the interval $x_p - x_0$ becomes:

$$\begin{aligned}\int_{x_0}^{x_p} f(x)\, dx &= \int_{x0}^{x_p} (ax+b)\, dx \\ &= \left[\frac{1}{2}ax^2 + bx\right]_{x_0}^{x_p}\end{aligned}$$

with:

$$\begin{aligned}a &= \frac{f(x_p) - f(x_0)}{x_p - x_0} \\ b &= f(x_0) - ax_0\end{aligned}$$

Integration over two intervals results into:

$$\int_{x_m}^{x_p} f(x)\, dx = \frac{(x_0 - x_m) f(x_m) + (x_p - x_m) f(x_0) + (x_p - x_0) f(x_m)}{2}$$

15.2.2 Second Degree Integrations

Second degree integrations - integrations carried out by Simpson's rule, see figure 15.1-b - have to be carried out over a set of two intervals. At each of the two intervals, the integrand is described by a second degree polynomial:

$$f(x) = ax^2 + bx + c$$

Then the integral becomes:

$$\begin{aligned}\int_{x_m}^{x_p} f(x)\, dx &= \int_{x_m}^{x_p} \left(ax^2 + bx + c\right) dx \\ &= \left[\frac{1}{3}ax^3 + \frac{1}{2}bx^2 + cx\right]_{x_m}^{x_p}\end{aligned}$$

with:

$$\begin{aligned}a &= \frac{\frac{f(x_p)-f(x_0)}{x_p-x_0} - \frac{f(x_0)-f(x_m)}{x_0-x_m}}{x_p - x_m} \\ b &= \frac{f(x_p) - f(x_0)}{x_p - x_0} - a(x_p + x_0) \\ c &= f(x_0) - ax_0^2 - bx_0\end{aligned}$$

Some algebra leads for the integration over these two intervals to:

$$\int\limits_{x_m}^{x_p} f(x)\,dx = \left\{ \frac{x_0 - x_m - \frac{x_p - x_0}{2}}{x_0 - x_m} f(x_m) + \frac{(x_p - x_m)^2}{2(x_0 - x_m)(x_p - x_0)} f(x_0) + \frac{x_p - x_0 - \frac{x_0 - x_m}{2}}{x_p - x_0} f(x_p) \right\} \cdot \frac{x_p - x_m}{3}$$

15.2.3 Integration of Wave Loads

The wave loads can be written as:

$$F_w = F_{wa} \cos\left(\omega_e t + \varepsilon_{F_w\zeta}\right)$$

The in-phase and out-phase parts of the wave loads have to be obtained from longitudinal integrations:

$$\begin{aligned} F_{w1} &= \int\limits_L F'_{w1}(x_b)\,dx = \int\limits_L f'_{w1}(x_b)\cos x_b\,dx_b \\ F_{w2} &= \int\limits_L F'_{w2}(x_b)\,dx = \int\limits_L f'_{w2}(x_b)\sin x_b\,dx_b \end{aligned}$$

Direct numerical integrations of F'_{w1} and F'_{w2} over the ship length, L, require integration intervals, Δx_b, which are much smaller than the smallest wave length, $\Delta x_b \leqslant \lambda_{\min}/10$. This means that a large number of cross sections are required.
This can be avoided by writing $F'_{w1,2}$ in terms of $f'_{w1,2}(x_b)\cos x_b$ and $f'_{w1,2}(x_b)\sin x_b$, in which the integrands $f'_{w1,2}(x_b)$ vary very slow over short wave lengths. These functions $f'_{w1,2}(x_b)$ can be approximated by second degree polynomials:

$$f(x) = ax^2 + bx + c$$

When making use of the general integral rules:

$$\begin{aligned} \int \cos x\,dx &= +\sin x \\ \int x\cos x\,dx &= +\cos x + x\sin x \\ \int x^2\cos x\,dx &= +2x\cos x + \left(x^2 - 2\right)\sin x \end{aligned}$$

and:

$$\begin{aligned}
\int \sin x \, dx &= -\cos x \\
\int x \sin x \, dx &= +\sin x - x \cos x \\
\int x^2 \sin x \, dx &= +2x \sin x - \left(x^2 - 2\right) \cos x
\end{aligned}$$

the following expressions can be obtained for the in-phase and out-phase parts of the wave loads, integrated from x_m through x_p, so over the two intervals $x_0 - x_m$ and $x_p - x_0$:

$$\begin{aligned}
\int\limits_{x_m}^{x_p} F(x)\, dx &= \int\limits_{x_m}^{x_p} f(x) \cos x \, dx \\
&= \int\limits_{x_m}^{x_p} \left(ax^2 + bx + c\right) \cos x \, dx \\
&= a \int\limits_{x_m}^{x_p} x^2 \cos x \, dx + b \int\limits_{x_m}^{x_p} x \cos x \, dx + c \int\limits_{x_m}^{x_p} \cos x \, dx \\
&= \left[+\left(f(x) - 2a\right) \sin x + (2ax + b) \cos x\right]_{x_m}^{x_p}
\end{aligned}$$

$$\begin{aligned}
\int\limits_{x_m}^{x_p} F(x)\, dx &= \int\limits_{x_m}^{x_p} f(x) \sin x \, dx \\
&= \int\limits_{x_m}^{x_p} \left(ax^2 + bx + c\right) \sin x \, dx \\
&= a \int\limits_{x_m}^{x_p} x^2 \sin x \, dx + b \int\limits_{x_m}^{x_p} x \sin x \, dx + c \int\limits_{x_m}^{x_p} \sin x \, dx \\
&= \left[-\left(f(x) - 2a\right) \cos x + (2ax + b) \sin x\right]_{x_m}^{x_p}
\end{aligned}$$

with coefficients obtained by:

$$\begin{aligned}
a &= \frac{\frac{f(x_p) - f(x_0)}{x_p - x_0} - \frac{f(x_0) - f(x_m)}{x_0 - x_m}}{x_p - x_m} \\
b &= \frac{f(x_p) - f(x_0)}{x_p - x_0} - a\left(x_p + x_0\right) \\
c &= f(x_0) - a x_0^2 - b x_0
\end{aligned}$$

15.3 Derivatives

First and second degree functions, of which the derivatives have to be determined, have been given in figure 15.2-a,b.

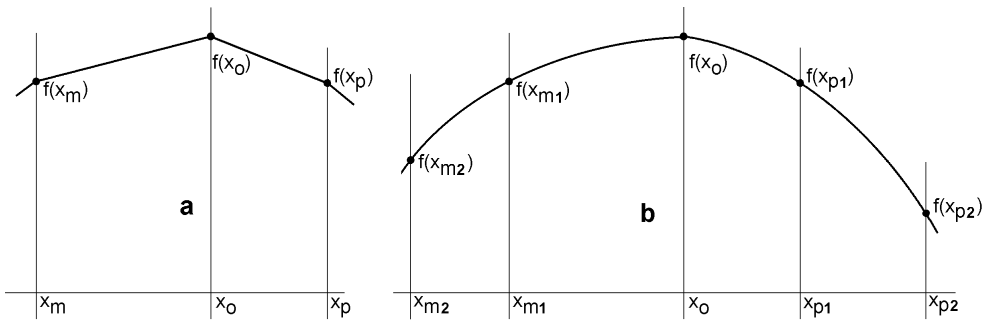

Figure 15.2: Determination of Longitudinal Derivatives

15.3.1 First Degree Derivatives

The two polynomials - each valid over two intervals below and above $x = x_0$ - are given by:

$$\begin{aligned} \text{for } x &< x_0: \quad f(x) = a_m x + b_m \\ \text{for } x &> x_0: \quad f(x) = a_p x + b_p \end{aligned}$$

The derivative is given by:

$$\begin{aligned} \text{for } x &< x_0: \quad \frac{df(x)}{dx} = a_m \\ \text{for } x &> x_0: \quad \frac{df(x)}{dx} = a_p \end{aligned}$$

It is obvious that, generally, the derivative at the left hand side of x_0 - with index m (minus) - and the derivative at the right hand side of x_0 - with index p (plus) - will differ:

$$\left[\left(\frac{df}{dx}\right)_{x=x_0}\right]_{m \text{ (minus or left of } x_0)} \neq \left[\left(\frac{df}{dx}\right)_{x=x_0}\right]_{p \text{ (plus or right of } x_0)}$$

A mean derivative $\frac{df}{dx}$ at $x = x_0$ can be obtained by:

$$\left(\frac{df}{dx}\right)_{x=x_0} = \frac{(x_0 - x_m)\cdot\left[\left(\frac{df}{dx}\right)_{x=x_0}\right]_m + (x_p - x_0)\cdot\left[\left(\frac{df}{dx}\right)_{x=x_0}\right]_p}{x_p - x_m}$$

15.3.2 Second Degree Derivatives

The two polynomials - each valid over two intervals below and above $x = x_0$ - are given by:

$$\begin{aligned} \text{for } x &< x_0: \quad f(x) = a_m x^2 + b_m x + c_m \\ \text{for } x &> x_0: \quad f(x) = a_p x^2 + b_p x + c_p \end{aligned}$$

A derivative of a second degree function:

$$f(x) = ax^2 + bx + c$$

is given by:

$$\frac{df(x)}{dx} = 2ax + b$$

This leads for $x < x_0$ to:

$$\begin{aligned}
\left(\frac{df}{dx}\right)_{x=x_{m2}} &= \frac{\left[\begin{array}{c} -2(x_{m1}-x_{m2})(x_0-x_{m1})\{f(x_{m1})-f(x_{m2})\} \\ -(x_{m1}-x_{m2})^2\{f(x_0)-f(x_{m1})\}+(x_0-x_{m1})^2\{f(x_{m1})-f(x_{m2})\} \end{array}\right]}{(x_{m1}-x_{m2})(x_0-x_{m1})(x_0-x_{m2})} \\
\left(\frac{df}{dx}\right)_{x=x_{m1}} &= \frac{(x_{m1}-x_{m2})^2\{f(x_0)-f(x_{m1})\}+(x_0-x_{m1})^2\{f(x_{m1})-f(x_{m2})\}}{(x_{m1}-x_{m2})(x_0-x_{m1})(x_0-x_{m2})} \\
\left(\frac{df}{dx}\right)_{x=x_0} &= \frac{\left[\begin{array}{c} +2(x_{m1}-x_{m2})(x_0-x_{m1})\{f(x_0)-f(x_{m1})\} \\ +(x_{m1}-x_{m2})^2\{f(x_0)-f(x_{m1})\}-(x_0-x_{m1})^2\{f(x_{m1})-f(x_{m2})\} \end{array}\right]}{(x_{m1}-x_{m2})(x_0-x_{m1})(x_0-x_{m2})}
\end{aligned}$$

and for $x > x_0$ to:

$$\begin{aligned}
\left(\frac{df}{dx}\right)_{x=x_0} &= \frac{\left[\begin{array}{c} -2(x_{p1}-x_0)(x_{p2}-x_{p1})\{f(x_{p1})-f(x_0)\} \\ -(x_{p1}-x_0)^2\{f(x_{p2})-f(x_{p1})\}+(x_{p2}-x_{p1})^2\{f(x_{p1})-f(x_0)\} \end{array}\right]}{(x_{p1}-x_0)(x_{p2}-x_{p1})(x_{p2}-x_0)} \\
\left(\frac{df}{dx}\right)_{x=x_{p1}} &= \frac{(x_{p1}-x_0)^2\{f(x_{p2})-f(x_{p1})\}+(x_{p2}-x_{p1})^2\{f(x_{p1})-f(x_0)\}}{(x_{p1}-x_0)(x_{p2}-x_{p1})(x_{p2}-x_0)} \\
\left(\frac{df}{dx}\right)_{x=x_{p2}} &= \frac{\left[\begin{array}{c} +2(x_{p1}-x_0)(x_{p2}-x_{p1})\{f(x_{p2})-f(x_{p1})\} \\ +(x_{p1}-x_0)^2\{f(x_{p2})-f(x_{p1})\}-(x_{p2}-x_{p1})^2\{f(x_{p1})-f(x_0)\} \end{array}\right]}{(x_{p1}-x_0)(x_{p2}-x_{p1})(x_{p2}-x_0)}
\end{aligned}$$

Generally, the derivative at the left hand side of x_0 - with index m (minus) - and the derivative at the right hand side of x_0 - with index p (plus) - will differ:

$$\left[\left(\frac{df}{dx}\right)_{x=x_0}\right]_{m \text{ (minus or left of } x_0)} \neq \left[\left(\frac{df}{dx}\right)_{x=x_0}\right]_{p \text{ (plus or right of } x_0)}$$

A mean derivative $\frac{df}{dx}$ at $x = x_0$ can be obtained by:

$$\left(\frac{df}{dx}\right)_{x=x_0} = \frac{d_m \cdot \left[\left(\frac{df}{dx}\right)_{x=x_0}\right]_m + d_p \cdot \left[\left(\frac{df}{dx}\right)_{x=x_0}\right]_p}{d_m + d_p}$$

with:

$$\begin{aligned}
d_m &= \frac{\left(x_0 - x_{m1} - \frac{x_{m1}-x_{m2}}{2}\right)(x_0 - x_{m2})}{3(x_0 - x_{m1})} \\
d_p &= \frac{\left(x_{p1} - x_0 - \frac{x_{p2}-x_{p1}}{2}\right)(x_{p2} - x_0)}{3(x_{p1} - x_0)}
\end{aligned}$$

15.4 Curve Lengths

Discrete points, connected by a first degree or a second degree polynomial, are given in figure 15.3-a,b.

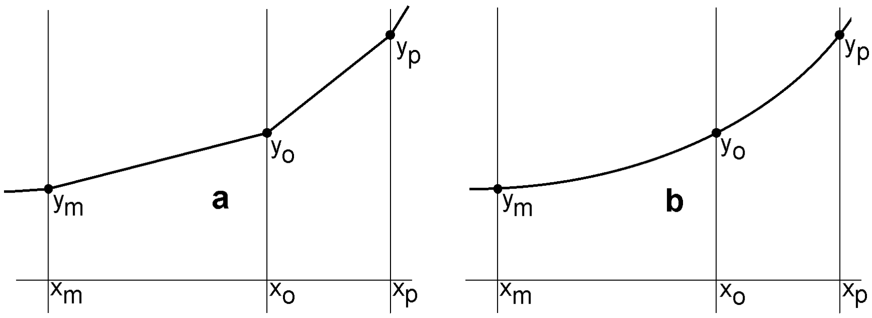

Figure 15.3: First and Second Order Curves

The curve length follows from:

$$\begin{aligned} s_{mp} &= \int_{x_m}^{x_p} ds \\ &= \int_{x_m}^{x_p} \sqrt{dx^2 + dy^2} \end{aligned}$$

15.4.1 First Degree Curves

The curve length of a first degree curve, see figure 15.3-a, in the two intervals in the region $x_m < x < x_p$ is:

$$s_{mp} = \sqrt{(x_0 - x_m)^2 + (y_0 - y_m)^2} + \sqrt{(x_p - x_0)^2 + (y_p - y_0)^2}$$

15.4.2 Second Degree Curves

The curve length of a second degree curve, see figure 15.3-b, in the two intervals in the region $x_m < x < x_p$ is:

$$s_{mp} = p_2 \left\{ p_0\sqrt{1+p_0^2} + \ln\left[p_0 + \sqrt{1+p_0^2}\right] - p_1\sqrt{1+p_1^2} - \ln\left[p_1 + \sqrt{1+p_1^2}\right] \right\}$$

with:

$$\begin{aligned} \cos\alpha &= \frac{x_p - x_m}{\sqrt{(x_p - x_m)^2 + (y_p - y_m)^2}} \\ \sin\alpha &= \frac{y_p - y_m}{\sqrt{(x_p - x_m)^2 + (y_p - y_m)^2}} \end{aligned}$$

$$
\begin{aligned}
p_0 &= \pi + 2 \cdot \frac{(x_p - x_m)\cos\alpha + (y_p - y_m)\sin\alpha}{(x_p - x_0)\cos\alpha + (y_p - y_0)\sin\alpha} \\
p_1 &= \frac{(y_0 - y_m)\cos\alpha - (x_0 - x_m)\sin\alpha}{(x_0 - x_m)\cos\alpha + (y_0 - y_m)\sin\alpha} - \frac{(x_0 - x_m)\cos\alpha + (y_0 - y_m)\sin\alpha}{(x_p - x_0)\cos\alpha + (y_p - y_0)\sin\alpha} \\
p_2 &= \frac{(x_p - x_0)\cos\alpha + (y_p - y_0)\sin\alpha}{4}
\end{aligned}
$$

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