Lecture Notes for

Engineering 5003 – Ship Structures I

Claude Daley, Professor, D.Sc., P.Eng.
Faculty of Engineering and Applied Science
Memorial University
St. John's, CANADA

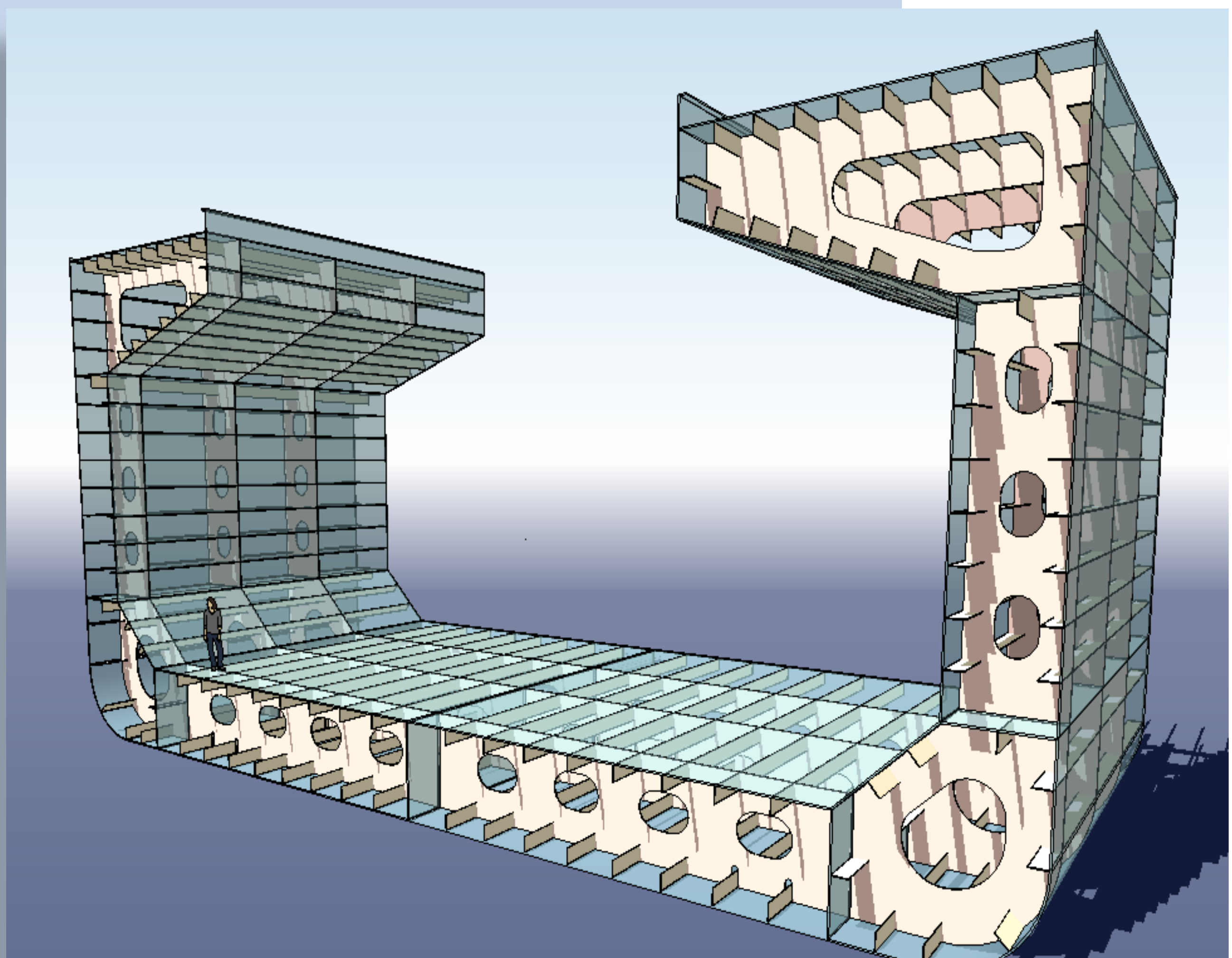


Table of Contents

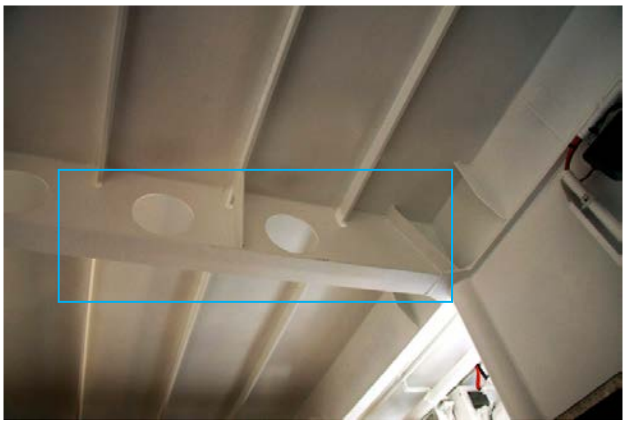
Cover image by CD using Sketchup

PART 1 : Introduction

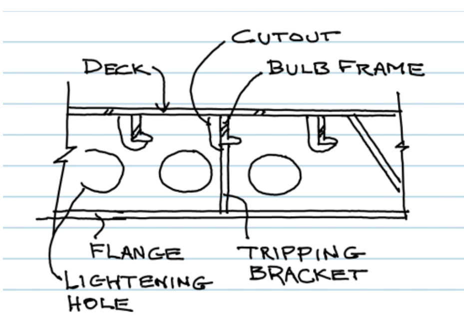
Church in Dubrovnik

Topic 1: Introduction to Ship Structures

The course is intended to develop the student's knowledge of ship structures. The focus is on various types of intact structural behavior, building upon concepts from mechanics of materials. The course project will involve the design, assessment, drawing and reporting on the mid-ship scantlings (hull girder design) of a large vessel. The follow-on course (6003) will move from the consideration of intact behavior to the mechanics of structural failure.



Cruise Ship Structure



hand drawn sketch

One of the aims of the course is for the students to develop the ability to make an **educated guess**. Such guesses are not wild or random. Educated guesses are based on sound reasoning, careful approximation and simplification of the problem. In most cases the 'guess' starts by forming an idea of the problem in its essential form, or in 'bounding' forms. Basic laws of mechanics are considered to determine what fundamental principle might govern the outcome. Most problems are governed by simple conservation laws, such as of forces, moments, momentum and/or energy.

A related aim of the project is for the students to develop the ability to sketch the problem at hand, by hand and clearly. Sketching is a form of symbolic communication, no less valuable than the alphabet or algebra.

Background

Humans have been constructing structures for a long time. A structure is a tool for carrying (carrying what is in or on the structure). Ship structures have evolved like all other types of structures (buildings, aircraft, bridges ...). Design was once purely a craft. Design is evolving as we understand more about the structure itself and the environment that we subject it to.

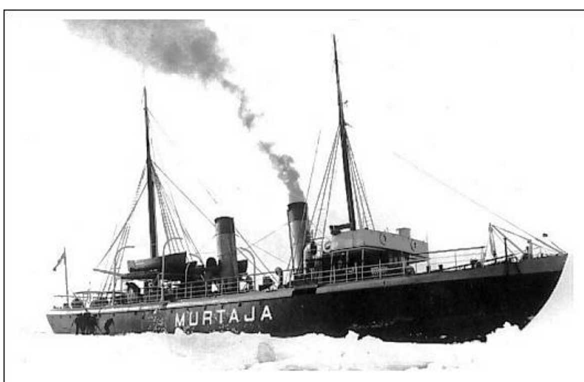
Traditional Design

Gondolas in Venice

- built by tradition (prior example)
- changes based primarily on experience (some analysis)
- essentially a builders "Craft"
- QA by proof test and use

Engineering Design

- incorporates analysis based on math/physics
- common designs are codified (building code, class rules..)
- new designs should follow the "Engineering Method"
- design, analysis, construction and regulation are separate specialties
- design practice is evolving: In the 1950 tabulated requirements were found in Class Rules. By the 70s all codes had changed to include prescriptive algebra. New trend are towards "LRFD - load and resistance factored design", "risk based design" and "goal based design". Current practice in large (novel) projects make extensive use of "scenario based" design, with HAZIDs (hazard identification and mitigation).
- The future of design will be "design by simulation" in which the many interacting process and systems will be simulated numerically. In some ways this will represent a return to the idea of proving a design by a "proof test", except it will be a numerical proof test and will simulate the life of the design.



early Finnish icebreaker (public domain - Wikipedia)

Purpose of Ship Structures

The structure of a ship or ocean platform has 3 principal functions:

- Strength (resist weight, environmental forces – waves +)
- Stiffness (resist deflections – allow ship/equipment to function)
- Water tight integrity (stay floating)

Warship (public domain - Wikipedia)

Bulk Carrier FLARE (from TSB report)

There are two other important functions

- provide subdivision (tolerance to damage of 1,3 above)
- support payloads

the beach at Chittagong (Naquib Hossain - Wikipedia)

These functions are all interrelated, but should be considered somewhat separately.

Structural Arrangement

The particular arrangement of the structure is done to suit a variety of demands;

- Hull is shaped (reduce resistance, reduce motions, reduce ice forces, increase ice forces, reduce noise)
- holds are arranged for holding/loading cargo
- holds are arranged for holding/installing engines
- superstructure is arranged for accommodation/navigation
- all structure is arranged for build-ability/maintainability
- all structure is arranged for safety
- all structure is arranged for low cost

Cruise ship Lifeboat

Types of Structural Work

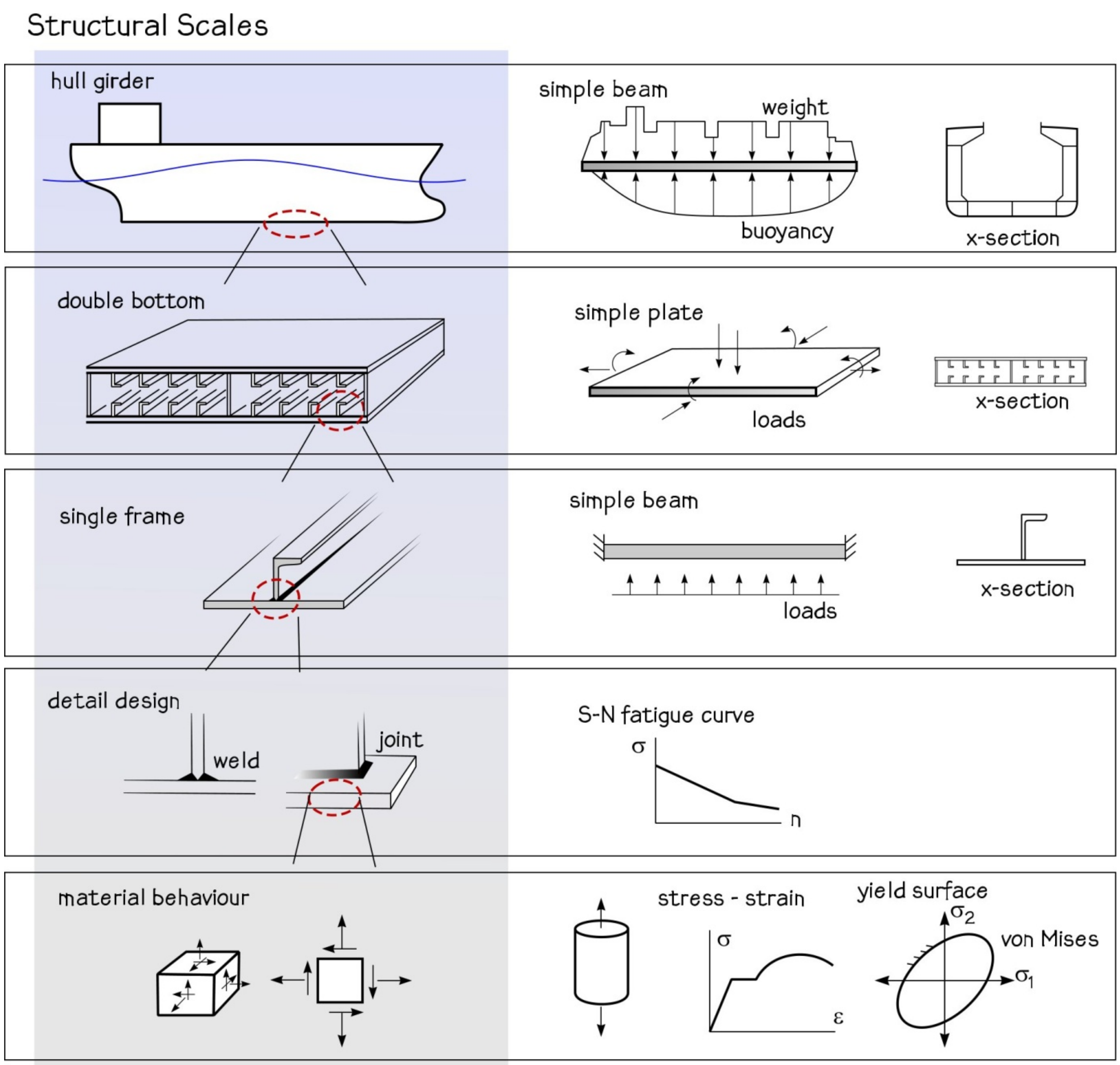
Ship structural specialists are involved in a variety of work;

- Design
- Analysis
- Construction
- Maintenance
- Repair
- Regulation

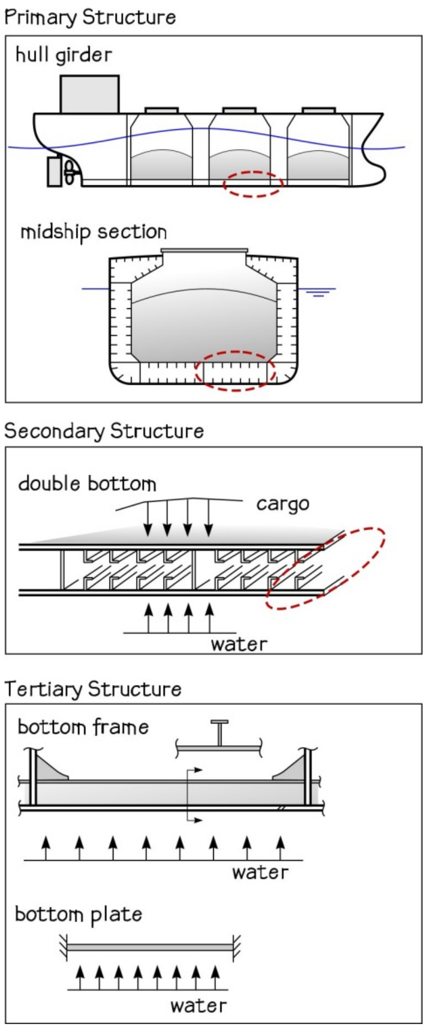
While almost all Naval Architects get involved in structural issues, as with most professions, a few focus on the area and tend to be involved in any advanced work. This course aims to have you develop your 'feel' as well as your knowledge of structures. In other words, you should work at developing you "Engineering Judgment" in the area of ship structures.

Structural Behavior

Ship structural behavior, as with all structural behavior is essentially very simple. Structures are an assemblage of parts. This distinguishes them from objects. A beam or plate is a structural element, but only a collection of structural elements is called a structure. The theory of structures builds upon the field of 'mechanics of materials' (also called mechanics of solids, or strength of materials), by considering the interactions and combined behaviors of collections of structural components. So, much of this course will focus on techniques for understanding collections of structural elements. We will also review and expand, somewhat, on the mechanics of individual elements.



Structural Hierarchy



Levels of Structure

As a structure, a ship is an assemblage of components. At the largest scale a ship is a simple beam, carrying weight and supported by buoyancy. The behavior or the whole ship as a single beam is referred to as the behavior of the **primary structure**.

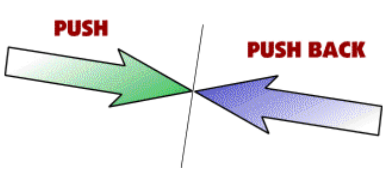
The primary structure is referred to as the hull girder. The strength and stiffness of the hull girder depend on the properties of the cross sections of the ship. The key section is the midship section.

Within the hull, as integral components of the hull, are large structural components that are themselves make of individual structural members, and yet act as individual systems. These are called **secondary structure**. For example, the whole double bottom, between bulkheads, is a unit that acts as a sandwich panel, behaving somewhat like a plate.

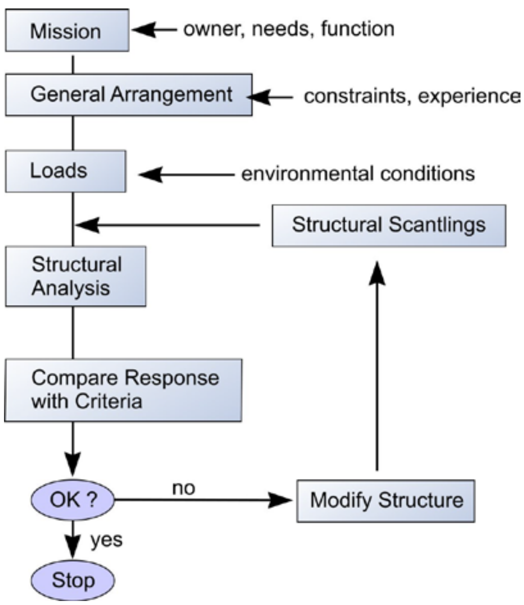
Locally a ship is comprised of frames and plate. These are called **tertiary structure**. The tertiary structure are individual structural members.

Ships are a class of structure called "semi-monocoque". In a pure monocoque, all the strength comes from the outer shell ("coque" in french). To contrast, in "skin-on-frame" construction, the loads are all borne a structure of framing under the skin. In ships, the skin is structurally integral with the framing which supports it, with the skin providing a substantial portion of the overall strength.

Newton's 3rd Law:
action = reaction



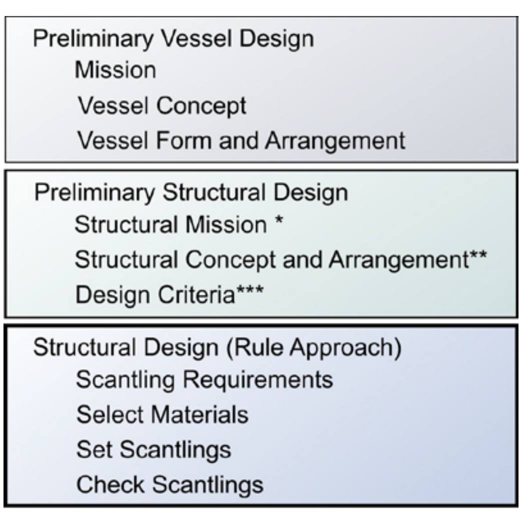
All the various parts and levels of a ship structure interact. Ships are "all-welded" structures, meaning that it is all one single, complex, solid elastic body. The main thing that structures (and all parts of structures) do is "**push back**". i.e. across any interface (across every patch of every plane, everywhere in the universe, always!) the force acts in both ways. This powerful idea is the key to following what happens in a structure.



Structural Design

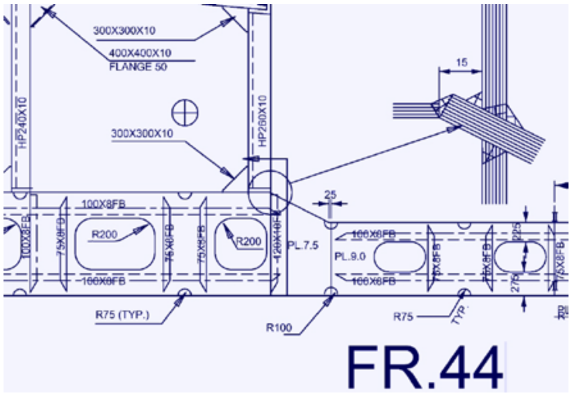
The process of ship structural design varies depending on the specific issues. Structural design occurs after the mission is set and a general arrangement is determined. The general arrangement allows us to determine both the environmental loads and the distribution of hull/outfit/cargo weights. The establishment of scantlings (structural dimensions) is iterative. We assume that a preliminary set of dimensions is settled upon from experience or by other choice. The loads will cause a set of responses (stresses, deflections). The response criteria are then compared to the responses. For any inadequacies we modify the structural dimensions and repeat the response analysis. When all responses are satisfactory, we are finished.

In cases where we wish to satisfy additional constraints (cost, performance..) we add checks for these items after we have checked the structural response. Again we loop until we have met the constraints, and reached optimal values for some measure.



* open ocean, inland waters, ice class?, special features
** main components, bulkhead spacing, tank sizes, frame spacing
*** rule, special rule, direct, mix

As stated above, the structural design can only occur after the overall vessel concept and arrangement is set, which is done during the preliminary design stage. The structural design itself is a process that is comparable to the overall design. Just as the vessels has a mission and a concept to satisfy that mission, so too does the structure have a mission and concept to satisfy the mission. Prior to deciding on the structural sizes (scantlings) , the designer must decide on the overall structural concept and arrangement. In rule based design (Classification Society rules), the loads and response criteria have been combined into standard scantling requirements formulae.



adapted for illustration from a design by Rolls Royce Marine

The user can use these formulae to determine minimum dimensions for members and components. There can then be the need to check additional criteria (e.g buckling, alternate loads). When this is complete the user has a complete structural design, but not yet a final detailed design. The final structural drawings also include detailed design features (e.g. bracket and weld specifications). The image at left is taken from a structural drawing of a web frame in an offshore supply vessel.

Load Types

We will define four general types of structural loads.

- Static Loads (e.g. fixed weights)
- Low Frequency Dynamic Loads (e.g. quasi static load, wave loads)
- High Frequency Dynamic Loads (e.g. vibrations)
- Impact Loads (e.g., blast, collisions)

With both static and quasi-static loads, we do not need to take inertial or rate effects into account in the structural response. With high frequency loads we need to consider structural vibrations which includes inertial effects and damping. For impact loads, we have both transient inertial effects and rate effects in material behavior. It is important to distinguish between loads affecting vessel rigid body motions and elastic structural response. Wave forces may cause the vessel as a whole to respond with inertial effects (heaving motions), but will seldom cause anything but quasi-static response of the structure. The important determinant is the relative frequency of the load and response. Local structure will respond elastically at frequencies in the 100hz to 3000hz range. The hull girder will flex at around the 1 hz rate. The vessel will heave and roll at around the 0.1 hz range. (large vessels/structures will respond more slowly).

launch of MEXOIL, by John N. Teunisson,
14 February 1918 (wikipedia)

In this course we will examine the structural response to quasi-static loads. The hull girder is sized to resist the combination of self weights and wave forces.

Topic 1: Problems

1.1 Longitudinal strength is a primary concern during the design of a ship. Describe the steps in the ship design process (in general terms) that must occur prior to consideration of the longitudinal strength.

1.2 What is the difference between “low frequency dynamic” and “high frequency dynamic” loads? Give examples.

1.3 Describe the types of loads that you would be concerned with during the launch of a vessel on a slipway.

1.4 Loads on ships
The following is a table of load types. Identify each load as static, quasi-static, dynamic or transient. Place a check mark ✓ to indicate which categories apply to each load type. If more than one type applies, explain why in the comments column.

LOAD	static	quasi-static	dynamic	transient	comments
Dry cargo					
Liquid cargo					
Engine					
Propeller					
Ice					
Waves					
Other: ______					
Other:______					

1.5 In preliminary design, when can the preliminary structural calculations be made?

1.6 List 5 purposes of structure in a ship.

1.7 When is a load considered to be quasi-static?

Topic 2: Ship Structural Features

lifeboat on the Battleship Texas

Introduction

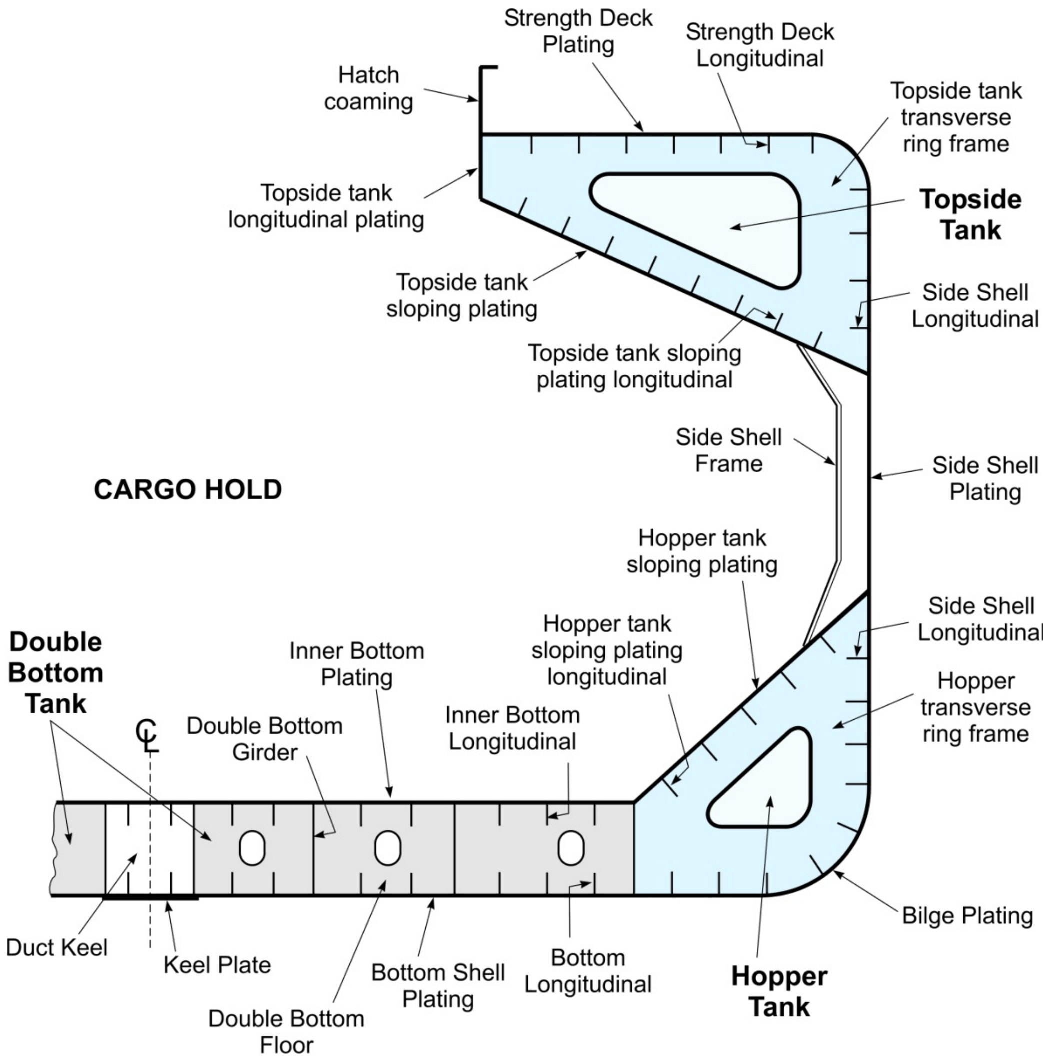
In this Chapter we will

Name and describe ships structural components.

Discuss some structural features and challenges for various vessels,

~~~~~~

Boats are made from a variety of materials, including wood, fiberglass, composites, aluminum, steel and cement. Ships are built mainly from steel. In this Chapter we will name and discuss the main structural features of steel ships. Ships are much longer than they are wide or deep. They are built this way in order to minimize resistance (fuel consumption), and yet maintain adequate stability and seaworthiness. This geometry results in the ship being a girder (a beam built from compound parts). The figures below show sketches of the structural details of the midship section of a bulk carrier.
~~~~~~



Single Side - Double Bottom Bulk Carrier

Figure 1.

This type of vessel is very common, and has many problems. Single side shell vessels are being replaced with double hull vessels. The FLARE had this type of construction.

Figure 2. Bow of Bulk Carrier FLARE

Bulk Carrier FLARE (from TSB report)

Figure 3 shows a 3D representation of the same x-section as show in Figure 1.

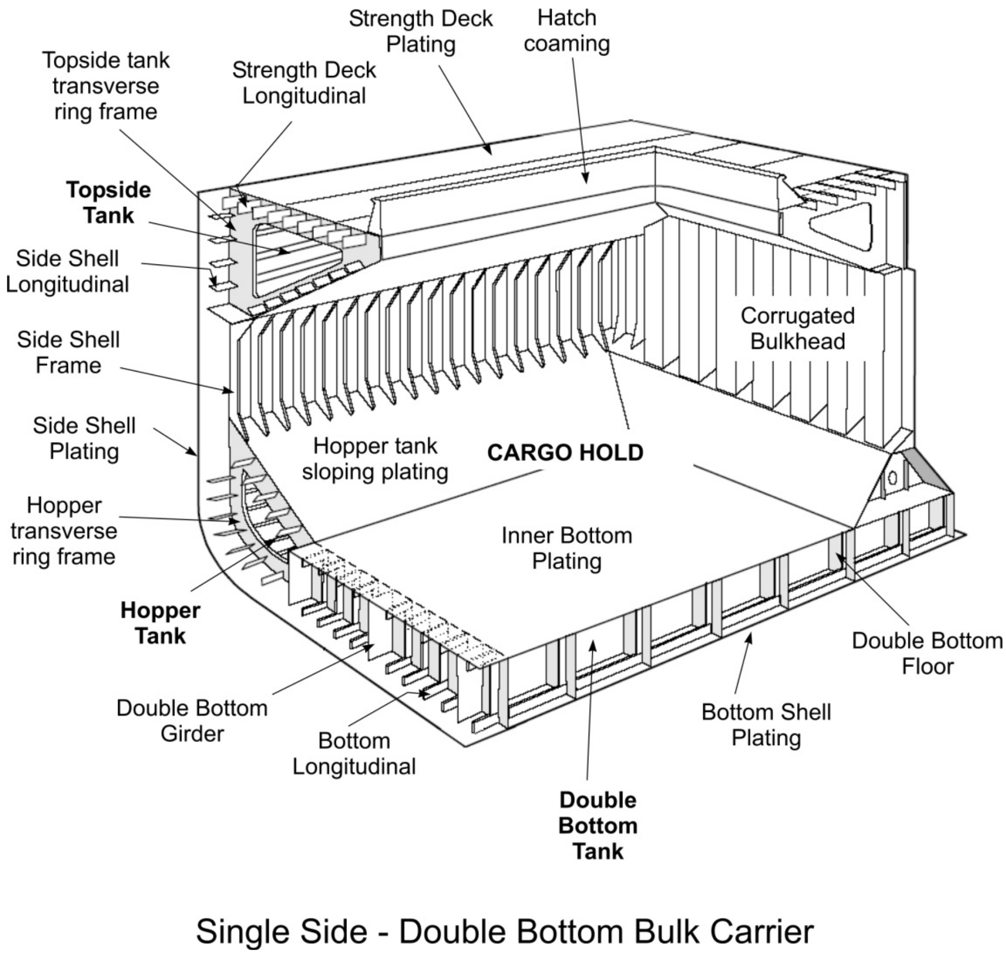


Figure 3.

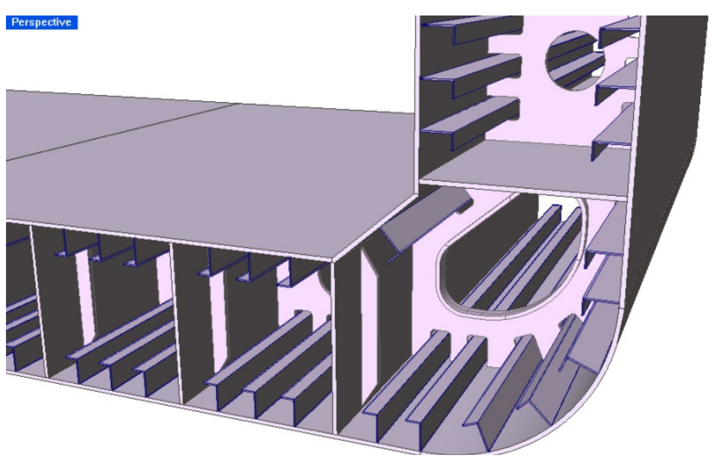


Figure 4 Rhino Sketch of section of longitudinally framed double hull Container vessel.

Ship Structural Photos

Terra Nova FPSO – Floating Production, Storage and Offloading vessel (from wikipedia)

Terra Nova Hull Framing

Terra Nova Structural Connection Details

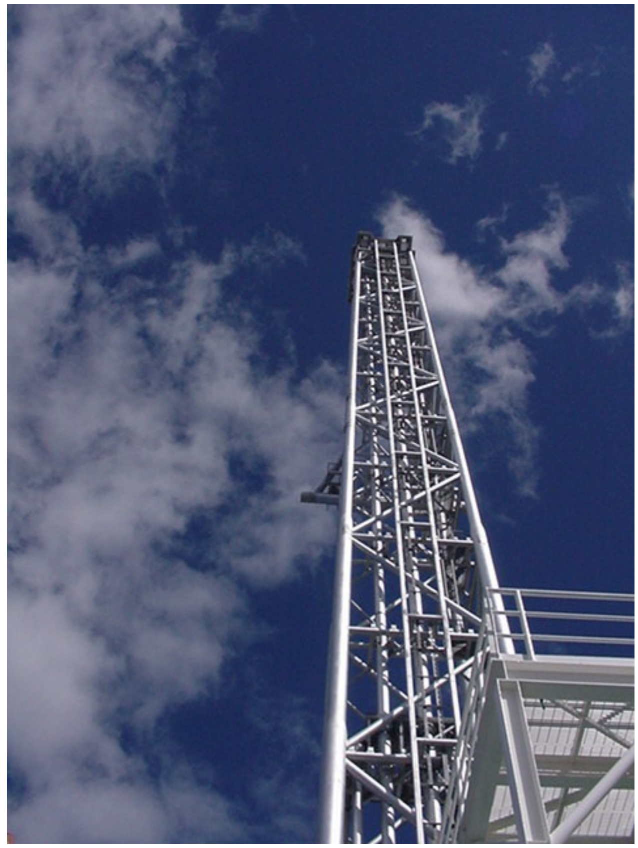
This structure is above the waterline, and so is quite light.


Terra Nova Stringer with web stiffener bracket

Terra Nova Longitudinal angle frames

Transverse flat bar frames through stringer


Terra Nova flat bar frames


Terra Nova Flare tower


Terry Fox – Icebreaker


Bow framing in Terry Fox (photo by R. Frederking)

The Terry Fox is ~7000 tons displacement and capable of ramming thick old ice. It has never been damaged.

Bow of Supply Boat

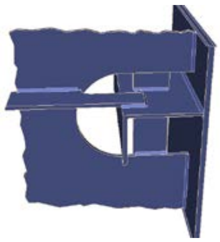
Reduta Ordona (Photo credit: Andrew Kendrick).

Local ice damage

CPF superstructure plating

Topic 2: Problems

2.1 Read the SSC Case Study V and name all the parts of the Rhino sketch shown below.



2.2 What was the basic cause of the "Recurring Failure of Side Longitudinal" in the SSC report?

2.3 Sketch a X-section of a ship at mid-ships and label all features/elements.

2.4 Sketch, free hand, the structure in the double bottom of a ship. Keep it neat and label the elements

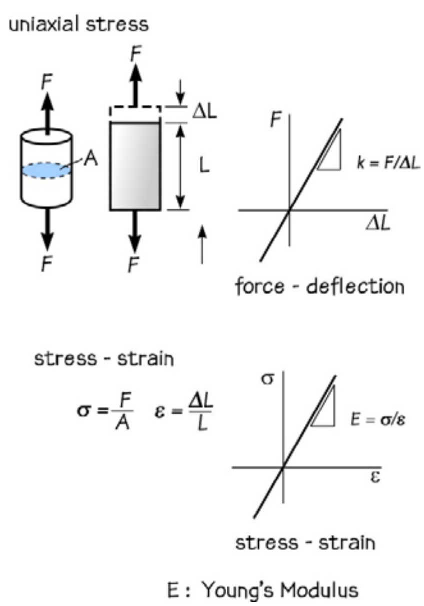
Topic 3: Material Behavior


plastic frame response to ice load test

Introduction

In this Chapter we will

- outline the material behavior models that are necessary to the analysis of structures.



<u>Hooke's Law</u>

Hooke's law is a very simple idea. It just states that there is a linear relationship between force F and deflection ΔL in an elastic body;

$$F = k\ \Delta L$$

where k is the 'spring constant' or the 'stiffness'

For a uni-axial state of stress we can also write Hooke's law in terms of stress (σ: normalized force) and strain (ε: non-dimensional deflection);

$$\sigma = E\ \epsilon$$

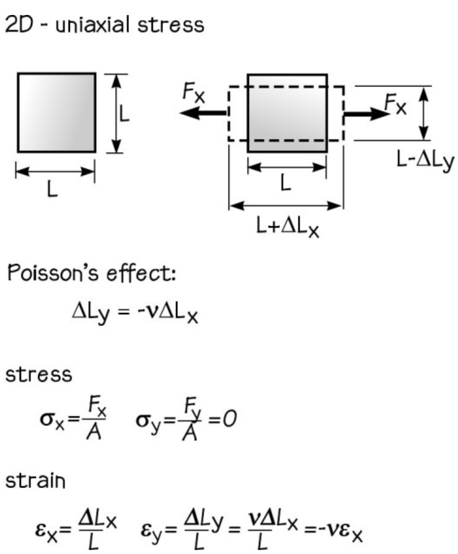
where E is Young's Modulus.

http://en.wikipedia.org/wiki/Robert_Hooke

This law may seem to be too simple to deserve the term 'law'. However, this idea was not easily found. The world, especially in the time of Hooke and before, was so full of variability, inaccuracy and non-linearity that this idea was not obvious. Many things were made from natural materials (stone and wood) and the idea of linear behavior was radical. Hooke was a contemporary, and rival, of Newton. He developed a coil spring for use in a pocket watch. In 1678 he published a discussion of the behavior of his spring, saying: "*ut tensio, sic vis*" meaning "*as the extension, so the force*". Hooke worked in many fields (architecture, astronomy, human memory, microscopy, palaeontology), but it is only in mechanics that his name is associated with a fundamental law.

How important is Hooke's contribution? For structural analysis it is the fundamental idea, as important to structural analysis as is Newton's 2nd law ($F = ma$) to the field of dynamics.

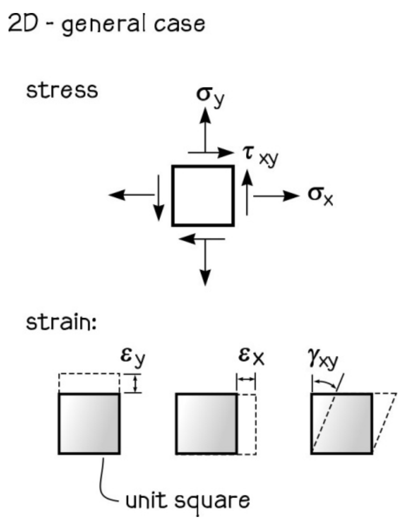
Hooke's law is important because linearity of behavior permits the use of superposition. And only with the idea of superposition can we divide problems up into parts, solve the parts and add them back together to get a total solution. The whole field of structural analysis depends on Hooke's law.



Hooke's law can be expanded to describe 2D and 3D behavior. Consider a 2D sample of elastic material. When a force is applied in one direction (x) the material stretches in that direction and contracts in the lateral direction(y). So for a stress in the x direction we get strains in x and y. This is Hooke's law in 2D for the case of uni-axial stress;

$$\varepsilon_x = \frac{\sigma_x}{E}$$

$$\varepsilon_y = -\nu\,\frac{\sigma_x}{E}$$



When we consider a general state of stress, comprised of a combination of x and y direction stresses (σ_x, σ_y), as well as shear stress (τ_{xy}) we can write the relationship amoung the stresses and strains Hooke's law in 2D for the general case;

$$\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} = \frac{1}{E} \begin{bmatrix} 1 & -\nu & 0 \\ -\nu & 1 & 0 \\ 0 & 0 & 2(1+\nu) \end{bmatrix} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix}$$

or equivalently;

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \dfrac{1-\nu}{2} \end{bmatrix} \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix}$$

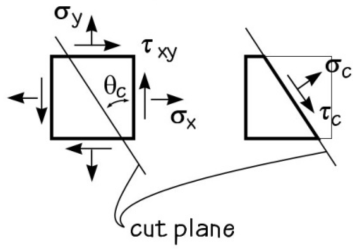
The above equations are used to describe isotropic materials (materials that are similar in all directions, such as steel), which have the same value of E and n in all directions.
Note: Anisotropic materials, such as wood and fiberglass have different values of E for each axis. Hooke's laws for anisotropic materials have many more terms.

Hooke's law can be expressed in 3D as well, but 2D is sufficient for the problems that we will examine.

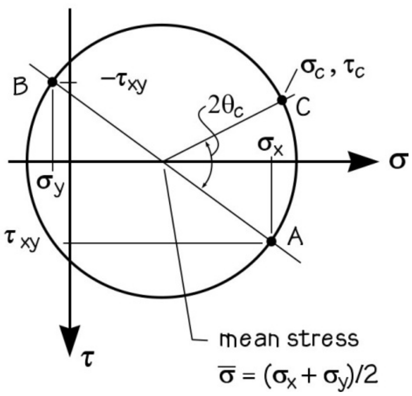
<u>state of stress in 2D</u>

Consider a small element of material with normal and shear stresses on vertical and horizontal planes. We refer to these stresses as engineering stresses, $\sigma_x, \sigma_y, \tau_{xy}$. Now consider what the stresses would be on any other plane, so one that is rotated by the angle θ_c from the vertical (from the plane for σ_x stress). Mohr showed that the stresses on all planes, when plotted, will form a circle in τ vs. σ coordinates.

The stresses on the vertical plane, σ_x and τ_{xy}, are plotted on the Mohr's circle (point A). The stresses on the horizontal plane, σ_y and $-\tau_{xy}$, are plotted at stresses on any plane



Mohr's circle



point B. These two planes are physically 90 degrees from each other, but are 180 degrees apart on the Mohr's circle.

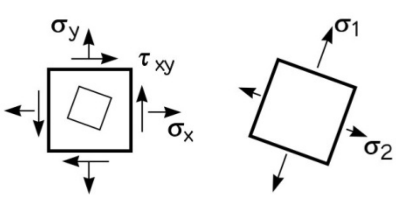
The line joining A, B is a baseline. To find the stresses on a cut plane at angle θ from the vertical plane (the plane of A), we must move 2θ from the 'A' direction around the Mohr's circle. This lands us at point C, where the stresses are , σ_c and τ_c.

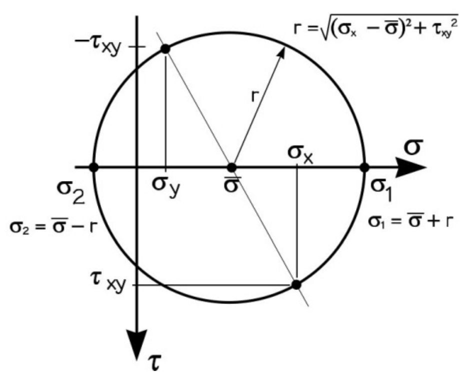
The general equations to find the stresses on a plane at angle θ from the plane of σ_x are;

$$\sigma_n = \frac{1}{2}(\sigma_x + \sigma_y) + \frac{1}{2}(\sigma_x - \sigma_y)cos2\theta + \tau_{xy}sin2\theta$$

$$\tau_n = -\frac{1}{2}(\sigma_x - \sigma_y)sin2\theta + \tau_{xy}cos2\theta$$

principal stresses:
- occur on planes of largest and smallest normal stess with zero shear stress
- always on planes at right angles





principal stresses

You can see from the drawing of Mohr's circle, that the largest value of σ occurs where τ is zero. The largest and smallest values of σ are called σ_1 and σ_2. They are sufficient to define the circle, and are called the principal stresses.

We do not need to solve for σ_1 and σ_2 graphically. We can use the following equations:

$$\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left[\sigma_x - \frac{\sigma_x + \sigma_y}{2}\right]^2 + \tau_{xy}^2}$$

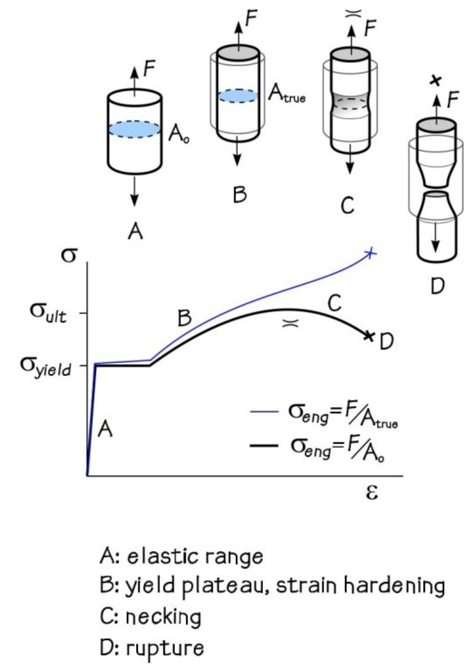
$$\sigma_2 = \frac{\sigma_x + \sigma_y}{2} - \sqrt{\left[\sigma_x - \frac{\sigma_x + \sigma_y}{2}\right]^2 + \tau_{xy}^2}$$

or

$$\sigma_1 = \overline{\sigma} + r$$

$$\sigma_2 = \overline{\sigma} - r$$

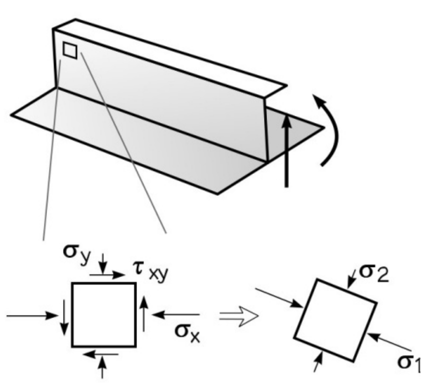
stress - strain (for structural steel)



<u>large strain behaviors</u>

At low strains steel is a linear elastic material. However, when steel is strained to large levels, the linear behaviour ends. Typical ship steels will follow a stress-strain curve as shown at the left. After yielding the stress plateaus while the strains increase significantly. At larger strains the stress begins to rise again, in a phenomenon called 'strain hardening'. At even larger strains the material starts to 'neck' and eventually ruptures. Typical yield stresses are in the range 225 to 400 MPa. Typical ultimate stresses are in the 350 to 550 MPa range.
The initial slope is the Young's modulus which is about 200,000 MPa (200 GPa). So the strain at yield is about 1200 to 2000 $\times 10^{-6}$ strain (μ-strain). Rupture occurs at around 25% strain (300,000 μ-strain).

biaxial stresses



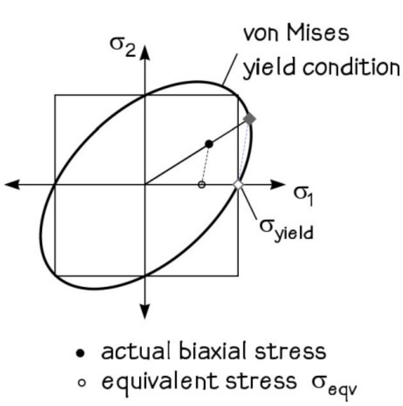
<u>yield criteria and equivalent stresses</u>

In ships structures, made almost entirely of plate steel, most stress states are essentially biaxial. In this case we need to have a criteria for any 2D state of stress.

The 2D von Mises criteria is plotted at left. The curve is normally represented in terms of principal stresses and forms an oval. The oval crosses the axes ay the uniaxial yield stress σ_{yield}. The equation for the yield condition is;

$$\sigma_1^2 - \sigma_1\sigma_2 + \sigma_2^2 = \sigma_{yield}^2$$

biaxial yield surface



The criteria can also be expresses in terms of engineering stresses;

$$\sigma_x^2 - \sigma_x\sigma_y + \sigma_y^2 + 3\tau_{xy}^2 = \sigma_{yield}^2$$

To show whether a general 2D stress is at yield, the concept of an equivalent stress is used (the von-mises equivalent stress). The equivalent stress is a uniaxial stress that represents the same % of yield as the biaxial stress. In this way any 2 states of stress can be compared. The equivalent stress is;

$$\sigma_{eqv} = \sqrt{\sigma_1^2 - \sigma_1\sigma_2 + \sigma_2^2}$$

or

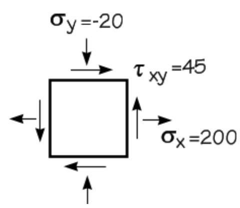
$$\sigma_{eqv} = \sqrt{\sigma_x^2 - \sigma_x\sigma_y + \sigma_y^2 + 3\tau_{xy}^2}$$

We will make use of equivalent stresses in the ANSYS labs.

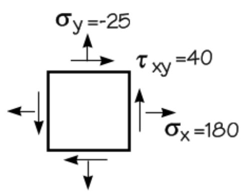
Topic 3: Problems

3.1 A column is made of steel pipe with OD of 8", and ID of 7". It is 8 feet tall. The column supports a weight of 300kips (300,000 lb). How much does the column shorten under load? (E for steel is 29,000,000 psi)

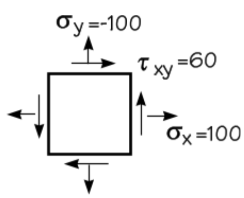
3.2 A 2D state of stress $(\sigma_x, \sigma_y, \tau_{xy})$ is (200, -20, 45) MPa. What are the strains $(\varepsilon_x, \varepsilon_y, \gamma_{xy})$?



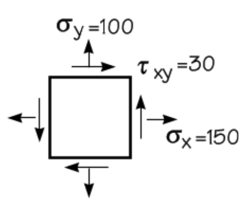
3.3 For a 2D state of stress $(\sigma_x, \sigma_y, \tau_{xy})$ of (180, -25, 40) MPa, plot the Mohr's circle. What are the principal stresses (σ_1, σ_2) ?



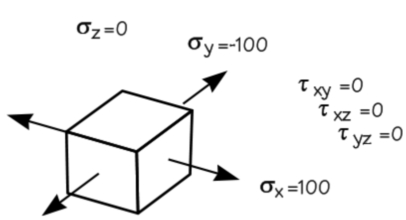
3.4 For a 2D state of stress $(\sigma_x, \sigma_y, \tau_{xy})$ of (100, -100, 60) MPa, what is the von-mises equivalent stresses σ_{eqv} ?



3.5 For a 2D state of stress $(\sigma_x, \sigma_y, \tau_{xy})$ of (150, 100, 30) MPa, what is the von-mises equivalent stresses σ_{eqv} ?



3.6 For a small cube of material with $(\sigma_x, = 100, \sigma_y = 100)$ what is the maximum shear on any plane?



PART 2 : Longitudinal Strength

St. John's Harbour

Topic 4: Longitudinal Strength: Buoyancy & Weight


Pompei

Introduction

In this Chapter we will

- Discuss Still water bending moments, bonjean curves, Prohaska's method and a similar method for non-parallel midbodys

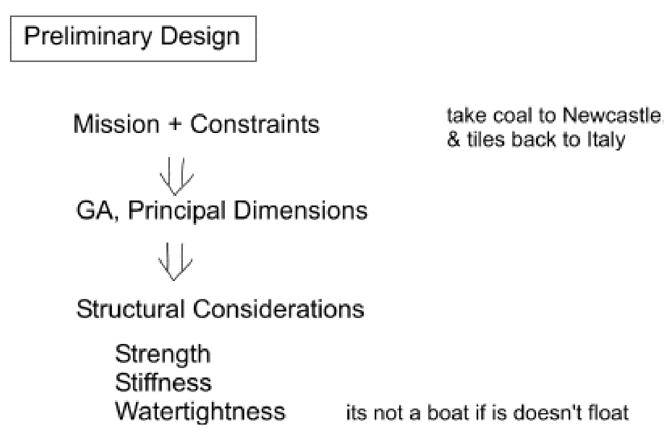
~~~~~~~~~~~~~~~~~~~~

Overview

Structural design starts from:

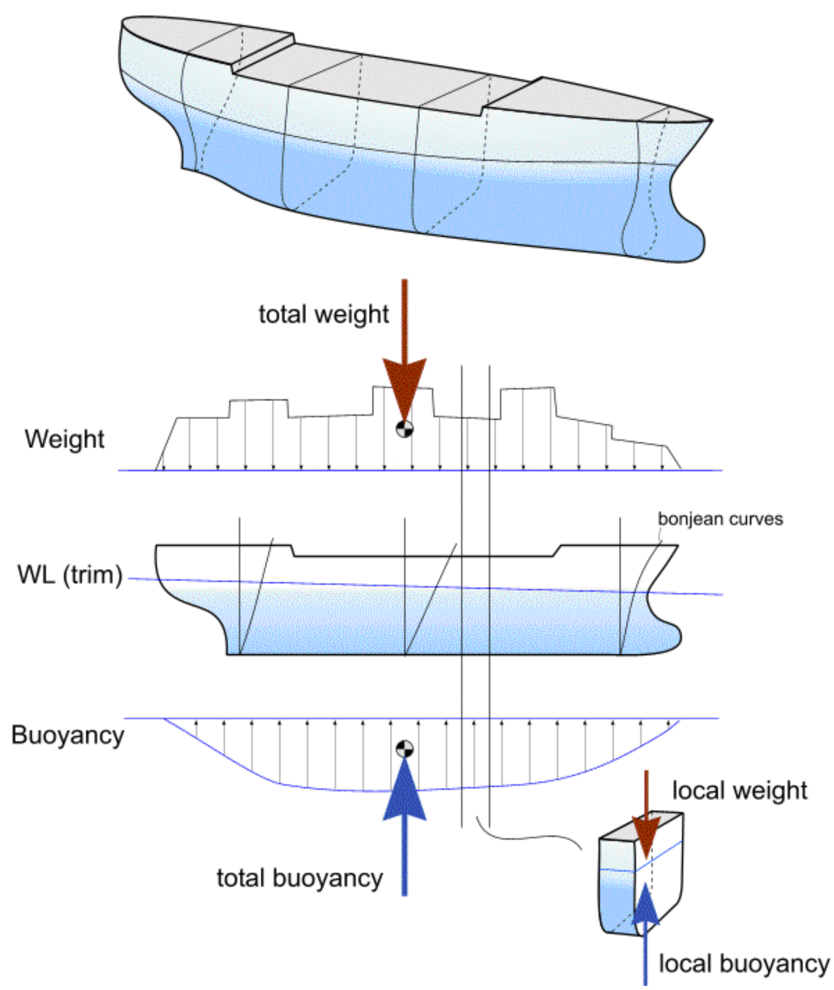
Principal Dimensions - L,B,T
Hull Form - C_B, C_{WP}, C_M
General Arrangement – decks and bulkheads

Which is called preliminary design:


~~~~~~~~~~~~~~~~~~~~

The first strength consideration is the longitudinal strength of the hull girder. The hull girder feels vertical forces due to weight and buoyancy. For any floating body the total weight must equal the total buoyancy, and both forces must act along the same line of action. However, at each location along the ship, the weight will not normally equal the buoyancy.

The weights are set by the combination of lightship and cargo weights. The locations of the weights are fixed (more or less). The buoyancy forces are determined by the shape of the hull and the location of the vessel in the water (draft and trim). The net buoyancy will adjust itself until is exactly counteracts the net weight force. However, this does not mean that each part of the vessel has a balance of weight and buoyancy. Local segments of the vessel may have more or less weight than the local buoyancy. The difference will be made up by a transfer of shear forces along the vessel.



Bending Moment Calculations

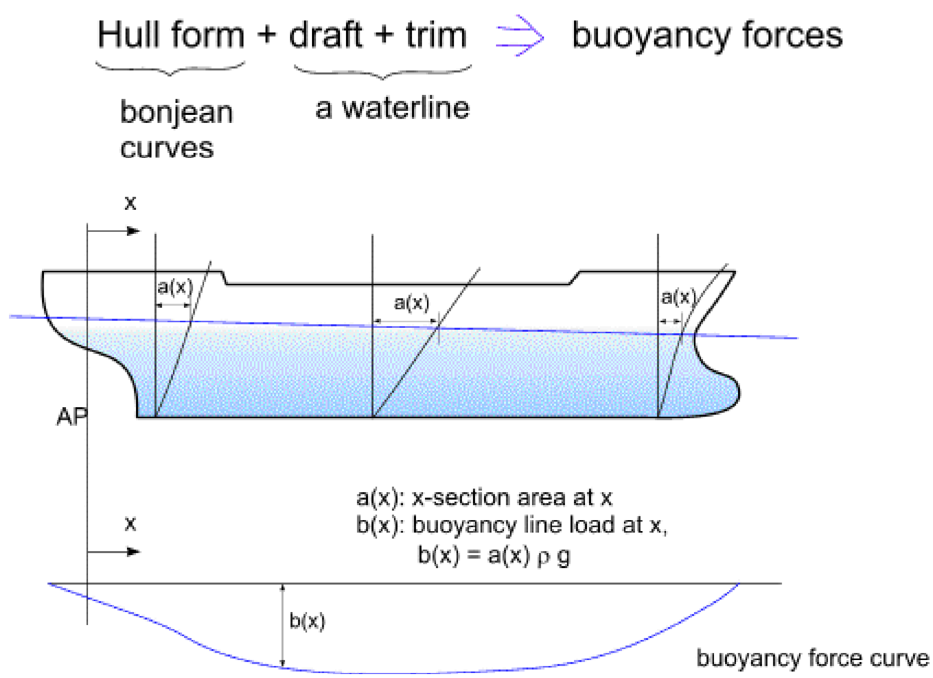
The 'design' bending moment is the combination of Stillwater bending and wave bending. To calculate these values we will make the following assumptions;

1. Ship is a beam
2. Small deflection theory
3. Response is quasi-static
4. Lateral loading can be superimposed

~~~~~~~~

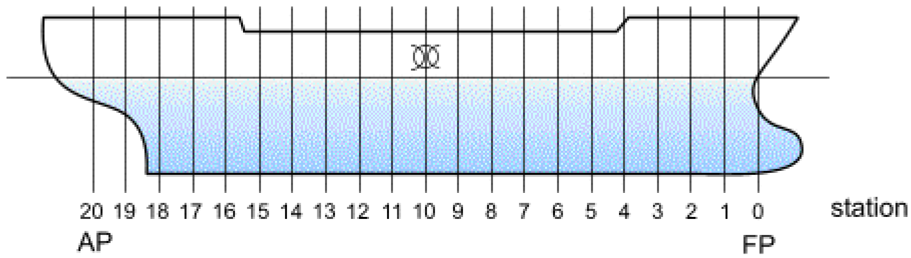
Still Water Bending Moment (SWBM)

The still water bending moment is calculated from the effect of the weights and buoyancy in calm water. The buoyancy force is a line load (e.g. kN/m). The local buoyancy per meter is found from the x-sectional area of the hull at each location. The x-sectional area depends on the local draft and are found from the 'bonjean' curves.

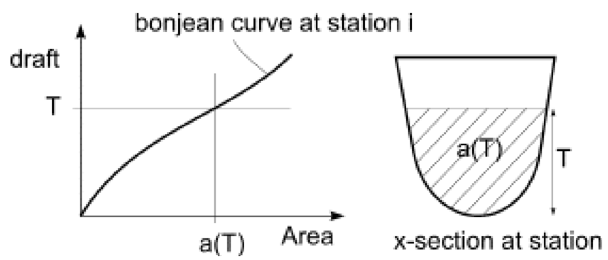

~~~~~~~~

Bonjean Curves – Calculating the Buoyancy Distribution

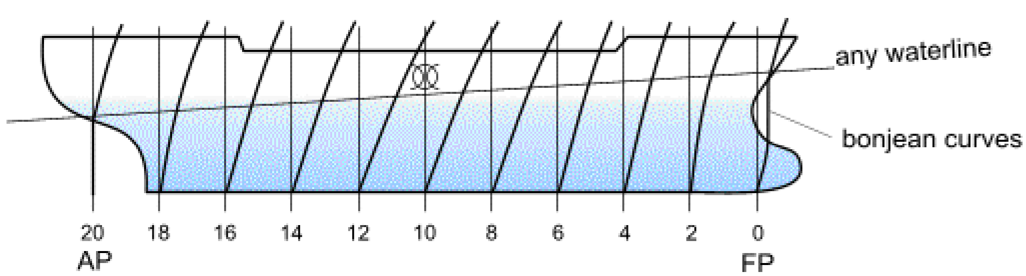
Bonjean curves show the relationship between local draft and submerged cross-sectional area. There is one bonjean curve for each station. There are typically 21 stations from the FP to the AP, with 0 being the FP. This divides the Lbp into 20 segments.



At each station we can draw a bonjean curve of the x-section area;



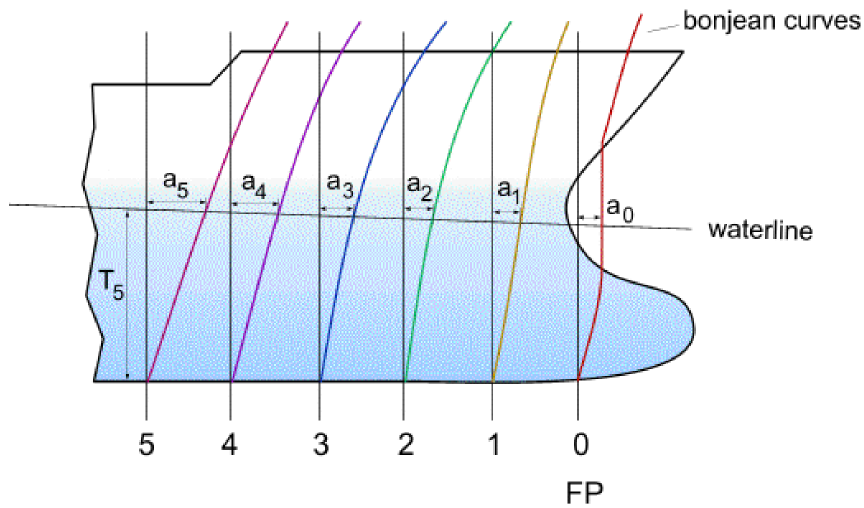
Bonjeans are drawn on the profile of the vessel. With these curves, we can find the distribution of buoyancy for any waterline (any draft, any trim).



For hydrostatic calculations we need to know the distribution of buoyancy along the ship. We need to be able to find this for every possible draft/trim. If we had a wall sided vessel, it would be relatively easy to solve for the draft/trim (as in Assignment #1). With shaped hulls, there is a non-linear relationship between buoyancy and position. We use bonjean curves to find the buoyancies as follows.

For the typical 21 station ship, we divide the ship into 21 slices, each extending fore and aft of its station. Using the bonjean curve for each station we calculate the total displacement at our draft/trim;

$$\nabla = \sum_{i=0}^{20} \left(a_i(\mathrm{T_i}) * \frac{L_{BP}}{20} \right) \quad [\mathrm{m}^3]$$



For example, the displacement for station 3 is;

$$\nabla_3 = A_3 \cdot \frac{L_{BP}}{20} \quad [\mathrm{m}^3]$$

The buoyant line load for station 3 is;

$$\Delta_3 = \nabla_3 \cdot \rho \cdot g \quad [\mathrm{N/m}]$$

(assuming that area is in m^2, g=9.81 m/s^2 and ρ = 1025 kg/m^3)

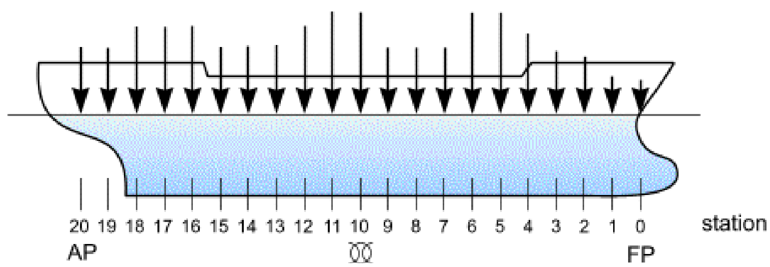
The above will provide a way of calculating the buoyant forces at each station. We will now discuss the weights.

Calculating the Weight Distribution

We will discuss three methods for determining weighs.

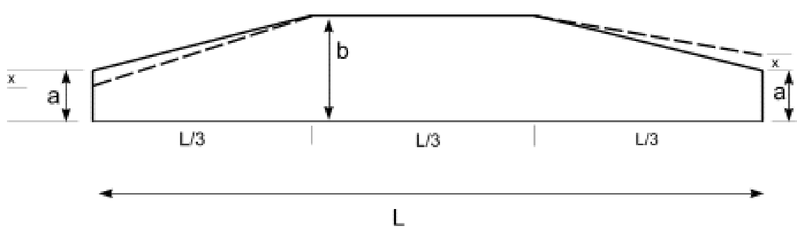
If the **weight distribution is known** (even preliminarily), we use them directly. The steps to follow are;

- Calculate the weight at each station (+- half station)
- (optionally) find the c.g. of weights for each segment
- (optionally) place the weights at the c.g.



~~~~~~~

If the **weight distribution is unknown** and we need to estimate the distribution, we can use the Prohaska method. Prohaska proposed a method for a ship with **parallel middle body** (i.e. most cargo vessels). The weight distribution is a trapezoid on top of a uniform distribution, as follows;

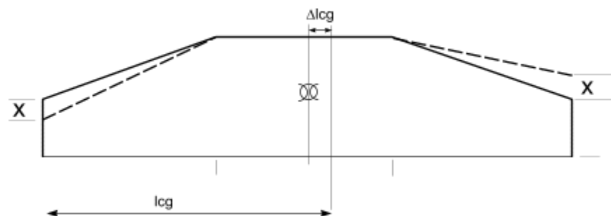


The weights are distributed according to the pattern above. With the average weight/meter of the hull : $\overline{W} = \frac{W_{hull}}{L}$ the values of a and b are ;

	$\frac{a}{\overline{W}}$	$\frac{b}{\overline{W}}$
Tankers	.75	1.125
Full Cargo Ships	.55	1.225
Fine Cargo Ships	.45	1.275
Large Passenger Ships	.30	1.35
~~~~~~~

Note that the values of a and b are related, so that the average is $\overline{W}$. This gives $\frac{b}{\overline{W}} = 1.5 - \frac{a}{2\overline{W}}$.

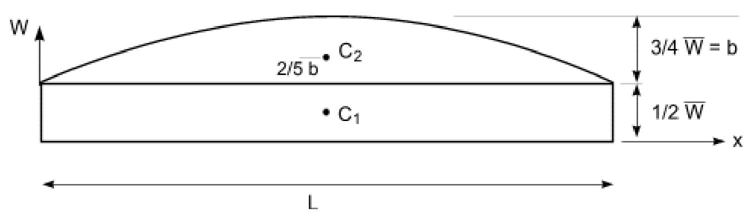
To move the position of the center of weight (the lcg) the fore and aft ends of the load diagram are adjusted by equal (and opposite) amounts.



$$\Delta lcg = x \cdot L^2 \frac{7}{54} \quad \text{or,} \qquad x = \frac{\Delta lcg}{L^2}\frac{54}{7}$$

~~~~~~

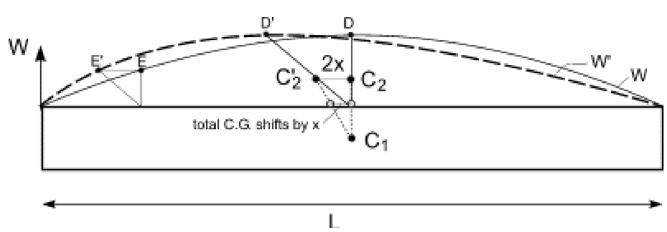
If the **weight distribution is unknown** and we have a vessel **without a parallel middle body** (i.e. most sail yachts), we need a smoother distribution. The method below uses a parabolic distribution on top of a uniform distribution. The two parts each have half the weight.



The equation for the weight is;

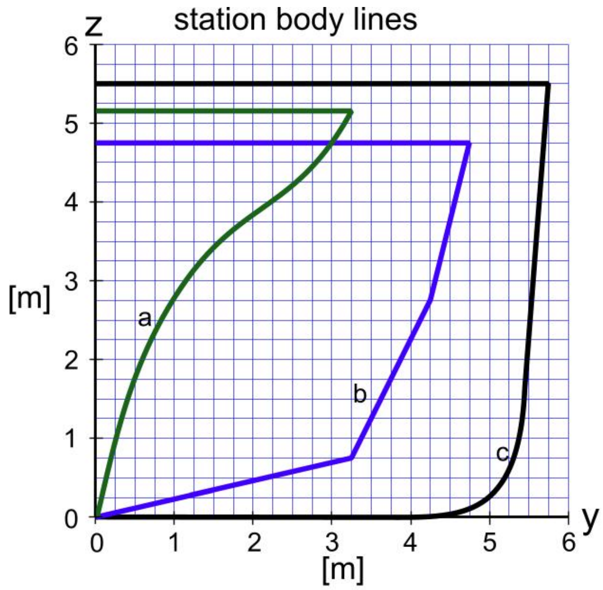
$$W = \frac{\overline{W}}{2} + \frac{3}{4}\overline{W}(1 - (\frac{2x}{L} - 1)^2)$$

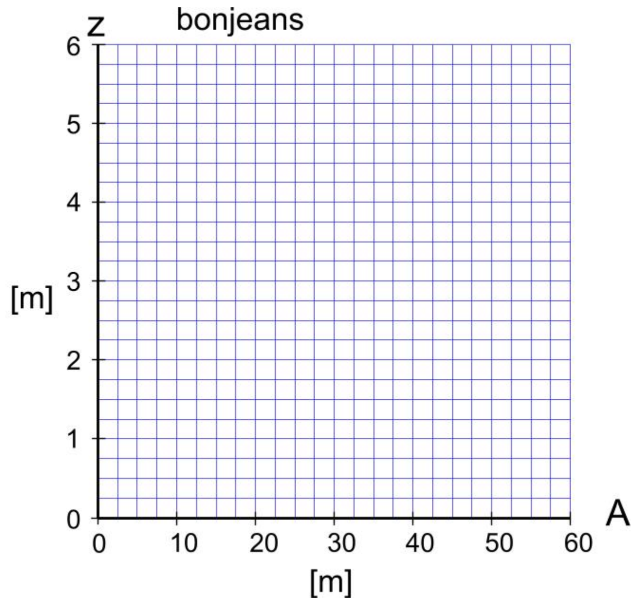
To shift the total center of weight by 'x' we shift the c.g. of the parabola by 2x. This is done by 'shearing' the curve, so that the top center, 'D', shifts by 5x. All other points shift proportionally.


~~~~~~

Topic 4: Problems

4.1. For the three station profiles shown below, draw the bonjean curves in the space provided.



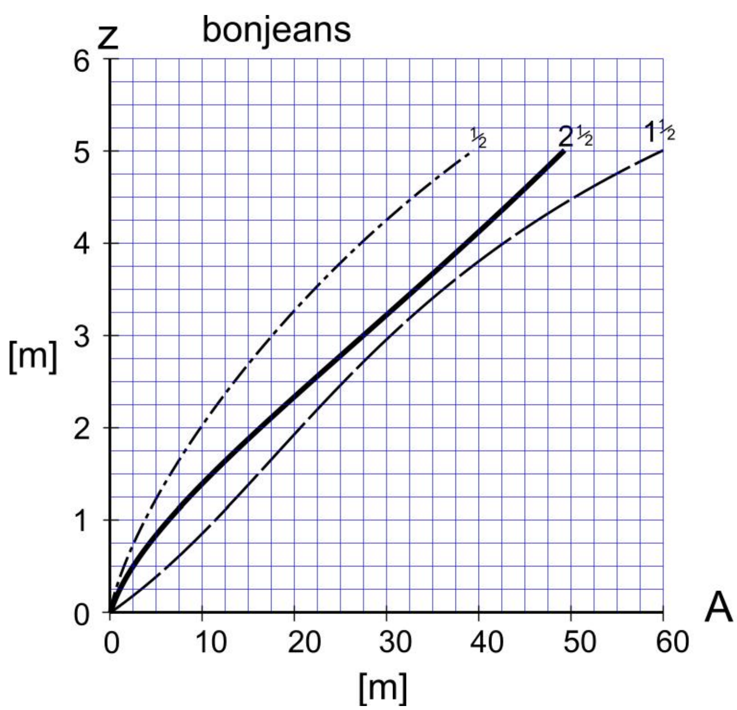


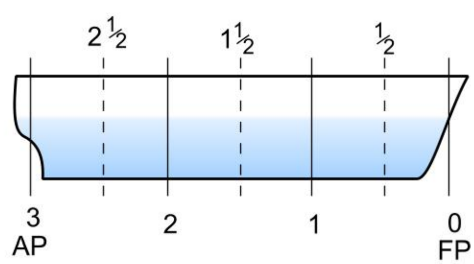
4.2. For a vessel with 4 stations, the bonjean curves are given at the 3 half stations. Lbp is 60m.

for the vessel to float level (no trim), at a 4.5 m draft, where is the C.G.?

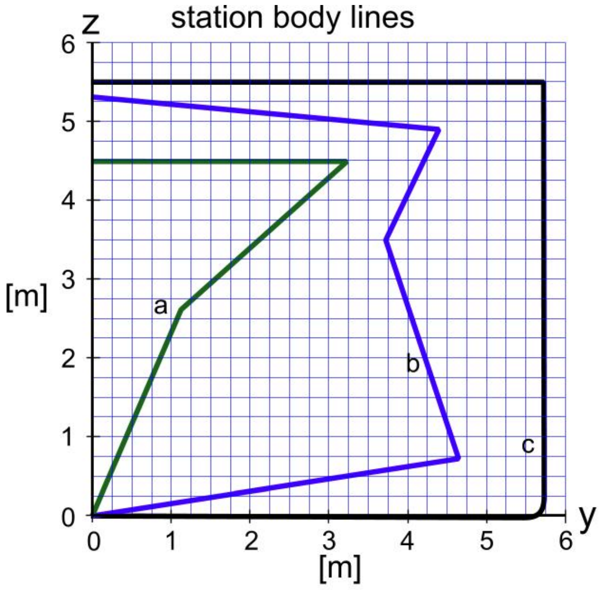
What would the Prohaska distribution of weight be to achieve this? (plot)

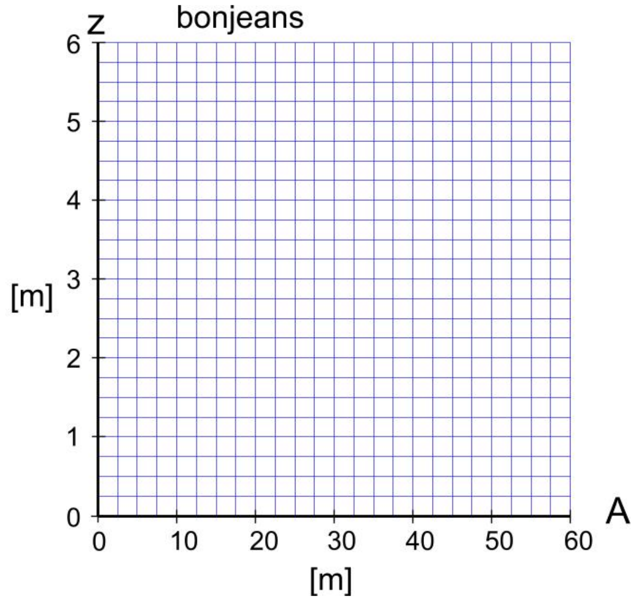
If the C.G is at midships, and the draft (at midships) is 4.5 m, what is the trim?



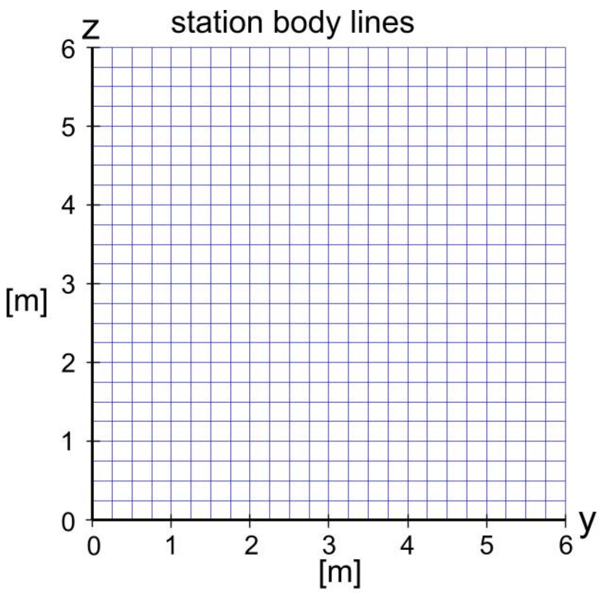


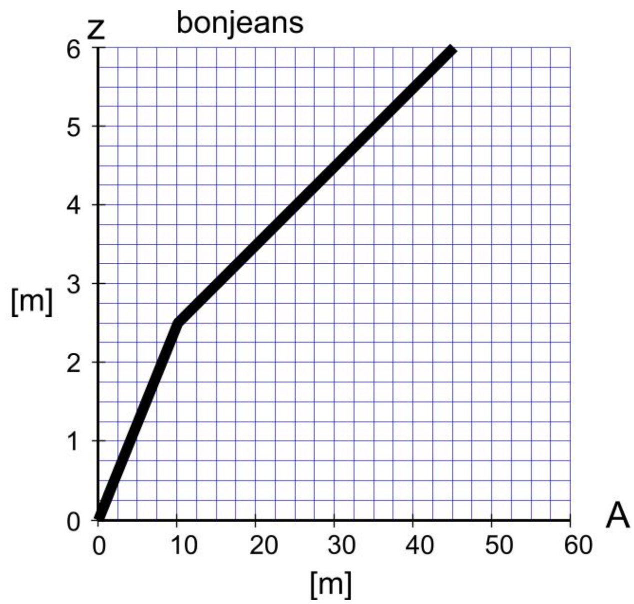
4.3. For the vessel body plan shown below (left), sketch the corresponding bonjean curves (on the right).



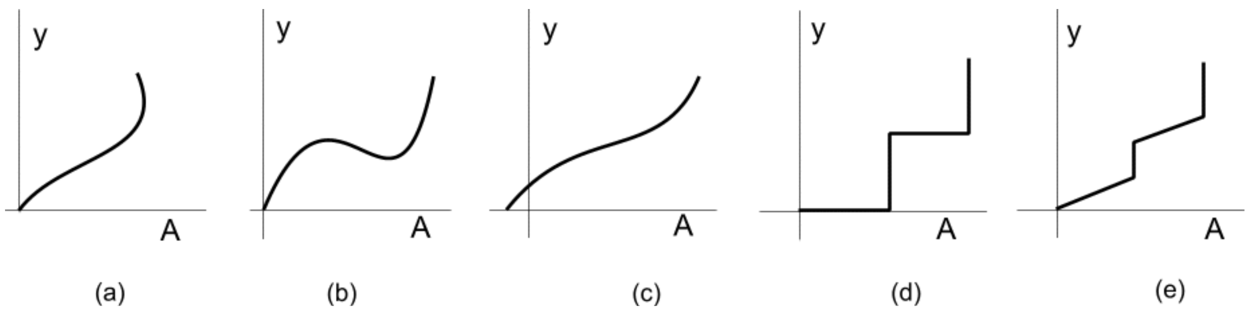


4.4. For the bonjean shown below (right), sketch the corresponding vessel body plan curve (on the left).

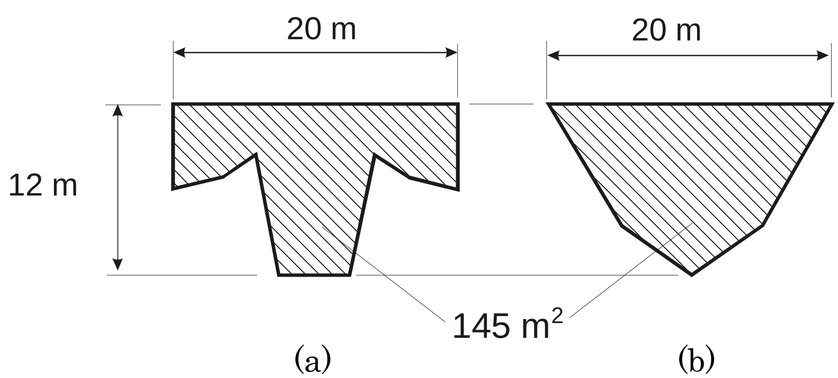


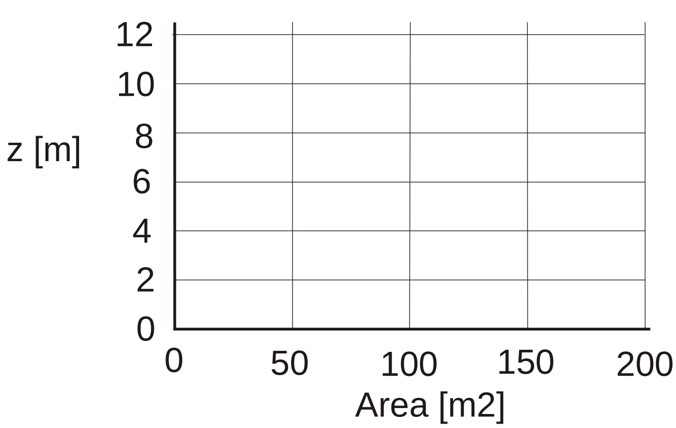


4.5. Bonjean Curves The following figure shows 5 potential Bonjean curves. Some of them are impossible. Identify the curves that can not be Bonjean curves and explain why. For the feasible Bonjeans, sketch the x-section that the Bonjean describes.

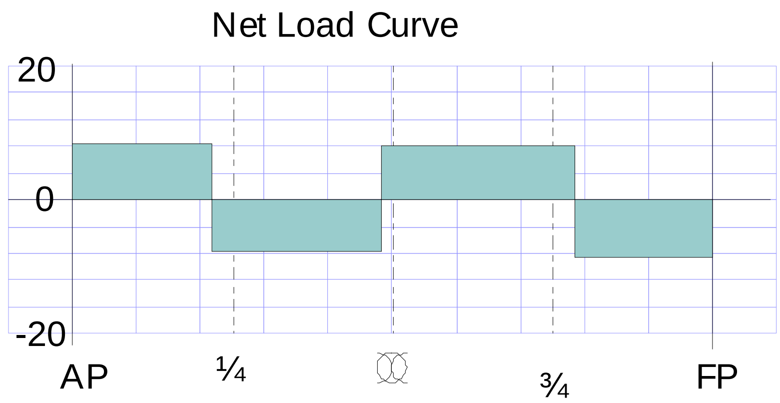


4.6. For the two ship stations shown below, sketch the corresponding bonjean curves on the grid below.



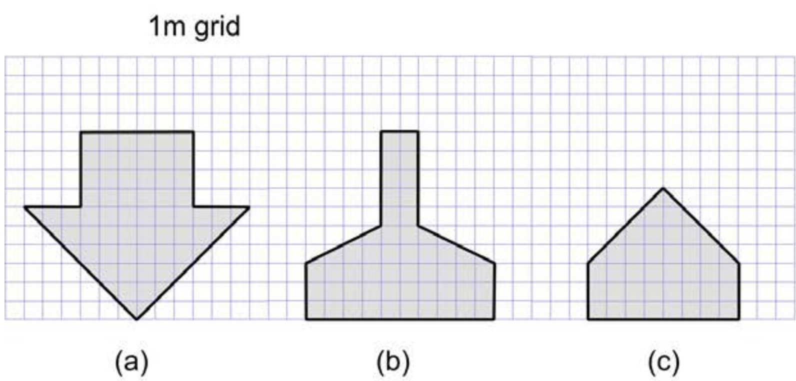


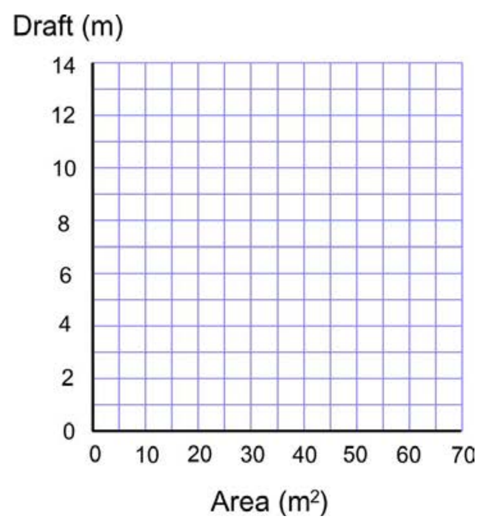
4.7. You are supervising a preliminary ship design project. You have asked one of your team to produce a net load (weight-buoyancy) diagram, so that bending moments can be calculated. The diagram you are given is ;



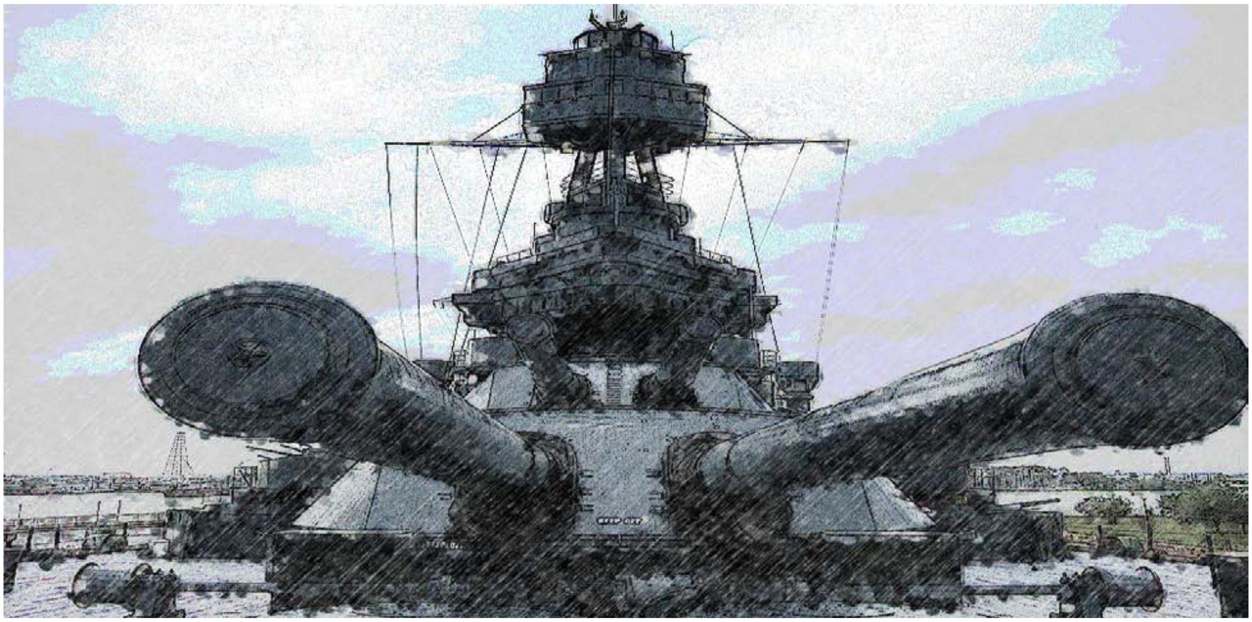
why is this diagram impossible? Justify your answer. (hint: use SFD and/or BMD)

4.8. For the three station profiles shown below, sketch the corresponding bonjean curves





Topic 5: Longitudinal Strength: Murray's Method


Battleship TEXAS

Introduction

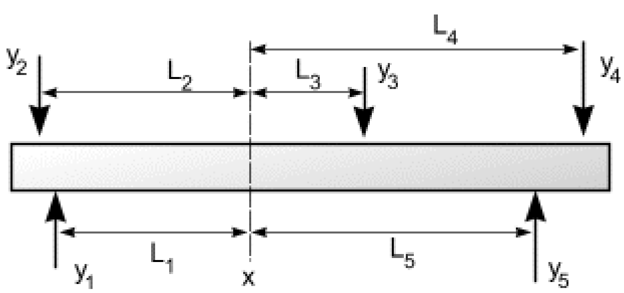
In this chapter we will

- Discuss Murray's Method to estimate still water bending moments

~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~

Murray's Method

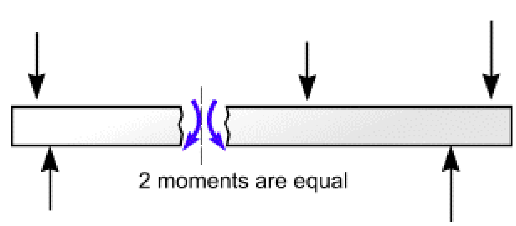
Murray's method is based on the idea that forces and moments in a ship are self-balancing (no net force or moment is transferred to the world). Any set of weight and buoyancy forces are in balance.



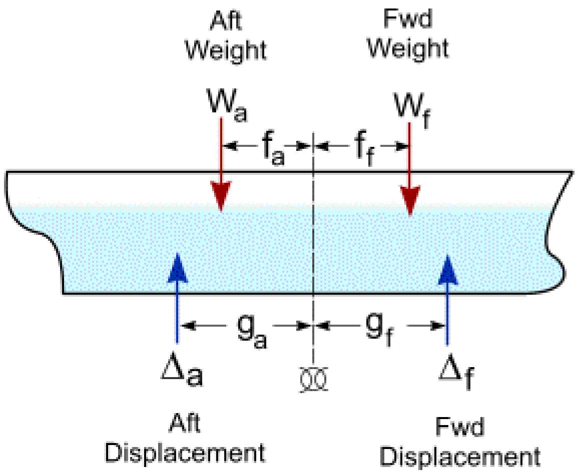
Also, for any cut at x, the moment at the cut can be determined in two ways;

$$BM(x) = y_1 L_1 - y_2 L_2$$

$$= y_5 L_5 - y_3 L_3 - y_4 L_4$$



Murray applied this idea to a ship:
~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~



where
f_f, f_a are the distances from the ⚯ to the centers of weight (fore and aft)
g_f, g_a are the distances from the ⚯ to the centers of buoyancy (fore and aft)

The bending moment at midships is;

$$BM_{\text{⚯}} = W_a f_a - \Delta_a g_a$$

or

$$BM_{\text{⚯}} = W_f f_f - \Delta_f g_f$$

These are two 'estimates' of the maximum bending moment. We can combine the two, and increase our accuracy, by taking the average of the two;

$$BM_{\text{⚯}} = \frac{1}{2}\left(W_a f_a + W_f f_f\right) - \frac{1}{2}\left(\Delta_a g_a + \Delta_f g_f\right)$$
$$= BM_W - BM_B$$

weight - buoyancy

To find the buoyancy part, Murray suggested

$$BM_B = \frac{1}{2}\left(\Delta_a g_a + \Delta_f g_f\right) = \frac{1}{2}\Delta \cdot \bar{x}$$

where $\bar{x}$ = average moment arm

Murray suggested a set of values for $\bar{x}$, as a function of the ship length, block coefficient and the ratio of draft to length;

$$\bar{x} = L(a \cdot C_B + b)$$

where

T/L	a	b
.03	.209	.03
.04	.199	.041
.05	.189	.052
.06	.179	.063

This table for a and b can be represented adequately by the equation;
$a = .239 - T / L$
$b = .1.1T / L - .003$

Example using Murray's Method

Ship: Tanker L=278m, B=37m, C_B=0.8

Assume wave bending moment is;
WBM_{sag} = 583800 t-m
WBM_{hog} = 520440 t-m

The vessel weights, and weight bending moments are as follows;

ITEM	Weight	lcg	Moment
	(t)	(m)	(t-m)
Fwd			
cargo	62000	40	2480000
fuel & water	590	116	68440
steel	12000	55.6	667200
			3,215,640
Aft			
cargo	49800	37	1842600
machinery	3400	125	425000
outfit	900	120	108000
steel	12000	55.6	667200
Σ	140690	t	3,042,800
		BM_w =	**3,129,220**

To find the buoyancy moment we need the draft;

$$W = \Delta = C_B \cdot L \cdot B \cdot T \cdot \gamma$$

$$T = \frac{\Delta}{C_B \cdot L \cdot B \cdot \gamma} = \frac{140690}{0.8 \cdot 278 \cdot 37 \cdot 1.025}$$
$$= 16.68 \quad \text{m}$$

$$\frac{T}{L} = \frac{16.68}{278} = 0.06$$

Murray's table gives;

a=0.179, b=0.063

$$\bar{x} = 278(.179 \cdot 0.8 + .063) = 57.32 \ \ m$$

$$BM_B = \frac{1}{2}\Delta \cdot \bar{x}$$
$$= \frac{1}{2} 140690 \cdot 57.32 = 4{,}032{,}428 \ \ t\text{-}m$$

SWBM = BM_W-BM_B

hog sag

$$= 3{,}129{,}220 - 4{,}032{,}428$$
$$= -903{,}145 \ \ t\text{-}m \ \ (\text{- is sag})$$

we need to add the wave bending moment in sag

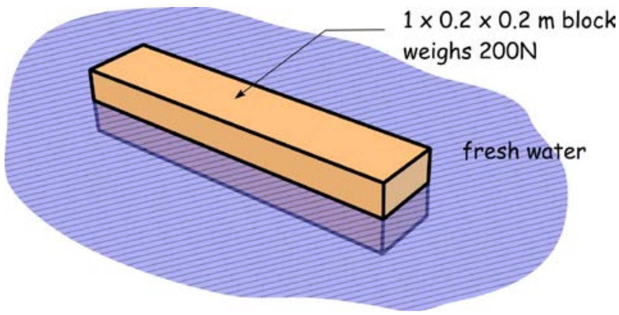
Total BM = 903,145 + 583,800
= 1,486,945 t-m (sag)

Note that in this case the ship will never get in the hogging condition, because the SWBM is so large.

Topic 5: Problems

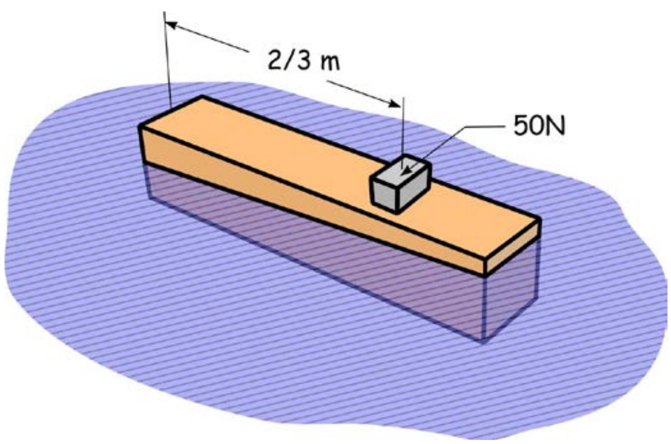
5.1. Longitudinal strength is a primary concern during the design of a ship. Briefly explain the idea behind Murray's Method.

5.2. There is a 'rectangular' shaped block of wood, as shown in the image below. The block weighs 200 N and has uniform density. It is 1 m long and 0.20 m wide. It is 20 cm thick and is floating in fresh water.



a) draw the shear force and bending moment diagrams for the block.

Now consider the addition of a small 50 N weight on the top of the block, at a distance 2/3m from one end. (hint - a right triangle has its centroid at 2/3 of its length)



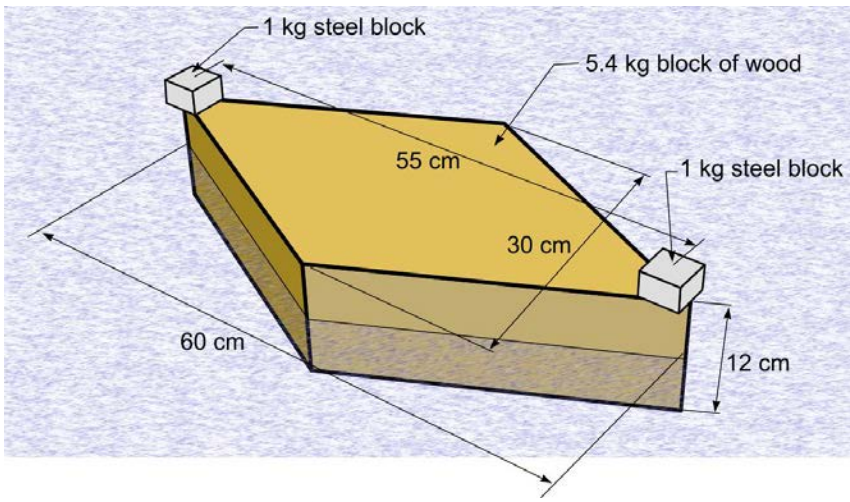
After the block settles to an equilibrium position -

b) Draw the bending moment and shear force diagrams
c) What is the max. bending stress on the transverse plane at the middle of the block (ie at 0.5 m from the end)?

5.3. There is a 'diamond' shaped block of wood, as shown in the image below. The block weighs 5.4 kg. and has uniform density. It is 60 cm long and 30 cm wide. It is 12 cm thick and is floating in fresh water. Resting on the block are 2 weights, each small blocks of steel weighing 1 kg. They are symmetrically placed and are 55cm apart.

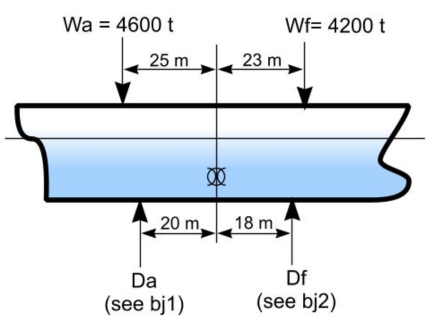
What is the midship bending moment in units of N-cm ?
What is the maximum bending stress in the wooden block?

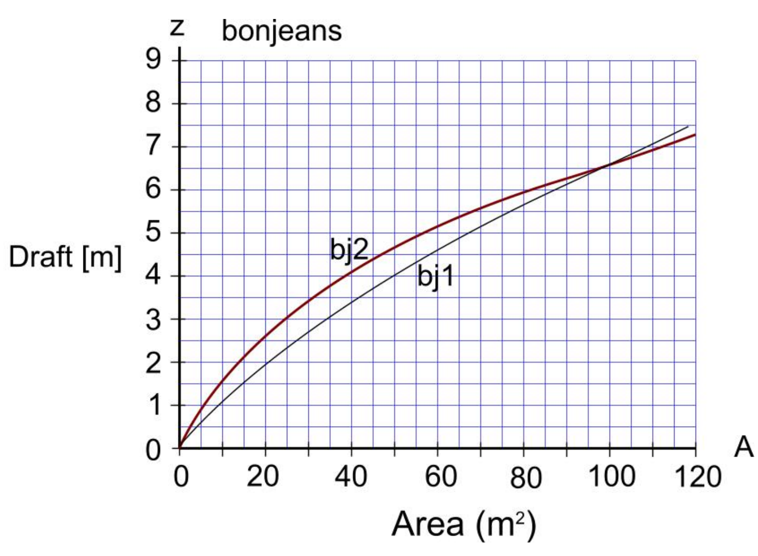
Draw the bonjean curve for a cross section of the wooden block at a point 15cm from the end. (show actual units).
What is the block coefficient for the block?



ANS: a) 171.5 N-cm (hog) b) 23.8 MPa c) Straight and then vertical d) 0.5

5.4. Consider a 100m vessel resting in sheltered fresh waters (see below). The CG of all weights fwd of midships is 23m fwd of midships (ff=23m). The CG of all weights aft of midships is 25m aft of midships (fa=25m). The weights fwd and aft are 4200 and 4600 t respectively. Two bonjean curves are given. Assume each refers to the average x-section area for 50m of ship (fore and aft). The (fore and aft) buoyancy forces act at the bonjean locations, which are 18m fwd and 20 aft (of midships). The buoyancy force aft is 4650 t.



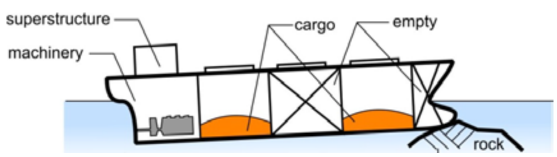


Using the bonjeans, find
The vessel drafts at the two bonjeans.
The buoyancy force fwd.
The still-water bending moment at midships

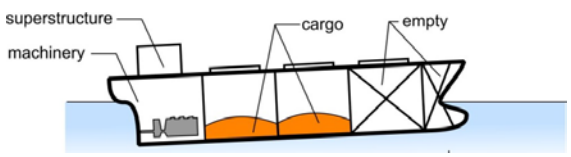
5.5. **Murray's Method** Consider a 100m long vessel resting in sheltered waters. The CG of all weights fwd of midships is 20m fwd of midships (ff=20m). The CG of all weights aft of midships is 25m aft of midships (fa=25m).
- Describe how you would use Murray's Method to determine the still water bending moment for this vessel.
- What other info, if any do you need?
Note: you don't need to remember the specific values for terms suggested by Murray.

5.6. **Hull girder strength** The hull girder can be viewed as a beam. When floating in still water, is the beam statically determinate or statically indeterminate? Provide reasons for your answer.

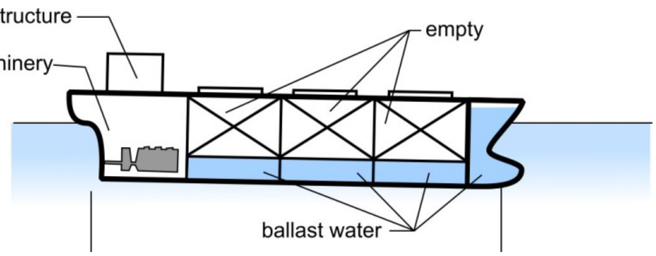
5.7. You see below a sketch of a ship that is 200 m long. The displacement is made up of the lightship plus the weight of cargo in two holds. The ship has stranded itself on a submerged rock. Draw the various curves of load and response for the vessel (weight, buoyancy, net load, shear, moment, slope and deflection) that are compatible with the information given. The numerical values don't matter. The intention is to draw a set of curves that are logical for the ship as shown.



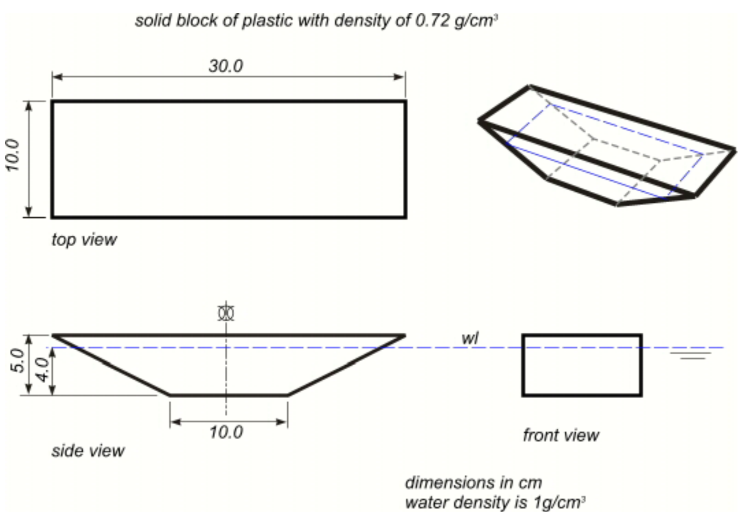
5.8. You see below a sketch of a ship that is 200 m long. The displacement is made up of the lightship plus the weight of cargo in two holds. The forward cargo hold is empty. Draw the various curves of load and response for the vessel (weight, buoyancy, net load, shear, moment, slope and deflection) that are compatible with the information given. The numerical values don't matter. The intention is to draw a set of curves that are logical for the ship as shown.



5.9. You see below a sketch of a ship that is 200 m long. The displacement is made up of the lightship plus the weight of ballast in 4 tanks. The cargo holds are empty. Draw the various curves of load and response for the vessel (weight, buoyancy, net load, shear, moment, slope and deflection) that are compatible with the information given. The numerical values don't matter. The intention is to draw a set of curves that are logical for the ship as shown.



5.10. Calculate the still water bending moment (in N-cm) for the solid block of plastic sketched below. Assume the block has density as given and is floating in fresh water (density also given). Is the moment hogging or sagging?



5.11. For the example of Murray's method in the Chapter, remove the cargo weight and add 4000 t of ballast, with a cg of 116m fwd of midship. Re-calculate the maximum sag and hog moments (both still water and wave).

5.12. For the example of Murray's method in the Chapter, instead of using the weight locations as given, assume that the weights are distributed according to Prohaska. Re-calculate the SWBM.

5.13. Consider a 100m long tanker resting on an even keel (same draft fore and aft) in sheltered waters. The CG of all weights is at midships and is 8000 tonnes.
Use Murray's Method and Prohaska's values to determine the still water bending moment for this vessel (i.e. get both the weight and buoyancy BMs about midships).

Topic 6: Longitudinal Strength: Wave Bending Moments

Cape Spear

Introduction

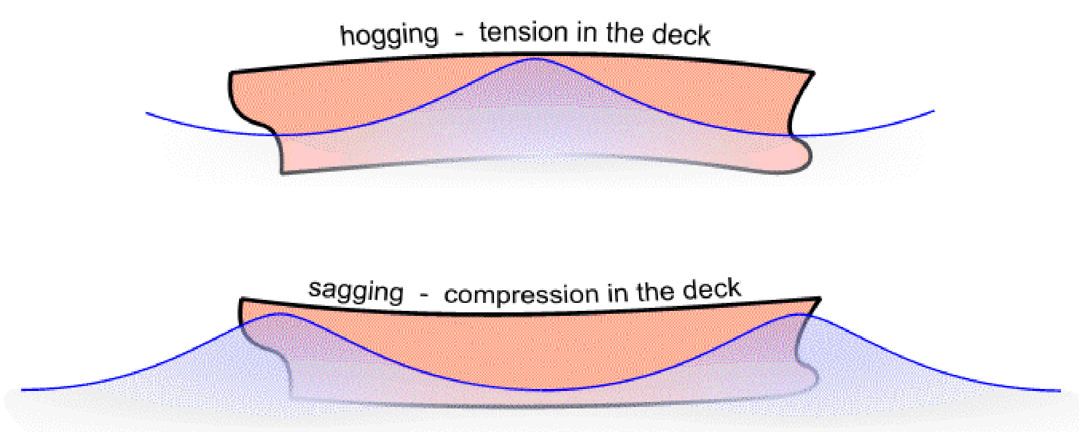
In this Chapter we will

- Discuss the shape of ocean design waves
- The moments caused by waves

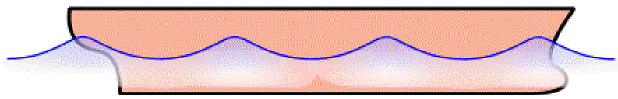
~~~~~~~~~~~~~~~~~~~~~

Design Waves

Design wave forces are considered to be quasi-static. As a wave passes by a vessel, the worst hogging moment will occur when the midbody is on the crest of a wave and the bow and stern are in the troughs. The worst sagging moment will happen when the bow and stern are on two crests, with the midbody in the trough between.



Whether for sagging or hogging, the worst condition will occur when the wavelength is close to the vessel length. If the waves are much shorter,
~~~~~~~~~~~~~~~~~~~~~



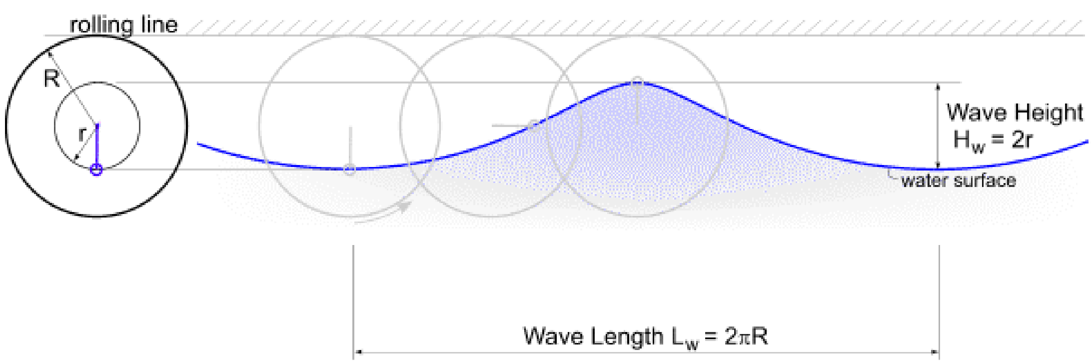
or much longer than the vessel, the bending moments will be less than if the wavelength equals the ship length.

Consequently, the design wave for any vessel will have a wavelength equal to the vessel length. The wave height is also constrained. Waves will have a limited height to length ratio, or they will break. This results in a standard design wave of L/20. In other words the wave height (peak to trough) is 1/20th of the wave length.

Trochoidal Wave Profile

Note that the waves sketched above did not look like sinusoids. Waves at sea tend to be trochoidal shaped, rather than simple sine waves. This has the feature that the crests are steeper and the troughs are more rounded.

A trochoidal wave is constructed using a rolling wheel.



In the case of the design wave;

$$\left.\begin{array}{l} L_W = L_{BP} \\ H_W = L_{BP}/20 \end{array}\right\}$$ for now we assume that this length and height or wave is possible

We can see that;

$L_W = 2\,\pi\,R$
$H_W = 2\,r$

Which gives;

$$R = \frac{L_{BP}}{2\pi}, \quad r = \frac{L_{BP}}{40}$$

$$\frac{r}{R} = \frac{\pi}{20}$$

To construct a plot of the wave, we start with a coordinate system at the crest of the wave.

$$\left.\begin{aligned} x &= R\theta - r\sin\theta \\ z &= r(1-\cos\theta) \end{aligned}\right\} \quad \theta = \text{rolling angle}$$

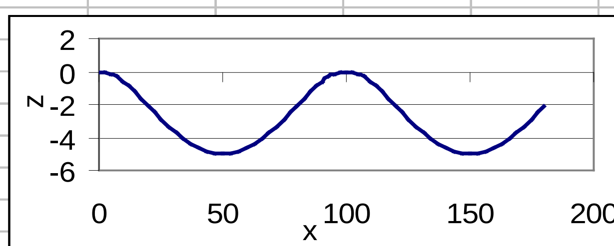
This is a parametric equation (θ is a parameter). We can write;

$$x = \frac{L}{2\pi}\theta - \frac{L}{40}\sin\theta$$

$$z = \frac{L}{40}(1-\cos\theta)$$

To plot the wave, it is a simple matter to calculate x and z as a function of θ and then plot z vs x. This is done in the spreadsheet below.

L	100	
H	5	
θ	x	z
0	0	0
10	2.343657	-0.03798
20	4.700505	-0.15077
30	7.083333	-0.33494
40	9.504142	-0.58489
50	11.97378	-0.89303
60	14.5016	-1.25
70	17.09521	-1.64495
80	19.7602	-2.06588
90	22.5	-2.5
100	25.31576	-2.93412
110	28.20632	-3.35505
120	31.16827	-3.75



$1.1\sqrt{L}$ Wave

L/20 waves have been found to be too conservative for large vessels, esp. for vessels >500 ft. A more modern version of the $1.1\sqrt{L}$ wave. In this case;

as before, $L_W = L_{BP}$

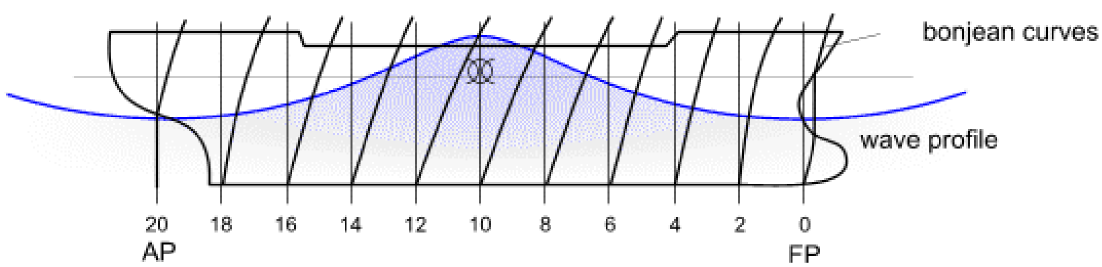
$H_w = 1.1\sqrt{L_{BP}}$ (in feet)
or
$H_w = 0.607\sqrt{L_{BP}}$ (in meters)

For trochoidal waves this gives;

$R = \frac{L_{BP}}{2\pi}$, $r = .55\sqrt{L_{BP}}$ (feet) or $r = .303\sqrt{L_{BP}}$ (meters)

Calculating Wave Bending Moments

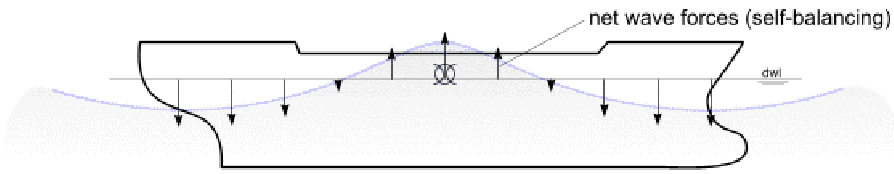
We can now calculate the wave bending moments by placing the ship on the design wave. We can use the bonjean curves to determine the buoyancy forces due to the quasi-static effects of the wave;

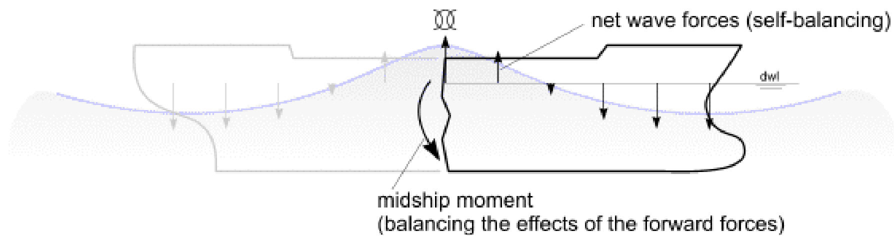


The steps to determine the wave bending moment are;

1. Obtain bonjeans
2. at each station determine the still water buoyancy forces, using the design draft. $F_{isw} = A_{isw}\, l_i\, \rho g$
3. at each station determine the total buoyancy forces, using the local draft in that portion of the wave. $F_{iwt} = A_{iwt}\, l_i\, \rho g$
4. The net wave buoyancy forces are the difference between wave and still water. $F_{iwave}=F_{iwt}-F_{isw}$

This gives us a set of station buoyancy forces due to the wave (net of still water). These forces should be in equilibrium (no net vertical force). We can calculate the moment at midships from either the net effect of all forces forward, or all forces aft (the two moments will balance).





There are other ways to do this kind of calculation. 3D cad programs such as Rhino can be used to find the still water and wave bending moments. Assuming that we have a hull modeled in Rhino, we can find the still water buoyancy forces for the fore and aft halves of the vessel by finding the volume and location of the centroids of the two submerged volumes.
The procedure would be as follows;

1. Produce solid model of hull
2. Cut the model at both the centerline and waterlines.
3. Find the volumes and centroids of the two halves.
4. Calculate the buoyant moments about midships.

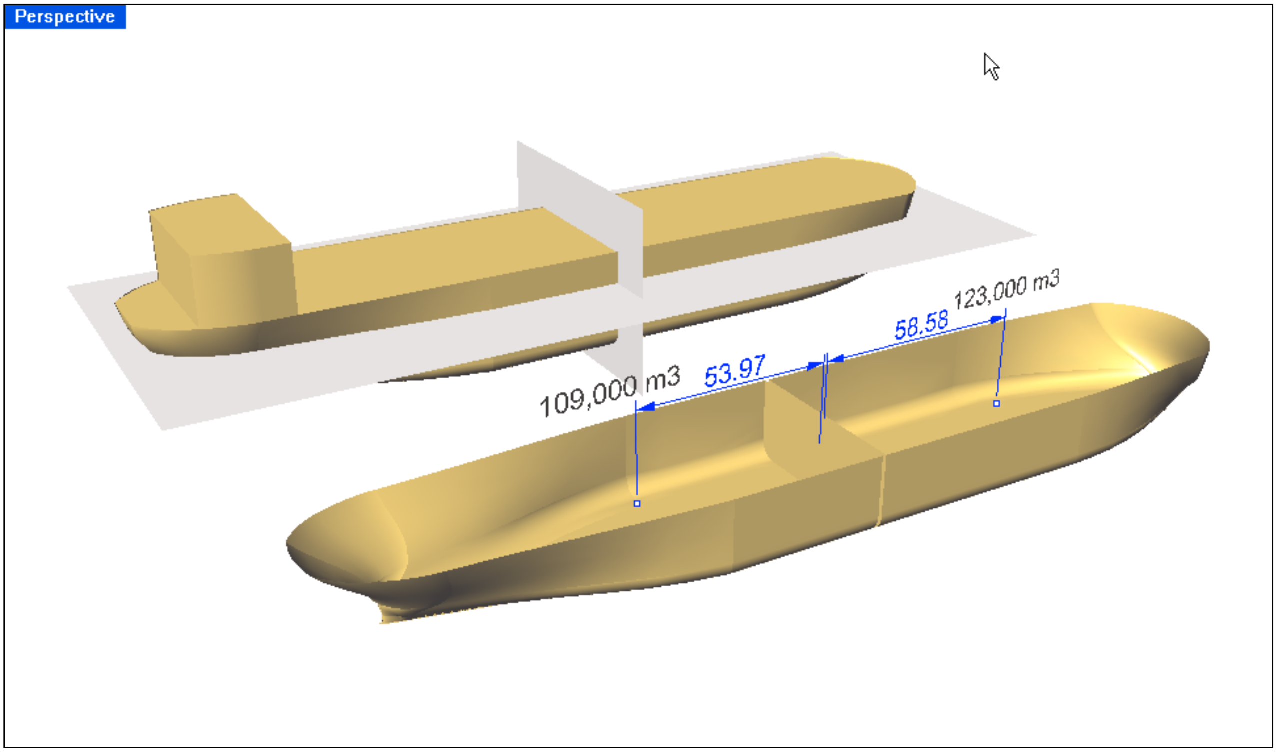
A similar procedure would determine the wave values. The only difference would be the need to draw the trochoidal wave as a surface.

The example below shows use of Rhino to calculate the Bouyant BM for a large vessel. The centroids of the two half volumes are shown.

BM_B = 109,000 x 1.025 x 53.97 (m3 x t/m3 x m = t-m)
= 6,029,798 t-m
or
BM_B = 123,000 x 1.025 x 58.58 (m3 x t/m3 x m = t-m)
= 7,385,473 t-m

average: BM_B = 6,707,376 t-m (sag)

The difference between this and the weight moment (hog) will give the SWBM.



Rhino model, showing slices and centroids

Topic 6: Problems

6.1. Using a spreadsheet, plot the design trochoidal wave for a 250m vessel, for the L/20 wave.

6.2. Using a spreadsheet, plot the design trochoidal wave for a 250m vessel, for the 1.1 $L^{.5}$ wave.

Topic 7: Longitudinal Strength: Inclined Bending / Section Modulus

a breaking wave in Lisbon

Introduction

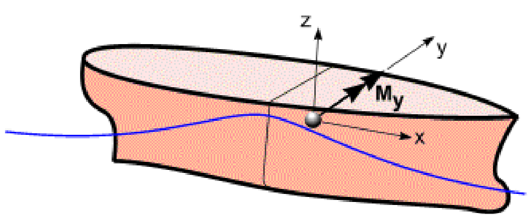
In this Chapter we will

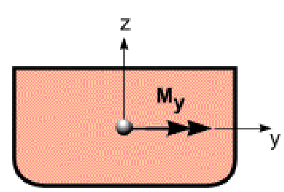
- Discuss the calculation of bending of an inclined vessel
- General calculation of hull section modulus/inertia

~~~~~~~~~~~~~~~~~~~~

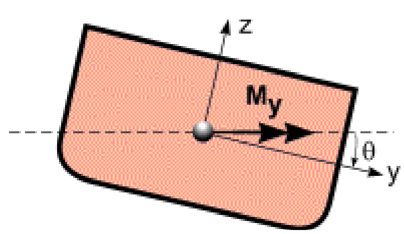
Inclined and Lateral Bending

When a ship rolls the weight and buoyancy forces cause lateral as well as vertical bending. Normally the bending moment vector is aligned with the ship's y axis. $\mathbf{M_y}$ is the bending moment that results from buoyancy and weight forces.



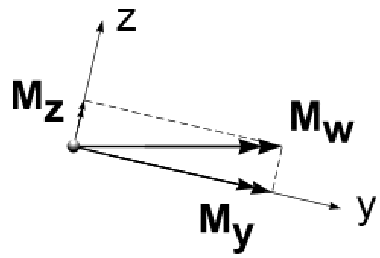


When the vessels rolls by an angle q, the moment vector remains horizontal. This is because the buoyancy and gravity forces are always vertical. This means that the bending moment is no longer aligned with the y,z axis of the vessel;


~~~~~~~~~~~~~~~~~~~~

Moments are vectors, adding in the same way that force vectors do.

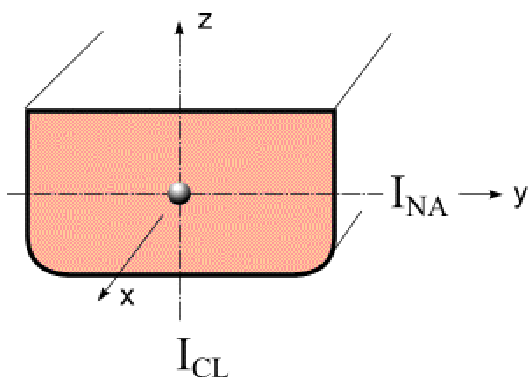
$$\mathbf{M}_z = \mathbf{M}_w \sin\theta \quad \text{lateral bending}$$
$$\mathbf{M}_y = \mathbf{M}_w \cos\theta \quad \text{vertical bending}$$



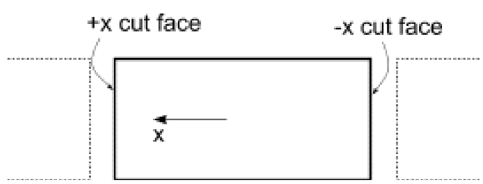
Stresses in the Vessel

Both $\mathbf{M_y}$ and $\mathbf{M_z}$ cause bending stresses in the x (along ship) direction.

$$\sigma_V = \frac{-M_y z}{I_{NA}} \qquad \sigma_H = +\frac{M_z y}{I_{CL}}$$



Note: Sign convention: R.H.R., moment acting on +x cut face, compression is positive.



In this case a $+\mathbf{M_y}$ causes tension (-) on the +z part of the vessel. A $+\mathbf{M_z}$ causes compression (+) on the +y side of the vessel.

The total axial stress at any point on the vessel is the sum of the stresses caused by the two directions of bending.

$$\sigma_X = \sigma_V + \sigma_H = \frac{-M_y z}{I_{NA}} + \frac{M_z y}{I_{CL}}$$
$$= \frac{-M_w z \cos\theta}{I_{NA}} + \frac{M_w y \sin\theta}{I_{CL}}$$

When we have bending moments in both y and z, there will be a line of zero axial stress that we call the heeled neutral axis. This is not necessarily aligned with the total moment. To find the heeled neutral axis we solve for the location of zero stress;

$$\sigma_X = 0 = \frac{-M_w z \cos\theta}{I_{NA}} + \frac{M_w y \sin\theta}{I_{CL}},$$

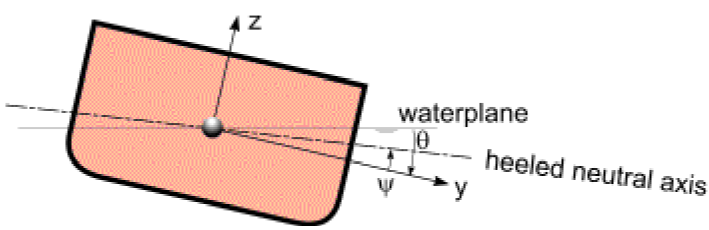
solving for z in terms of y , we get;

$$z = \frac{I_{NA}}{I_{CL}} \tan\theta \cdot y ,$$

where we define: $\tan\psi = \frac{I_{NA}}{I_{CL}} \tan\theta$

$$z = \tan\psi \cdot y$$

ψ is the angle of the heeled neutral axis from the y axis;



Peak Stresses

The highest stresses will occur @ y=∀B/2 , Z=Z_{deck}

There are 2 section modulus values;

$$Z_{NA} = \frac{I_{NA}}{z_{deck}}, \quad Z_{CL} = \frac{I_{CL}}{B/2}$$

So that we can write;

$$\sigma_{\max} = M_w \left(\frac{\cos\theta}{Z_{NA}} + \frac{\sin\theta}{Z_{CL}} \right)$$

This leads to the question: What is the worst angle of heel (θ_{cr})?

To find it we use;

$$\frac{d\sigma_{\max}}{d\theta} = 0 = M_w \left(\frac{-\sin\theta_{cr}}{Z_{NA}} + \frac{\cos\theta_{cr}}{Z_{CL}} \right),$$

which gives;

$$\tan\theta_{cr} = \frac{Z_{NA}}{Z_{CL}}$$

Typically $Z_{NA}/Z_{CL} \cong 0.5$ so $\theta_{cr} = 26.6^\circ$

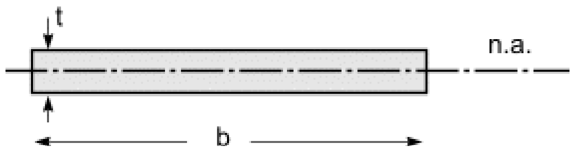
For example, if $\sigma_{\theta=0} = \frac{M_w}{Z_{NA}}$ then $\sigma_{\theta=26.6} = M_w \left(\frac{-\sin 26.6}{Z_{NA}} + \frac{\cos 26.6}{2 \cdot Z_{NA}} \right)$

$$= \frac{M_w}{Z_{NA}} 1.12$$

i.e. for this vessel, there is a **12% increase in stress** during the worst roll.

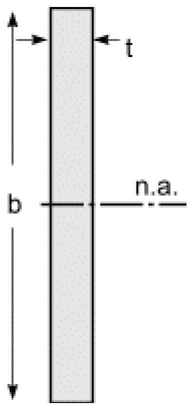
Section Modulus Calculations

Ships are largely built of plates. This means that the moment or inertia and section modulus calculations normally involve a collection of rectangular parts. For any individual plate:



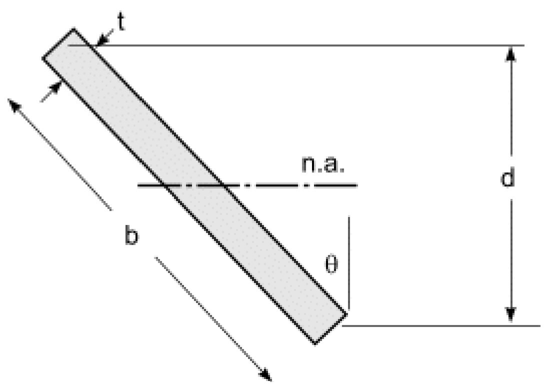
$Ina = 1/12\ b\ t^3$

$= 1/12\ a\ t^2$



$Ina = 1/12\ t\ b^3$

$= 1/12\ a\ b^2$

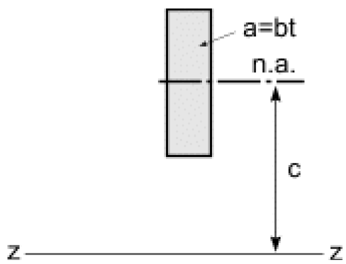


$Ina = 1/12\ a\ d^2$

$= 1/12\ t\ b^3\ \cos^2\theta$

~~~~~~

For compound sections we need to be able to find the inertia about other axes. We use the transfer of axis theorem:
~~~~~~



$$Izz = Ina + a\ c^2$$

The overall neutral axis (NA) is found by equating 2 expressions for the 1st moment of area;

$A\ h_{NA} = \Sigma\ a_i\ h_i$

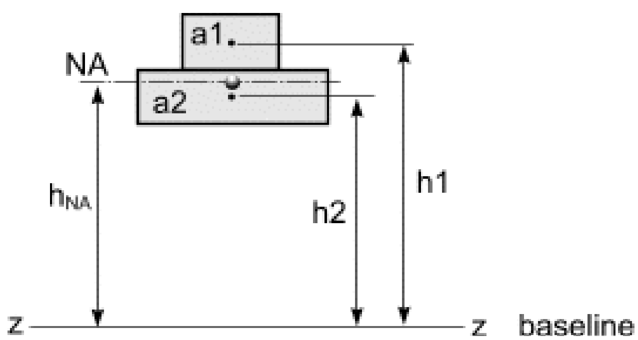
The total area A is just the sum of areas.

$A = \Sigma\ a_i$

This gives;

$h_{NA} = \Sigma\ a_i\ h_i\ /\ \Sigma\ a_i = (a1\ h1 + a2\ h2)/(a1+a2)$

The overall NA goes through the centroid of the compound area.



Moment of Inertia Calculation

$Izz = \Sigma\ a_i\ h_i^2\ + \Sigma\ Ina_i$

$I_{NA} = Izz - A\ h_{NA}^2$

or

$I_{NA} = \Sigma(Ina_i + a_i\ (h_i - h_{NA})^2)$

A simple spreadsheet, as shown below, can be used to find the moment of inertia of a ship;

#	item (desc.)	scantlings (desc.)	area a	height h	1st mom ah	2nd mom ah^2	local 2nd i_{na}
1							
2							
n							
			$A = \Sigma a$		Σah	Σah^2	Σi

$$\underbrace{\Sigma ah^2 \quad \Sigma i}_{Izz}$$

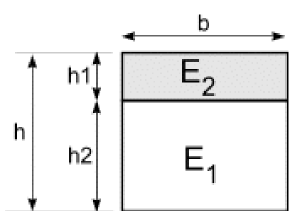
$$h_{NA} = \frac{\Sigma ah}{A}$$

$$I_{NA} = Izz - A\, h_{NA}^2$$

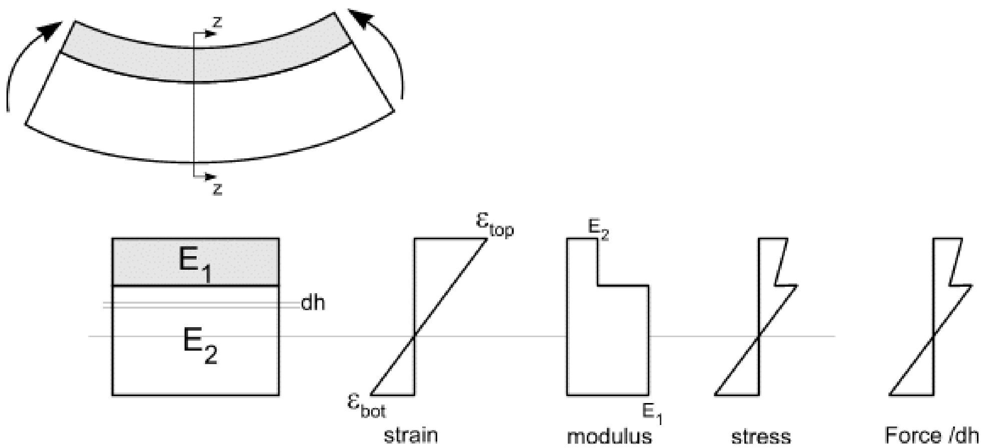
See Assignment #2 for an application.

Section Modulus for Material Combinations
(e.g. Steel Hull, Al Superstructure)

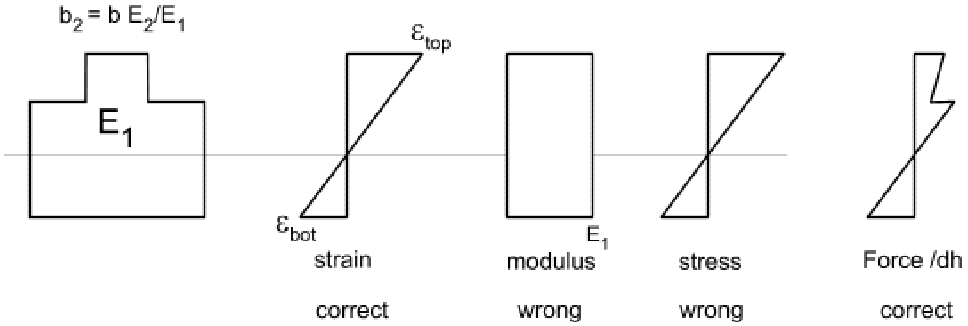
Consider a section with 2 materials



When the section bends the sections remain plane, meaning that the strain field is linear.



To determine the stress/strain/deflection relationships, we convert the x-section to an equivalent section. The idea is to modify the section so that it is all made of one material, but retains the distribution of axial force (and bending stiffness). We do this by adjusting the width of one of the materials, in accordance with the ratio of Young's Modulus. For example, Aluminum is converted to steel, but made thinner by E_{al}/E_{st}.



For the modified section, I_{TR} is calculated in the usual way. The strains and deflections for any vertical bending moment will be correct.

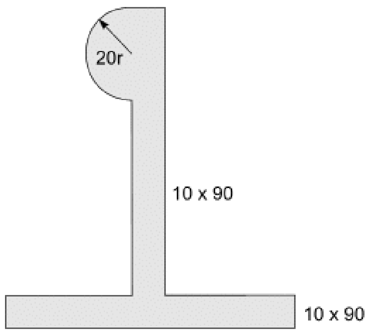
i.e. $v'' = \dfrac{M}{EI_{TR}}$

The only error will be the stresses in the transformed region. The stresses in the unmodified region will be correct, but the modified region will be wrong by the ratio of modulii. We can correct this as follows;

$\sigma \neq \dfrac{My}{I_{TR}} \Rightarrow \sigma_1 = \dfrac{My}{I_{TR}}$ and $\sigma_2 = \dfrac{E_2}{E_1}\dfrac{My}{I_{TR}}$

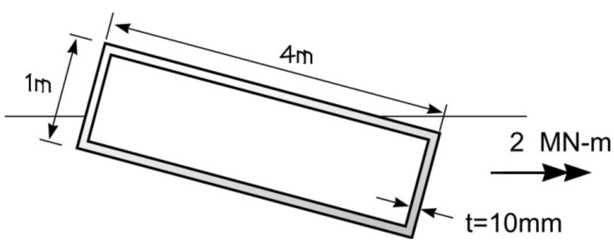
Topic 7: Problems

7.1. Find the moment of inertia of this compound section:

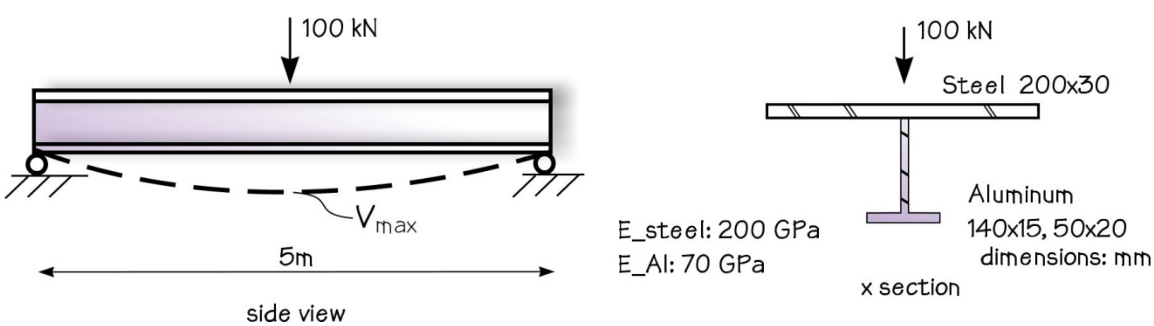


dimensions in mm

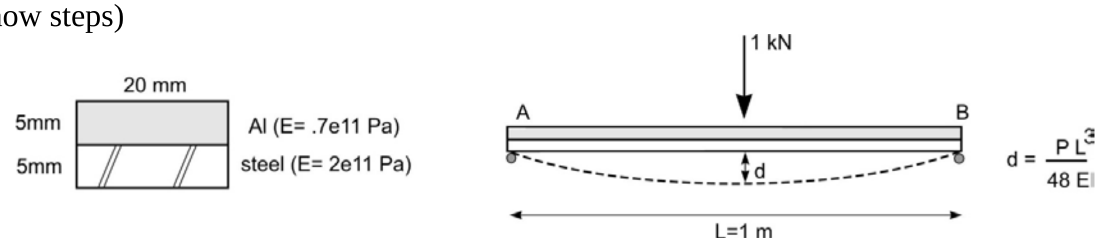
7.2. A box steel hull is 4m x 1m with a shell thickness of 10mm. It is inclined at 15 degrees, and subject to a vertical bending moment of 2 MN-m. Find the bending stress at the emerged deck edge.



7.3. For a composite beam (Steel plate with Aluminum web/Flange) loaded as show below
a) find the central deflection.
b) find the maximum stress in the Aluminum



7.4. Consider a compound steel-aluminum beam, shown below. Calculate the deflection d (show steps)



Ans: 0.112m

PART 3: Beams and Indeterminate structures

Sintra Tile Mosaic

Topic 8: Beam Theory

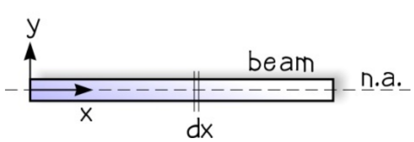
Test Grillage at Memorial University

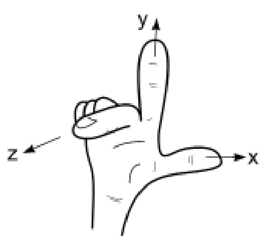
Introduction

In this Chapter we will

- Develop the elastic behavior of beams
- Show the relationship among load, shear, bending, slope and deflection

~~~~~~~~~~~~~~~~~~~~~~~

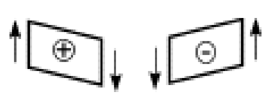




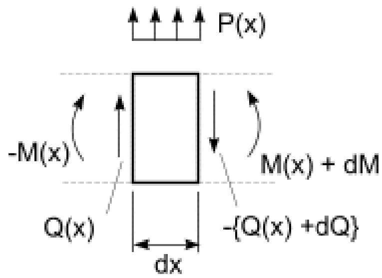
Coordinate System and Sign Convention

The standard coordinate system has the x axis along the neural axis of the beam. The positive y axis is pointed up. The sign convention for force and moment vectors follows the right hand rule;

+ Forces and deflections follow the axes.
+ Moments and rotations follow the curl of the fingers (on the right hand) when the thumb is pointing along the axis.

Shear strain: 

Bending moment: 
~~~~~~~~~~~~~~~~~~~~~~~



in SI units:
P : N/m
Q : N
M : Nm
dx: m

To determine the equations for beam bending we take a small section of the beam (which represents any part) as a free body. We look at all the forces and moments on the section and assuming that the net force and net moment are zero (Newton!) we derive the equations.

At this point we haven't specified P,Q or M. They can have any values. We will examine equilibrium conditions and see how these result in relationships among P,Q,M.

We start by summing vertical forces, which must sum to zero for equilibrium;

$$Q(x) - (Q(x) + dQ) + p(x)\,dx = 0$$

[N]

which is simplified to;

$$dQ = p(x)dx$$

and rearranged to give;

$$p(x) = \frac{dQ}{dx}$$

This is a differential equation that states that the line load on a beam is equal to the rate of change (slope) of the shear force. Next we sum moments about the right hand end, which must also sum to zero to show equilibrium of the free body.

$$-M(x) - Q(x)dx - p(x)dx\frac{dx}{2} + (M(x) + dM) = 0$$

which is simplified to;

$$-Q(x)dx - p(x)\frac{dx^2}{2} + dM = 0$$

note that *dx* is not just small, it is <u>vanishingly</u> small, so that dx^2 is vanishingly small by comparison (i.o.w. we can remove the second order terms, in this case with no loss of accuracy). Therefore;

$$-Q(x)dx + dM = 0$$

or;

$$Q(x) = \frac{dM}{dx}$$

This is our second (related) differential equation, which states that the shear in a beam is the rate of change (slope) of the bending moment.

We now have two differential equations;

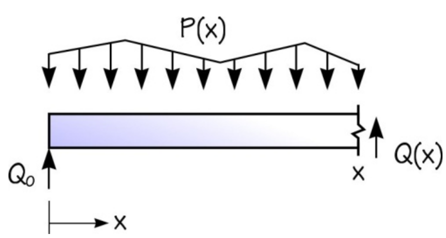
$$p(x) = \frac{dQ}{dx}$$

and

$$Q(x) = \frac{dM}{dx}$$

We can re-express these relationships as integral equations. The shear is;

$$Q(x) = \int p(x)dx$$



In the form of a definite integral with a constant of integration the shear is;

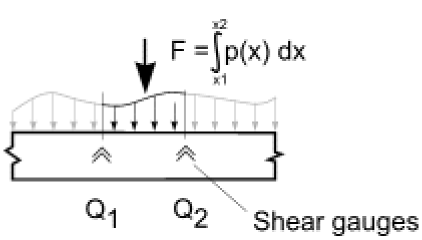
$$Q(x) = Q_o + \int_0^x p(x)dx$$

In words, this equation means: shear is the sum of all loads from the start to x. Similarly, the moment is;

$$M(x) = \int Q(x)dx$$

which becomes;

$$M(x) = M_o + \int_0^x Q(x)dx$$



Aside: The shear difference between any two points on a beam will be exactly equal to the load applied to the beam between these two points, for any pattern of load. This leads to a very easy and accurate way to measure force;

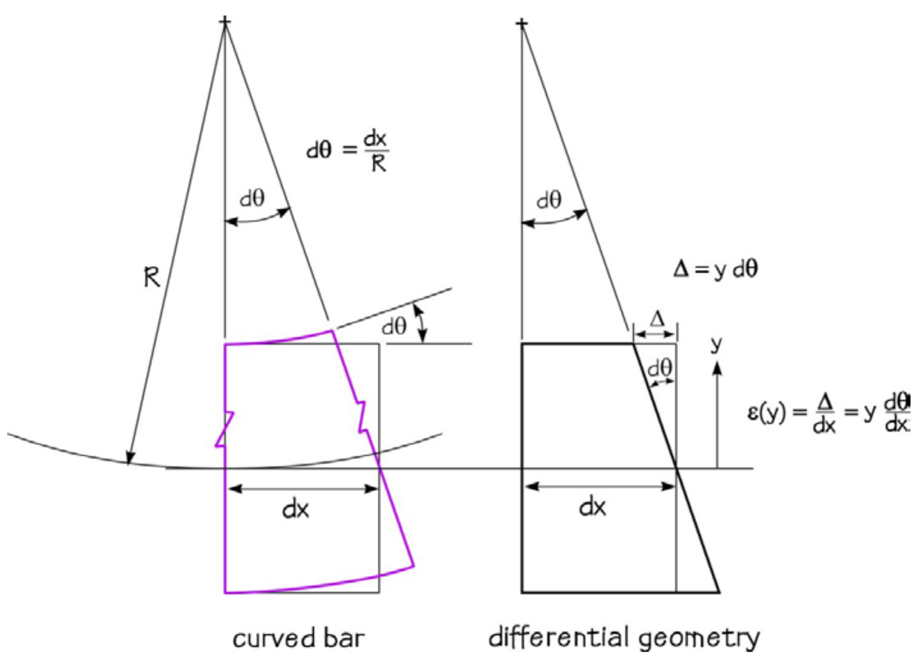
$$F = \int_{x_1}^{x_2} p(x)dx = Q_2 - Q_1$$

This principle has been used to design load cells, and to instrument ship frames to measure contact loads from ice or slamming.

Adding Deformations

So far we have differential equations for load/shear/bending relationships. Now we add deformations.

The shear force and bending moments are causing stresses and strains in the beam. We make the assumption that we can ignore the shear deformations (this is part of what we call simple beam theory), so that only the bending moments cause distortions. This means that only consider the shortening of the compression side of the beam and the lengthening of the tension side. When this happens, the beam deforms from being straight to being a curve. The curve shape for any short length is an arc of a circle, with a radius R. The local radius, as we can show, turns out to depend only on the local bending moment. The figure below show a short length of a bending beam. The curved shape is also presented in differential form, meaning essential or limit shape for a very small value of dx.



The neutral axis (NA) does not stretch or contract. The upper and lower parts of the beam compress and/or stretch. We can use the two 'known' relationships, the stress-moment equation;

$$\sigma = \frac{M\,y}{I}$$

and 1D Hooke's law;

$$\sigma = E\varepsilon$$

For the top fiber (in the figure above) we see that the strain is;

$$\varepsilon = \frac{\Delta}{dx} = y\frac{d\theta}{dx}$$

from the above we have;

$$\varepsilon = \frac{M\,y}{EI} = y\frac{d\theta}{dx}$$

which can be re-arranged to give;

$$\frac{d\theta}{dx} = \frac{M}{EI}$$

or

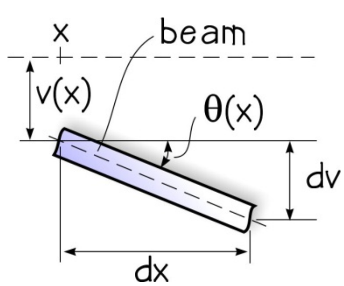
$$d\theta = \frac{M}{EI}dx$$

We also have

$$d\theta = \frac{dx}{R}$$

$$\frac{d\theta}{dx} = \frac{1}{R} = \kappa$$

Where R is the 'radius of curvature' and κ is called the 'curvature' (note the odd naming).



Note also that dθ is both the change in relative angle of two cross sections separated by dx and also the change in slope between two points separated by dx along the beam. θ(x) is the slope of the beam.

This gives us;

$$\theta = \theta_0 + \int_0^x \frac{M}{EI} dx$$

For prismatic sections, EI is constant, so;

$$\theta = \theta_0 + \frac{1}{EI} \int_0^x M \, dx$$

Similarly, to find deflections v, we use the relationship, assuming small deflections;

$$\frac{dv}{dx} = \theta$$

and

$$dv = \theta \, dx$$

which lets us write;

$$v(x) = \int \theta(x) \, dx$$

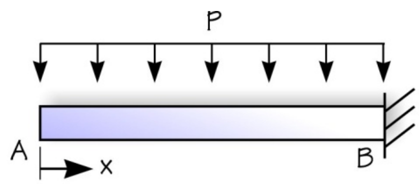
and;

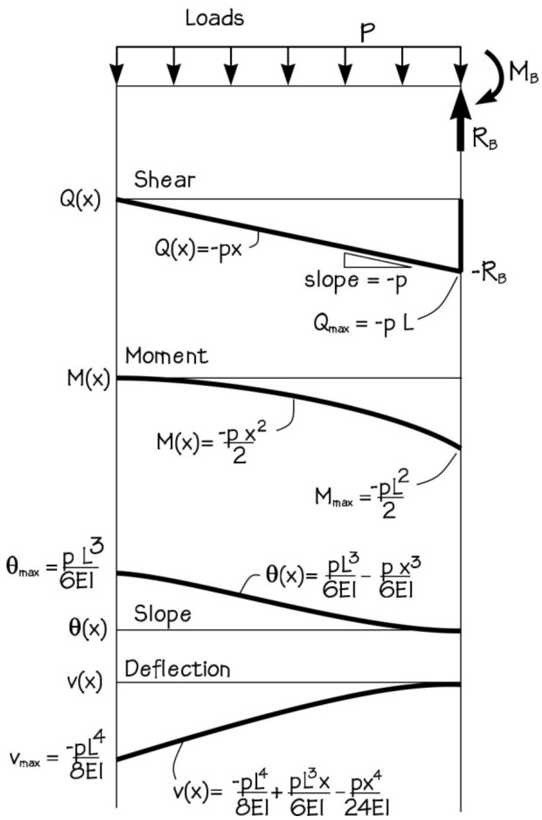
$$v(x) = v_o + \int_0^x \theta(x) dx$$

This completes the development of the differential and integral equations for beams.

Beam Example 1: Cantilever beam with left end free

The cantilever beam is sketched at the left. The left end is free and the right end is fixed. The shear force is found by integrating the load. In this case the initial shear is zero, because there is no reaction at the left had end (it's a free end) ;





$$Q(x) = Q_o + \int_0^x p(x)dx$$

$$Q(x) = 0 + \int_0^x -p \ dx$$

$$Q(x) = -px$$

The bending moment is similarly found by integrating the shear. And again there is no initial value of moment because the boundary condition has no moment;

$$M(x) = M_o + \int_0^x Q(x)dx$$

$$M(x) = 0 + \int_0^x -p\,x\,dx$$

$$M(x) = \frac{-px^2}{2}$$

The shear is a straight line. We did not solve for the right hand vertical reaction R_B , but it is $p\,L$ and it opposes the shear in the end of the beam (which we can see is $-p\,L$). The moment is a quadratic function with a maximum value of $-p\,L^2/2$ as is easily found from summing moments about the right hand end.

Next we solve the equation for the slope.

$$\theta(x) = \theta_o + \frac{1}{EI}\int_0^x M(x)dx$$

by inserting the expression for bending moment we get ;

$$\theta(x) = \theta_o + \frac{1}{EI}\int_0^x \frac{-p\,x^2}{2}dx$$

which becomes;

$$\theta(x) = \theta_o - \frac{p\,x^3}{6EI}$$

At this point we can either carry forward the unknown initial slope or solve for it. We know the slope at L is zero, so we can write;

$$\theta(L) = 0 = \theta_o - \frac{p\,L^3}{6EI}$$

which can be solved to get;

$$\theta_o = \frac{p\,L^3}{6EI}$$

therefore the complete equation for slope is;

$$\theta(x) = \frac{p\,L^3}{6EI} - \frac{p\,x^3}{6EI}$$

Now we can find the deflection. The integral equation is;

$$v(x) = v_o + \int_0^x \theta(x)dx$$

which becomes;

$$v(x) = v_o + \int_0^x \frac{p\,L^3}{6EI} - \frac{p\,x^3}{6EI}dx$$

which becomes;

$$v(x) = v_o + \frac{p\,L^3 x}{6EI} - \frac{p\,x^4}{24EI}$$

The deflection at L is zero, letting us write;

$$v(L) = 0 = v_o + \frac{p\,L^4}{6EI} - \frac{p\,L^4}{24EI}$$

which gives;

$$v_o = -\frac{p\,L^4}{8EI}$$

so the total equation for the deflection is;

$$v(x) = \frac{-p\,L^4}{8EI} + \frac{p\,L^3 x}{6EI} - \frac{p\,x^4}{24EI}$$

which completes the solution.

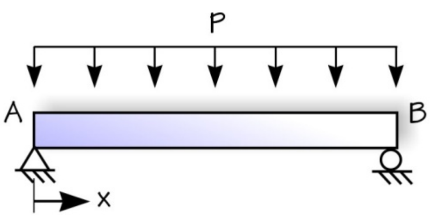
Example 2: Pinned-pinned beam

In this case the initial value of shear is the reaction at the left end. We can solve for this from static equilibrium at the start. So the shear is;

$$Q(x) = Q_o + \int_0^x p(x)dx$$

$$Q(x) = pL/2 + \int_0^x -p \ dx$$

$$Q(x) = \frac{pL}{2}$$

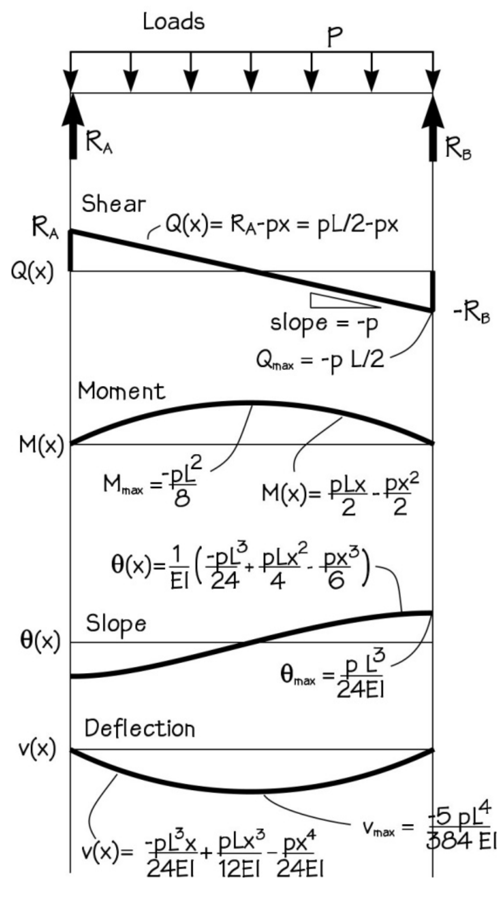


The bending moment is;

$$M(x) = M_o + \int_0^x Q(x)dx$$

$$M(x) = 0 + \int_0^x \frac{pL}{2} - p\ x\ dx$$

$$M(x) = \frac{pLx}{2} - \frac{px^2}{2}$$



The plot at the left shows the shear and bending solutions. In this case, we were able to use statics to solve for one unknown at the start, which simplified the problem.
Next we solve the equation for the slope, as before, by inserting the expression for bending moment we get ;

$$\theta(x) = \theta_o + \frac{1}{EI}\int_0^x \frac{pLx}{2} - \frac{px^2}{2} dx$$

which becomes;

$$\theta(x) = \theta_o + \frac{1}{EI}\left(\frac{pLx^2}{4} - \frac{px^3}{6}\right)$$

At this point we can either carry forward the unknown initial slope or solve for it. We know, from symmetry, that the slope at $x = L/2$ is zero, so we can write;

$$\theta(L/2) = 0 = \theta_o + \frac{1}{EI}\left(\frac{pL^3}{16} - \frac{pL^3}{48}\right)$$

which can be solved to get;

$$\theta_o = \frac{-p\ L^3}{24EI}$$

therefore the complete equation for slope is;

$$\theta(x) = \frac{1}{EI}\left(\frac{-p\,L^3}{24EI} + \frac{pLx^2}{4} - \frac{px^3}{6}\right)$$

Now we can find the deflection. The integral equation is;

$$v(x) = v_o + \int_0^x \theta(x)dx$$

which becomes;

$$v(x) = v_o + \frac{1}{\mathrm{EI}}\int_0^x \frac{-p\,L^3}{24EI} + \frac{pLx^2}{4} - \frac{px^3}{6}\,dx$$

which becomes;

$$v(x) = v_o + \frac{-p\,L^3 x}{24EI} + \frac{pLx^3}{12EI} - \frac{p\,x^4}{24EI}$$

The deflection at L is zero, letting us write;

$$v(x) = \frac{-p\,L^3 x}{24EI} + \frac{pLx^3}{12EI} - \frac{p\,x^4}{24EI}$$

which completes the solution.

Topic 8: Problems

8.1. Consider a beam made of steel joined to aluminum. The steel is 10 x 10 mm, with 5 x 10 mm of Aluminum attached. E_{steel} = 200,000 MPa, E_{Al} = 80,000 MPa. The beam is fixed as a simple cantilever, with a length of 100mm and a vertical force at the free end of 2 kN.

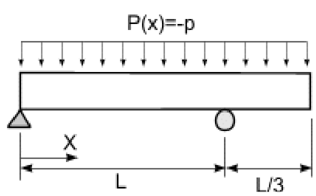
convert the section to an equivalent section in steel and calculate the equivalent moment of inertia.
What is the deflection of the end of the beam (derive from 1st principles).
What is the maximum bending stress in the Aluminum at the support?

8.2. For elastic beam bending, derive the equation:

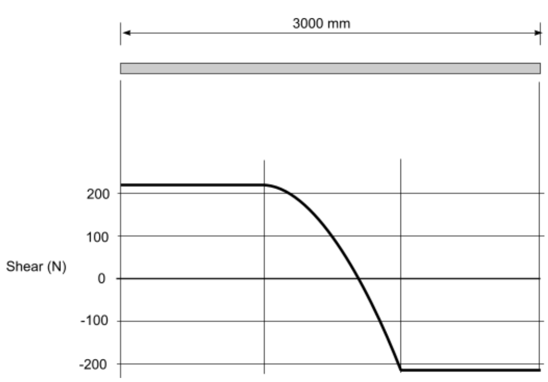
$$\frac{d\theta}{dx} = \frac{M}{EI}$$

where θ is the slope of the deflected shape, M is the moment, E is Young's Modulus, I is the moment of inertia. You can assume the σ=εE and σ=My/I. Use at least one sketch.

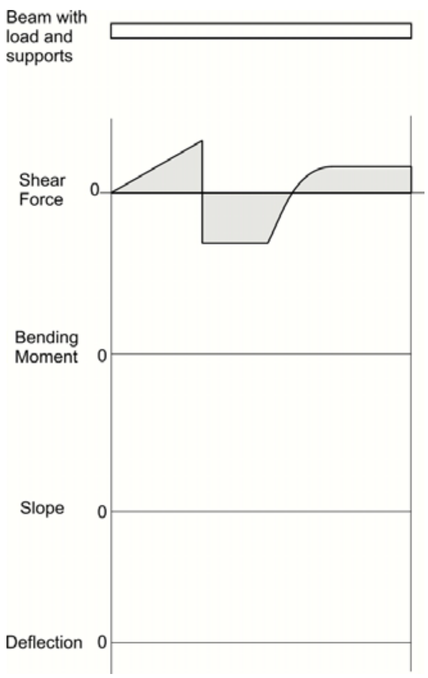
8.3. Find and draw the shear force and bending moment diagrams for the following beam. Find the values at supports and other max/min values.



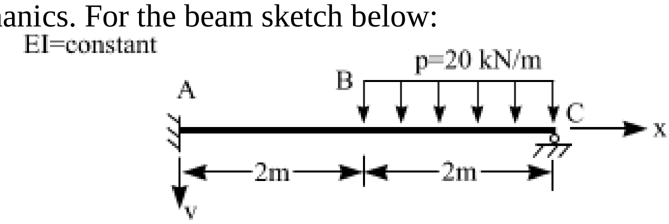
8.4. There is a 3m beam. The shear force diagram is sketched below.
Sketch the load, moment, slope and deflection diagrams (9)
What are the boundary conditions and discuss whether there can be more than one option for the boundary conditions.(6)



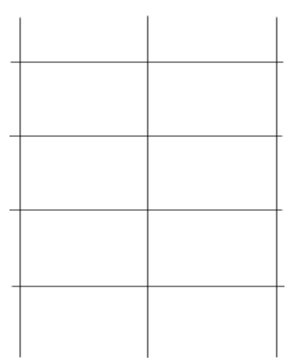
8.5. For elastic beam bending, complete Figure 1. The shear force diagram is sketched. You need to infer from the shear what the load (including support reactions) may be, as well as an estimate of the bending moment diagram, the slope diagram and the deflected shape. Draw the support conditions and the applied load on the beam, and sketch the moment, slope and deflection is the areas given.



8.6. Beam Mechanics. For the beam sketch below:



a) sketch by hand the shear, moment, slope and deflection diagrams

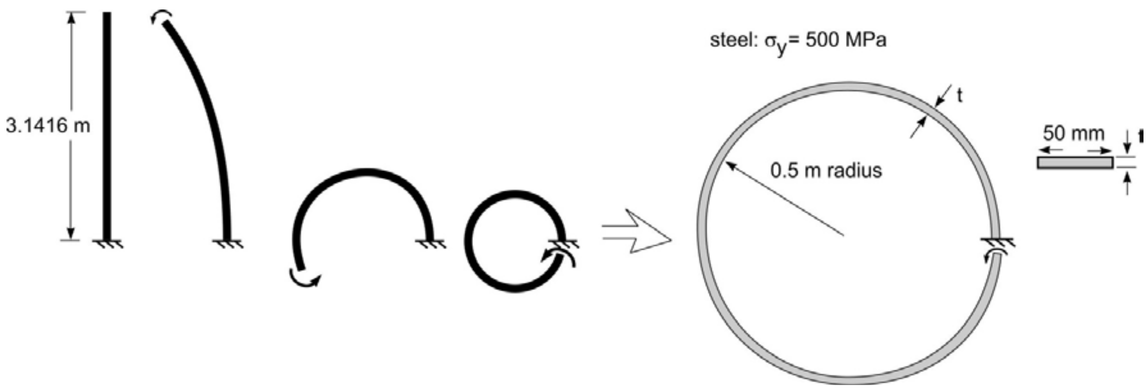


b) Assuming the beam is a 10cm x 10cm square steel bar, solve the problem to find the bending stress at the fixed support. Use any method you like.

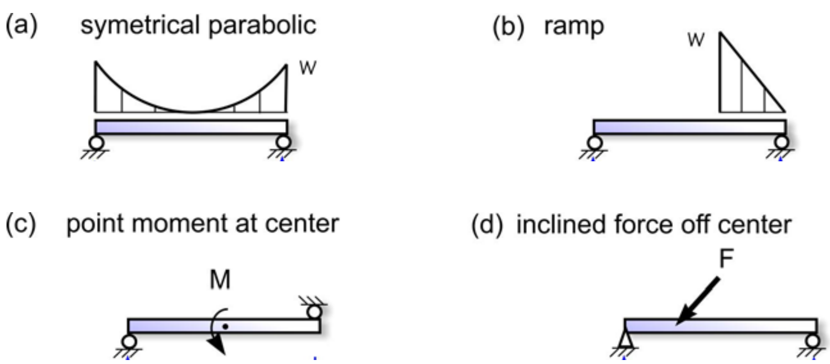
8.7. There is a length of steel that is 3.1416 m long, 50mm wide. It has a yield strength of 500 MPa (N/mm^2), and a Young's Modulus of 200 GPa. If the steel is thin enough it can be bent into a perfect circle without yielding.

a) What is the maximum thickness 't' for the steel to be bent elastically (and not yield)?
b) If the steel thickness is 1mm, what is the stress when it is bent into a 1m Dia circle.
c) What would the shear force diagram look like?

(Hint :this relates directly to the derivation of the differential equations for beam bending)



8.8. Sketch the shear, bending, slope and deflection patterns for the four cases shown below. No numerical values are required.



Topic 9: Solving Beam Equations

A Train Station in Lisbon

Introduction

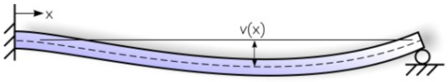
In this Chapter we will

- Review the differential equation set derived in the last Chapter and discuss solutions using Macaulay functions and Maple.

~~~~~~~~~~~~~~~~~~~~~~

Family of Differential Equations

Simple beam behavior considers only the deflections due to bending, and only in 2 dimensions. Torsion, shear and other elastic distortions are neglected (for now).



Consider a beam between two supports. We describe the deflections with the variable v(x).
~~~~~~~~~~~~~~~~~~~~~~

The analysis of bending in Chapter 8, developed the following differential equations;

$$p(x) = \frac{d\,Q(x)}{dx}$$

$$Q(x) = \frac{d\,M(x)}{dx}$$

$$M(x) = EI\frac{d\,\theta(x)}{dx}$$

$$\theta(x) = \frac{d\,v(x)}{dx}$$

These can be re-expressed into a set of related (not coupled) differential equations, of increasingly higher order;

$$v(x) = deflection[m]$$

$$\theta(x) = \frac{dv(x)}{dx} = v'(x) = slope\ [rad]$$

$$M(x) = EI\ \frac{d^2v(x)}{dx^2} = EI\ v''(x) =\ moment\ [Nm]$$

$$Q(x) = EI\ \frac{d^3v(x)}{dx^3} = EI\ v'''(x) = shear[N]$$

$$p(x) = EI\ \frac{d^4v(x)}{dx^4} = EIv''''\,(x) = load\ [N/m]$$

Seen in this way, the key behavior is deflection, with all other quantities being derived from it. There is a similar set of equations, expressed in integral form, starting from load;

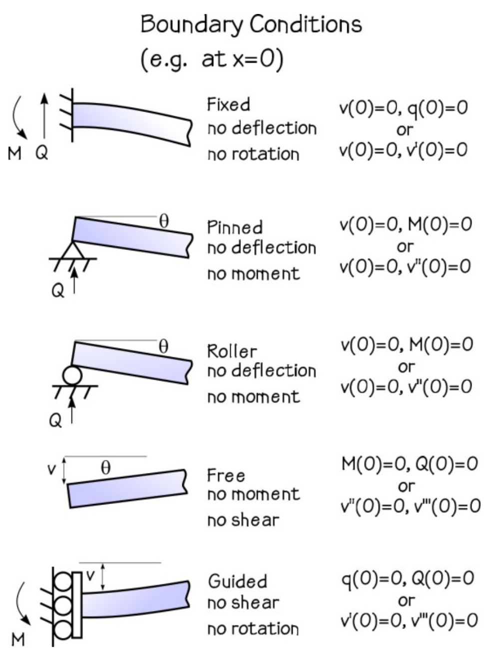
$$p(x) = load\ [N/m]$$

$$Q(x) = Q_o + \int_0^x p(x)dx = shear\ force\ [N]$$

$$M(x) = M_o + \int_0^x Q(x)dx = moment\ [Nm]$$

$$\theta = \theta_0 + \frac{1}{EI}\int_0^x M\,dx = \text{slope [rad]}$$

$$v(x) = v_o + \int_0^x \theta(x)dx = deflection\ [m]$$

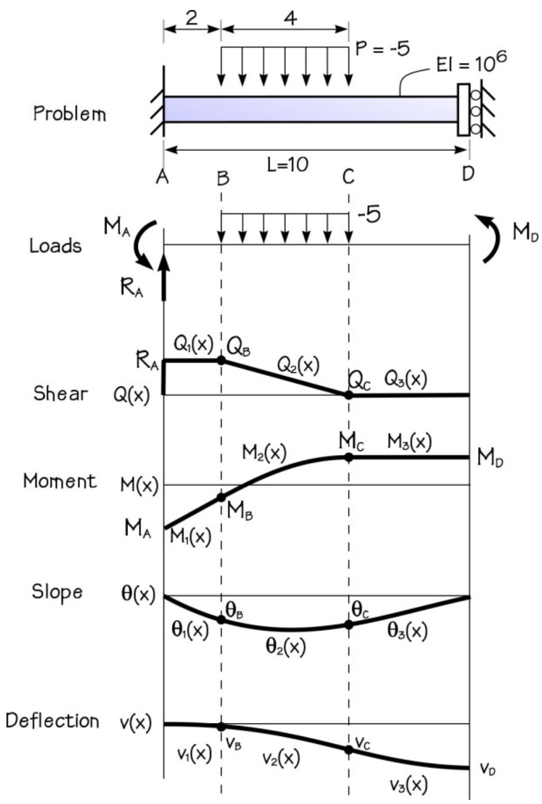


The set of derivative equations show that if the deflected shape is known, all other quantities can be determined. In such a case there is no need for any boundary conditions. (to do: think of a situation where the deflected shape is fully known, while other quantities are not.)

Normally we would not know the deflected shape. Instead we would know the load and would want to determine the deflected shape. In that case we would employ the integral equations. One significant issue with the integral equations is that the 'constants of integration' must be found. These are found from the boundary conditions. All types of end conditions can be represented as some derivative of deflection being zero. More specifically, two of the derivatives will be zero at each end of the beam. This gives four known boundary conditions for any beam (2 ends!), and so the four integral equations can be solved.

At this level of consideration, there is no difference between a determinate and an indeterminate beam. All beams have 4 integral equations and 4 boundary equations (or it could be said that all beams are represented by a fourth order ordinary differential equation with four boundary condition equations, regardless of the type or loading or supports).

In the previous chapter we solved two beams by progressively solving the integral equations. Those cases were relatively simple, both because they were determinant systems, and they had simple load patterns, and in one case was symmetric. Solving non-symmetric cases of indeterminate beams with discontinuous loads (patch loads) can involve a lot of algebra. We will solve one such system in three different ways; 1) directly with the integral equations, 2) with Macaulay functions and 3) with the help of the Maple program.



<u>Example 3: Solving Piecewise Beam Equations</u>

The beam sketched at left is fixed on the left end, guided on the right and with the loading and properties shown. A qualitative sketch of the solution is plotted, indicating that the solution is in three parts. The load is a patch load, so the solution must be in parts. The points labeled 'B' and 'C' represent break points in the solution. The various quantities at these points represent ending values for the partial solution to the left of the point and starting values for the solution to the right of that point.

The boundary conditions create a set of unknown loads on the ends of the beam, which are sketched in the 'Loads' diagram. For a fixed end we know that the deflection and rotation are zero. For a guided end we know that the shear (reaction) and rotation are zero. These conditions give two unknown loads at the left end of the beam. There are two known movements (deflection and slope are zero) at the left end of the beam. At the right end the moment and deflection are unknown while the shear and slope are both zero (recall that there are always 2 known and 2 unknown values at each end, in some combination of loads and displacements). In this particular beam we know that R_A is the only vertical support and must balance all the applied load (which is 4x5=20). We also know that there is no shear in the right end of the beam (the vertical force must be zero because the roller has released it). So the shear solution is as follows;

part 1:

$$Q_1(x) = 20$$

part 2:

$$Q_2(x) = C - 5\,x$$

$$Q_2(2) = C - 5\ 2 = 20 \ => C = 30$$
$$Q_2(x) = 30 - 5\,x$$

part 3:

$$Q_3(x) = 0$$

The moment solution is;

part 1:

$$M_1(x) = M_A + \int_0^x 20\,dx$$
$$M_1(x) = M_A + 20\,x$$
$$M_B = M_A + 40$$

part 2:

$$M_2(x) = M_B + \int_2^x (30 - 5\,x)\,dx$$
$$M_2(x) = M_A - 10 + \ 30x - \frac{5}{2}x^2$$
$$M_C = M_2(6) = M_A + 80$$

part 3:

$$M_3(x) = M_C + \int_6^x 0\,dx = M_A + 80$$

The slope solution is;

part 1:

$$\theta_1(x) = \theta_A + \frac{1}{EI}\int_0^x M_A + 20\,x\,dx$$
$$\theta_1(x) = 0 + \ (M_A x + 10\,x^2)\,10^{-6}$$
$$\theta_B = \theta_1(2) = (2\,M_A + 40)10^{-6}$$

part 2:

$$\theta_2(x) = \theta_B + \frac{1}{EI}\int_2^x M_A - 10 + 30\,x - \frac{5}{2}x^2\ \ dx$$
$$\theta_2(x) = (\frac{20}{3} + M_A x - 10\,x + 15\,x^2 - \frac{5}{6}x^3)10^{-6}$$
$$\theta_C = \theta_2(6) = (6\,M_A + \frac{920}{3})10^{-6}$$

part 3:

$$\theta_3(x) = \theta_C + \frac{1}{EI}\int_6^x M_A + 80\ \ dx$$
$$\theta_3(x) = (-\frac{520}{3} + M_A x + 80\,x)10^{-6}$$
$$\theta_D = \theta_3(10) = 0 = (10\,M_A + \frac{1880}{3})10^{-6}$$

Therefore

$$M_A = -\frac{188}{3}$$

$$\theta_3(x) = (-\frac{520}{3} + \frac{52}{3}x)10^{-6}$$

The deflection solution is;

part 1:

$$v_1(x) = v_A + \int_0^x (\frac{-188}{3}x + 10\,x^2)\;10^{-6}\;x\,dx$$

$$v_1(x) = (\frac{-94}{3}x^2 + \frac{10}{3}\,x^3)\;10^{-6}$$

$$v_B = v_1(2) = \left(\frac{-94}{3}x^2 + \frac{10}{3}\,x^3\right)10^{-6} = \frac{-296}{3}\;\;10^{-6}$$

part 2:

$$v_2(x) = (\frac{-296}{3} + \int_2^x \frac{20}{3} - \frac{218}{3}\,x + 15\,x^2 - \frac{5}{6}x^3\;\;dx)\;10^{-6}$$

$$v_2(x) = (\frac{-10}{3} + \frac{20}{3}\,x - \frac{109}{3}x^2 + 5\,x^3 - \frac{5}{24}x^4)10^{-6}$$

$$v_C = v_2(6) = (\frac{-1384}{3})10^{-6}$$

part 3:

$$v_3(x) = (\frac{-1384}{3} + \int_6^x -\frac{520}{3} + \frac{52}{3}\,x\;\;dx)\;10^{-6}$$

$$v_3(x) = (\frac{800}{3} - \frac{520}{3}\,x + \frac{26}{3}x^2)10^{-6}$$

$$v_D = v_3(10) = (-600)\;10^{-6}$$

Summary of solution:

$$Q(x) \;=\; \begin{cases} 20 & 0 \le x < 2 \\ 30 - 5x & 2 \le x < 6 \\ 0 & 6 \le x < 10 \end{cases}$$

$$M(x) \;=\; \begin{cases} -62.67 + 20x & 0 \le x < 2 \\ -72.67 + 30x - 2.5\,x^2 & 2 \le x < 6 \\ 17.33 & 6 \le x < 10 \end{cases}$$

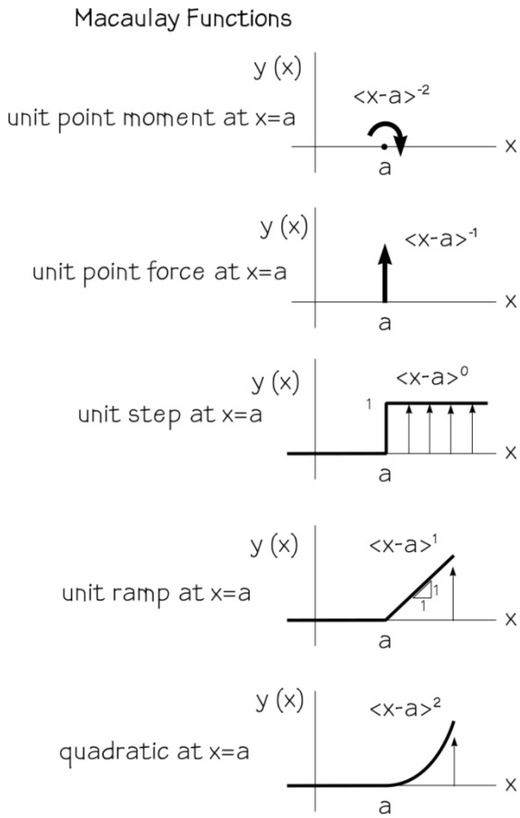
$$\theta(x) \;=\; 10^{-6}\begin{cases} -62.67x + 10x^2 & 0 \le x < 2 \\ 6.67 - 72.67x + 15\,x^2 - .83\,x^3 & 2 \le x < 6 \\ 17.33\,x - 173.3 & 6 \le x < 10 \end{cases}$$

$$v(x) \;=\; 10^{-6}\begin{cases} -31.33x^2 + 3.33x^3 & 0 \le x < 2 \\ -3.33 + 6.67x - 36.33\,x^2 + 5\,x^3 - .208x^4 & 2 \le x < 6 \\ 266.7 - 173.3\,x + 8.67\,x^2 & 6 \le x < 10 \end{cases}$$

$R_A = 20 \quad M_A = -62.67 \quad M_D = 17.33 \quad v_D = (-600)\;10^{-6}$

This completes the manual integration method for solving example 3. To check this we will be solving the same problem in 2 other ways.

Macaulay Functions



Macaulay functions (also called singularity functions) are simply a generalization of the idea of a step function. These functions provide a convenient way of describing point forces, moments and piece-wise continuous functions. And when a few special rules of integration are employed, it becomes very easy to use Macaulay functions to solve beam problems.

The fundamental Macaulay functions are shown on the left. Each function in the sequence represents the integral of the previous function (with the small exception noted later). Any of the functions can be multiplied to a constant to change the magnitude.

For example, a unit moment at $x = a$ is described as;

$$< x - a >^{-2}$$

and a moment of magnitude M at $x = a$ is;

$$M < x - a >^{-2}$$

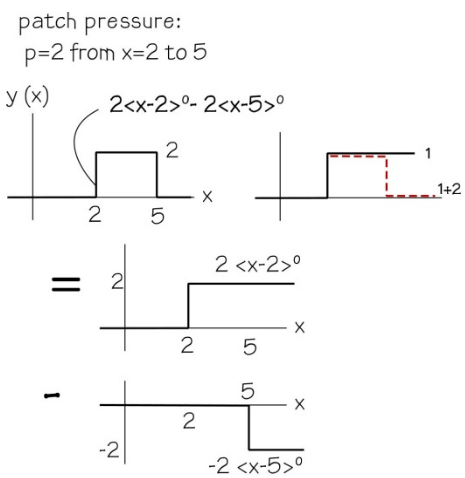
Similarly, a point for of magnitude F at $x = a$ is;

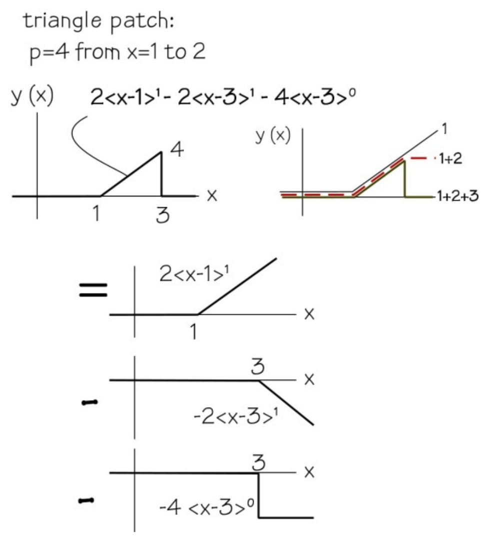
$$F < x - a >^{-1}$$

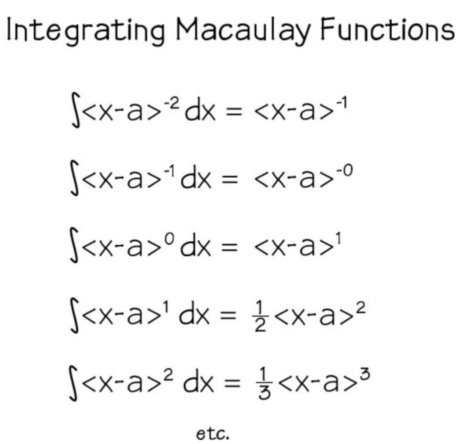
The triangular brackets are just a way of saying that the function is meant to be seen as "one sided". In simple terms :

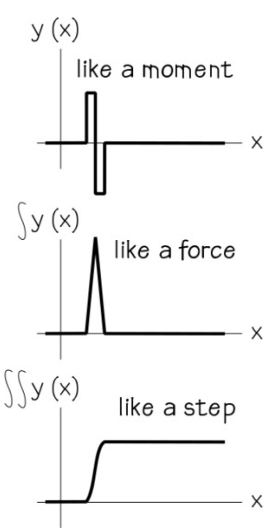
$$F < x - a >^{n} = \begin{cases} F\,(x-a)^{n} & if\ x \geq a \\ 0 & if\ x < a \end{cases}$$

Two examples of how Macaulay functions can be combined to describe various piecewise curves are shown below;









Integrating Macaulay Functions

The integration of Macaulay functions is very similar to normal functions with an exception. If the exponent is positive then the normal rules of integration apply. If the exponent is negative, then we just add one to the exponent. The rules are shown at the left.

So for example;

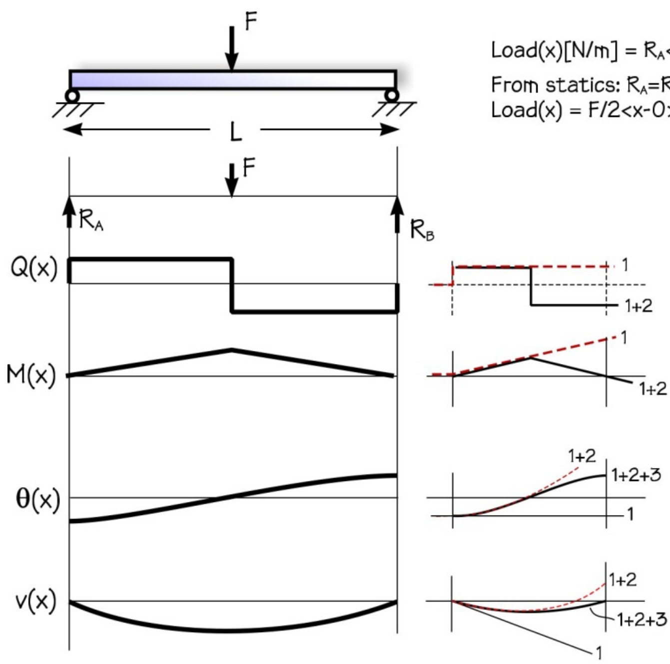
$$\int < x - a >^{-2} = < x - a >^{-1}$$

but

$$\int < x - a >^{2} = \frac{1}{3} < x - a >^{3}$$

It likely makes sense to the reader that the integral of a point force is a step and the integral of a step is a ramp. Does it make sense that the integral of a point moment is a force? To explore this idea, consider the functions sketched at the left. In the first case we have function with a small patch of load in one direction followed by a small patch of load in the opposite direction we have no net force but we do create a small point moment (in the limit). When we integrate this we get a small triangle, which when integrated again gives a step.

To Illustrate Macaulay functions, we start with an example of a pinned-pinned beam with a central force:



Load(x)[N/m] = $R_A\langle x-0\rangle^{-1} - F\langle x-L/2\rangle^{-1} + R_B\langle x-L\rangle^{-1}$

From statics: $R_A=R_B=F/2$

Load(x) = $F/2\langle x-0\rangle^{-1} - F\langle x-L/2\rangle^{-1} + F/2\langle x-L\rangle^{-1}$

We will leave out terms with <x-L> as they show no useful info

Shear $Q(x)[N] = \overbrace{F/2\langle x-0\rangle^{0}}^{1}\ \overbrace{- F\langle x-L/2\rangle^{0}}^{2}$

Moment $M(x)[Nm] = \cancel{M(0)} + \overbrace{F/2\langle x-0\rangle^{1}}^{1}\ \overbrace{- F\langle x-L/2\rangle^{1}}^{2}$

Slope $\theta(x)[Nm^2/Nm^2= rad] = \theta(0)+1/EI(F/4\langle x-0\rangle^{2} - F/2\langle x-L/2\rangle^{2}$

From symmetry: $\theta(L/2)=0$ $\theta(0)=- F\,L^2/(16\,EI)$ (ok)

$\theta(x)[Nm^2/Nm^2= rad] = 1/EI(\overbrace{- F\,L^2/16}^{1}\ \overbrace{+ F/4\langle x-0\rangle^{2}}^{2}\ \overbrace{- F/2\langle x-L/2\rangle^{2}}^{3})$

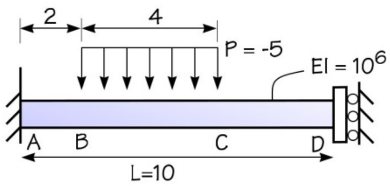
Deflection

$v(x)[Nm^3/Nm^2=m] = \cancel{v(0)} + 1/EI(\overbrace{- F\,L^2/16\,x}^{1}\ \overbrace{+ F/12\langle x-0\rangle^{3}}^{2}\ \overbrace{- F/6\langle x-L/2\rangle^{3}}^{3})$

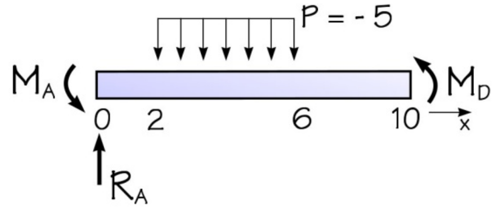
From symmetry: $v(L/2) = v_{max}$

$v_{max} = - F\,L^3/(32\,EI) + F(L/2)^3/(12\,EI) = FL^3/(48\,EI)$ (ok)

Problem



Loads (actions and reactions)



Load(x)[N/m] = $\overbrace{R_A\langle x-0\rangle^{-1} - 5\langle x-2\rangle^0 + 5\langle x-6\rangle^0}^{\text{vertical forces}}\ \overbrace{(-M_A\langle x-0\rangle^{-2} - M_D\langle x-10\rangle^{-2})}^{\text{point moments}}$

Shear $Q(x)[N] = R_A\langle x-0\rangle^0 - 5\langle x-2\rangle^1 + 5\langle x-6\rangle^1$ (integrate just the vertical forces)

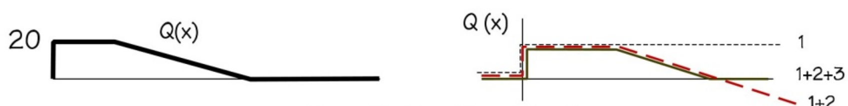
Moment $M(x)[Nm] = -M_A\langle x-0\rangle^0 + R_A\langle x-0\rangle^1 - 2.5\langle x-2\rangle^2 + 2.5\langle x-6\rangle^2 - M_D\langle x-10\rangle^0$ (now include the moments)

Slope $\theta(x)[Nm^2/Nm^2 = rad] = \cancel{\theta(0)} + 1/EI(-M_A\langle x-0\rangle^1 + R_A/2\langle x-0\rangle^2 - 5/6\langle x-2\rangle^3 + 5/6\langle x-6\rangle^3 - M_D\langle x-10\rangle^1)$

Deflection $v(x)[Nm^3/Nm^2 = m] = \cancel{v(0) + \theta(0)x} + 1/EI(-M_A/2\langle x-0\rangle^2 + R_A/6\langle x-0\rangle^3 - 5/24\langle x-2\rangle^4 + 5/24\langle x-6\rangle^4 - M_D/2\langle x-10\rangle^2$

verical reaction can be found from force equilibrium (statics) : $\Sigma F_V +\uparrow \quad R_A - 5 * 4 = 0 \Rightarrow R_A = 20$

$Q(x) = 20\langle x-0\rangle^0 - 5\langle x-2\rangle^1 + 5\langle x-6\rangle^1$



Shear Plot (and breakdown)

with R_A, Moment and Slope can be simplified to:

Moment $M(x) = -M_A\langle x-0\rangle^0 + 20\langle x-0\rangle^1 - 2.5\langle x-2\rangle^2 + 2.5\langle x-6\rangle^2 - M_D\langle x-10\rangle^0$

Slope $\theta(x) = 1/EI(-M_A\langle x-0\rangle^1 + 20/2\langle x-0\rangle^2 - 5/6\langle x-2\rangle^3 + 5/6\langle x-6\rangle^3 - M_D\langle x-10\rangle^1)$

unfortunately, equating initial slope to zero doesn't help. It will always be 0=0, regardless of the magnitude of M_A

To find M_A, we equate final slope to zero (guided support) :

$\theta(10) = 0 = 1/EI(-M_A\langle 10\rangle^1 + 20/2\langle 10\rangle^2 - 5/6\langle 8\rangle^3 + 5/6\langle 4\rangle^3 - M_D\langle 0\rangle^1)$

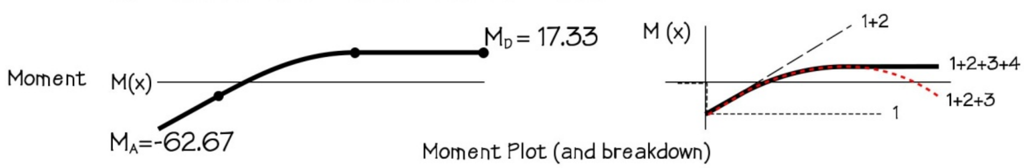
solve for M_A:

$M_A = \langle 10\rangle^2 - 5/60\langle 8\rangle^3 + 5/60\langle 4\rangle^3 = 62.67$

so:

Moment $M(x) = -62.67\langle x-0\rangle^0 + 20\langle x-0\rangle^1 - 2.5\langle x-2\rangle^2 + 2.5\langle x-6\rangle^2$

$M_D = -62.67 + 20\langle 10\rangle^1 - 2.5\langle 8\rangle^2 + 2.5\langle 4\rangle^2 = 17.33$



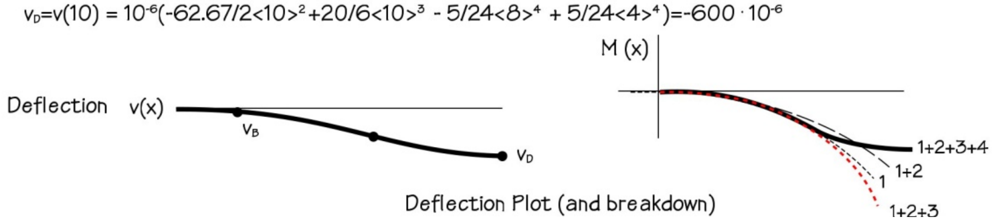
Moment Plot (and breakdown)

with M_A, Slope and deflection can be solved:

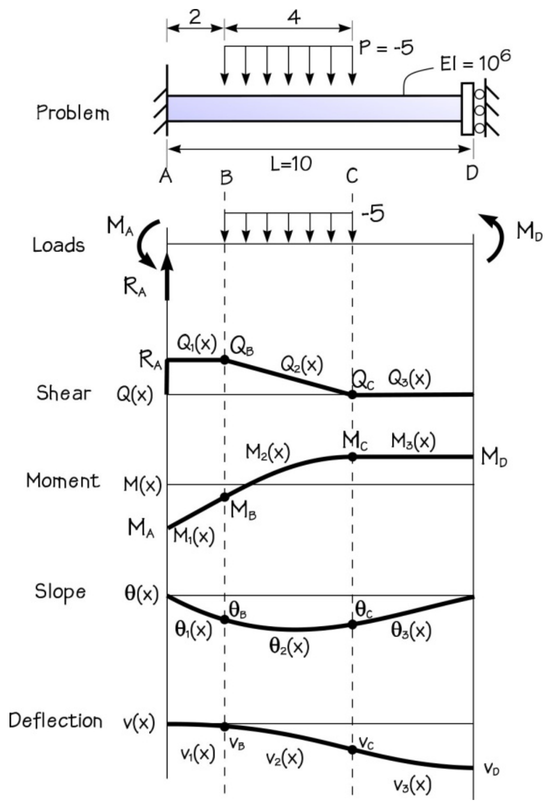
Slope $\theta(x) = 1/EI(-62.67\langle x-0\rangle^1 + 20/2\langle x-0\rangle^2 - 5/6\langle x-2\rangle^3 + 5/6\langle x-6\rangle^3)$

Deflection $v(x) = 1/EI(-62.67/2\langle x-0\rangle^2 + 20/6\langle x-0\rangle^3 - 5/24\langle x-2\rangle^4 + 5/24\langle x-6\rangle^4)$

$v_D = v(10) = 10^{-6}(-62.67/2\langle 10\rangle^2 + 20/6\langle 10\rangle^3 - 5/24\langle 8\rangle^4 + 5/24\langle 4\rangle^4) = -600 \cdot 10^{-6}$



Deflection Plot (and breakdown)



<u>Solving Example 3 using Maple</u>

Maple is a computer program that is capable of solving a wide variety of mathematical problems, including differential equations.

Here is a very simple example of Maple's ability to solve and plot differential equations. This is the solution of a cantilever beam (EI=1, L=10) under uniform load (p=-1).

The basic differential equation;

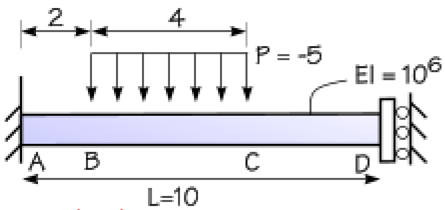
$$p(x) = EI\,\frac{d^4v(x)}{dx^4} = EIv''''(x) = load\ [N/m]$$

The boundary conditions are;

$$v(x=0)=0$$
$$v'(x=0)=0$$
$$v'(x=L)=0$$
$$v'''(x=L)=0$$

Below is the full Maple input and result, showing the shape of a deflected cantilever;

Example 3 using MAPLE 14 to solve differential equations for beam
by: Claude Daley




```
> restart;
```

The following aliases simplify the definition of loads.

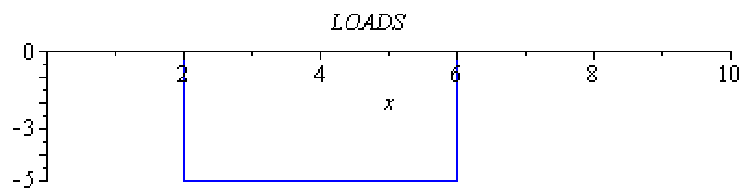
```
> dist_load := (w,a,b) -> w*Heaviside(x-a)- w*Heaviside(x-b):  # distributed force
```

Length, Stiffness, Load at "a", Load at end, Location of "a"

```
> L := 10:  EI := 10^6:
> wa:=5:we:=5:a:=2: b:=6:
> loads := -dist_load(wa,a,b)-(x-a)/(b-a)*dist_load(we-wa,a,b);
```

$$loads := -5\,\mathrm{Heaviside}(x-2)+5\,\mathrm{Heaviside}(x-6)$$

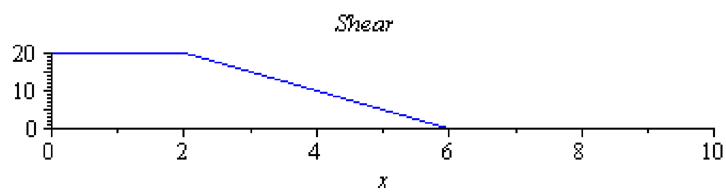
```
> plot(loads,x=0..L,title=`LOADS`, color=blue);
```



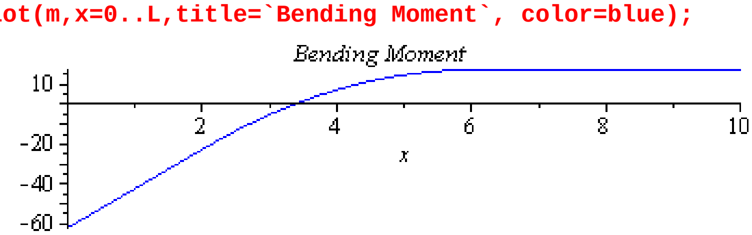

```
> supports := {y(0)=0, D(y)(0) = 0, D(y)(L) =0, D(D(D(y)))(L)=0}:
> de := EI*diff(y(x),x$4) = loads;    # Form differential equation
```

$$de := 1000000\left(\frac{d^4}{dx^4}\,y(x)\right) = -5\,\mathrm{Heaviside}(x-2)+5\,\mathrm{Heaviside}(x-6)$$

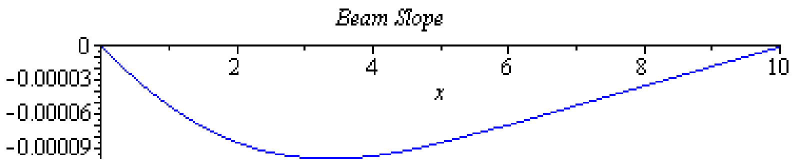
```
> dsolve({de}union supports ,y(x)):    # Solve boundary value problem
>     yy := rhs(%):                      # Extract deflection
>     th := diff(yy,x):                  # Extract slope
>     m  := EI*diff(yy,x$2):             # Extract moment
>     v  := EI*diff(yy,x$3):             # Extract shear
> plot(v,x=0..L,title=`Shear`, color=blue);
```



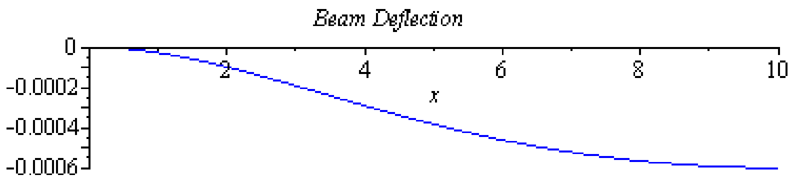

```
> plot(m,x=0..L,title=`Bending Moment`, color=blue);
```




```
> plot(th,x=0..L,title=`Beam Slope`, color=blue);
```




```
> plot(yy,x=0..L,title=`Beam Deflection`, color=blue);
```




```
> evalf(subs(x=0,m));evalf(subs(x=L,m));evalf(subs(x=L,yy));
```

−62.66666667

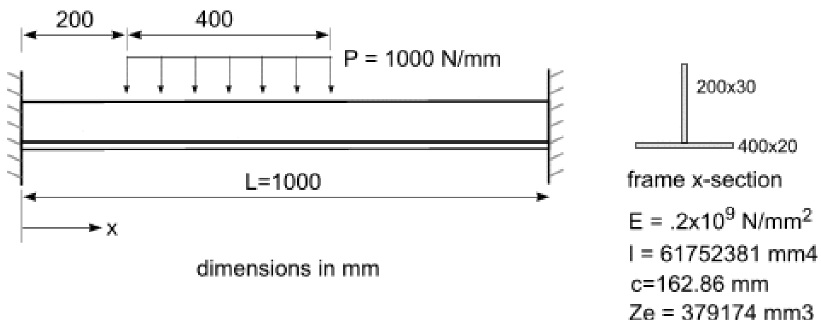
17.3333333

−0.0006000000000

The manual, Macaulay and Maple solutions are all the same, as expected.

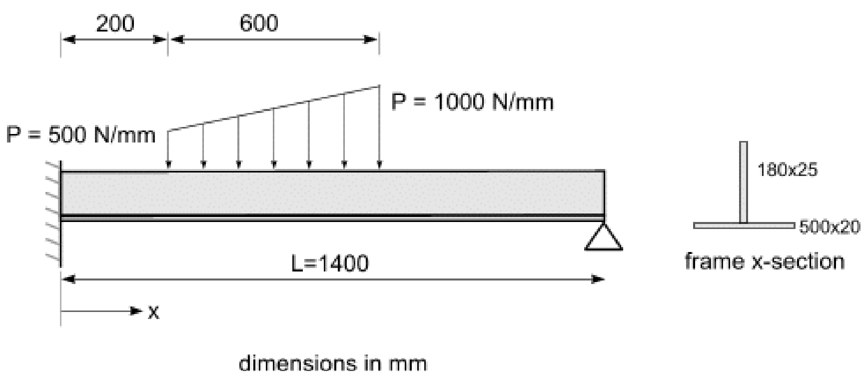
Topic 9: Problems

9.1. Solve the following beam by direct integration. What is the maximum deflection (mm)? What is the maximum stress (MPa) ?



ANS: .000136mm, 140 Pa

9.2. Solve the following beam using Macaulay functions. What is the maximum deflection (mm)? What is the maximum stress (MPa) ?



ANS: .000484mm, 253 Pa

Topic 10: Indeterminate Beams – Force Method

part of the superstructure on an FPSO

Introduction

In this chapter we will

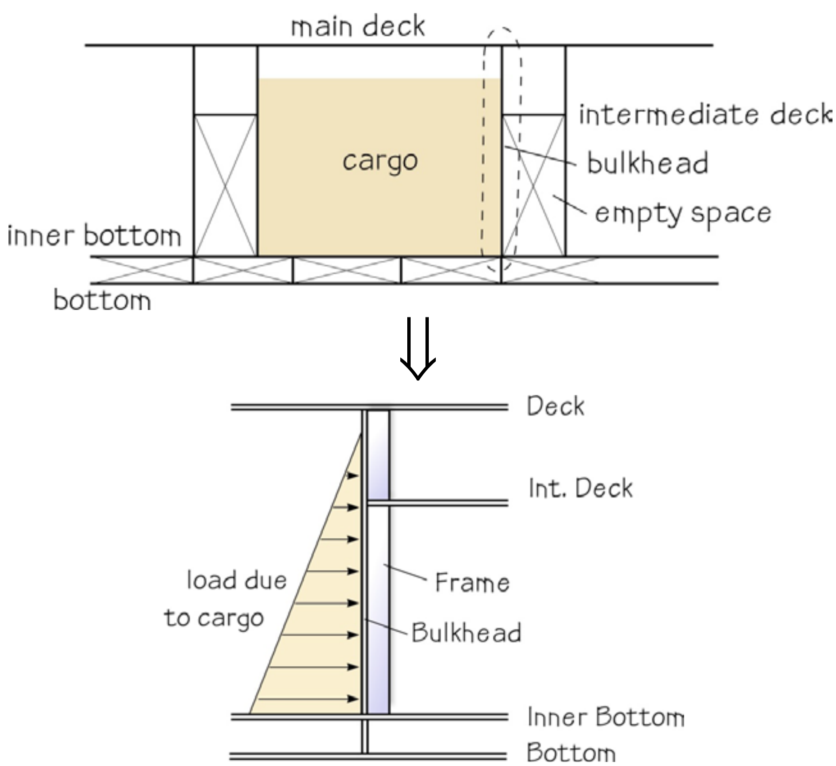
- Review the idea of indeterminate beams and one way to solve them

~~~~~~~~~~~~~~~~~~~~~

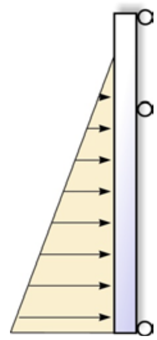
Transverse and Local Strength

Most of the local structure in a ship exists to resist lateral loads.

Example: The sketch below shows a bulkhead between the deck and inner bottom, supported by one intermediate deck. The bulk cargo (liquid or granular) will exert a lateral pressure on the bulkhead.


~~~~~~~~~~~~~~~~~~~~~

We can model the bulkhead frame as a pinned frame over 3 supports, subject to a lateral load;

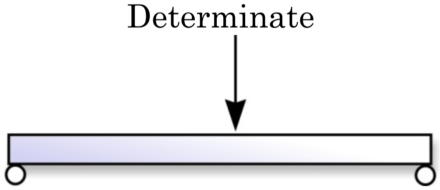


To solve this type of structure we need a method to solve indeterminate structures.

What does indeterminate mean? Determinate structures have a simple set of supports, such that the support reactions can be found from considerations of rigid body equilibrium alone. This means that there are just enough supports for equilibrium to exist. This is normally 3 for 2D structures and normally 6 for 3D structures. The number of supports is also the number of equilibrium conditions that need to be satisfied.

The sketch below illustrates the difference between determinate and indeterminate for a 2D beam.

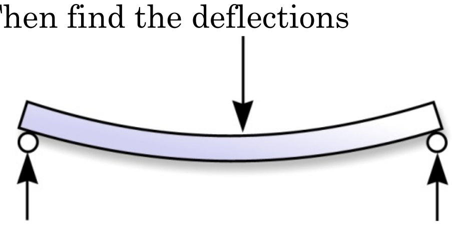
Determinate



① Find the Reactions

⇓

② Then find the deflections



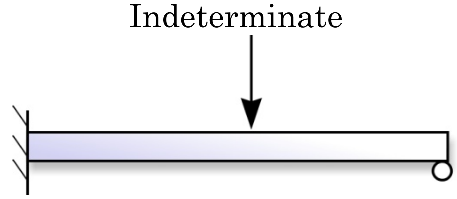
Reactions <u>don't</u> depend on deflections

① Equations for Reactions

⇓

Equations for Deflections

Indeterminate



① ~~Find the Reactions~~

⇓

② ~~Then find the deflections~~

The reactions depend on the deflections

Equations for Reactions

⇑⇓ coupled

Equations for Deflections

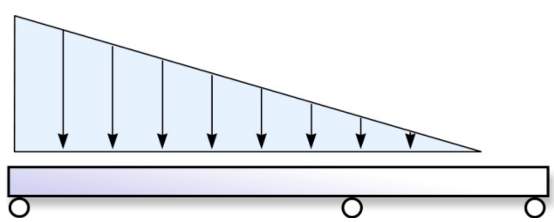
There are two approaches for solving indeterminate systems. Both approaches use the principle of superposition, by dividing the problem into two simpler problems, soling the simpler problems and adding the two solutions.

The first method is called the **Force Method** (also called the Flexibility Method). The idea for the force method is;

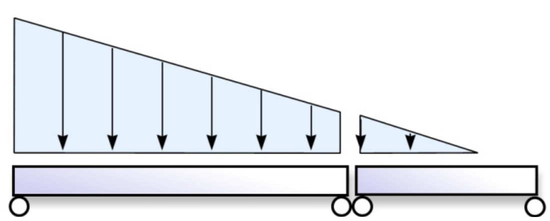
step ① release internal forces* or external reactions until we have one or more determinate systems
step ② solve each determinate system, to find all reactions and deflections. Note all incompatible deflections
step ③ re-solve the determinate structures with only a set of self-balancing internal unit forces* (at internal releases) or unit reaction forces at removed reactions. This solves the system for the internal or external forces removed in ①. Observe the magnitude of incompatible deflections that occur per unit force.
step ③a scale the unit forces to cause the opposite of the incompatible deflections noted in ②
step ④ Add solutions (everything: loads, reactions, deflections…) from ② and ③a. Note that this will result in no incompatible deflections.
*note: forces include both forces and moments

Overview of Force Method

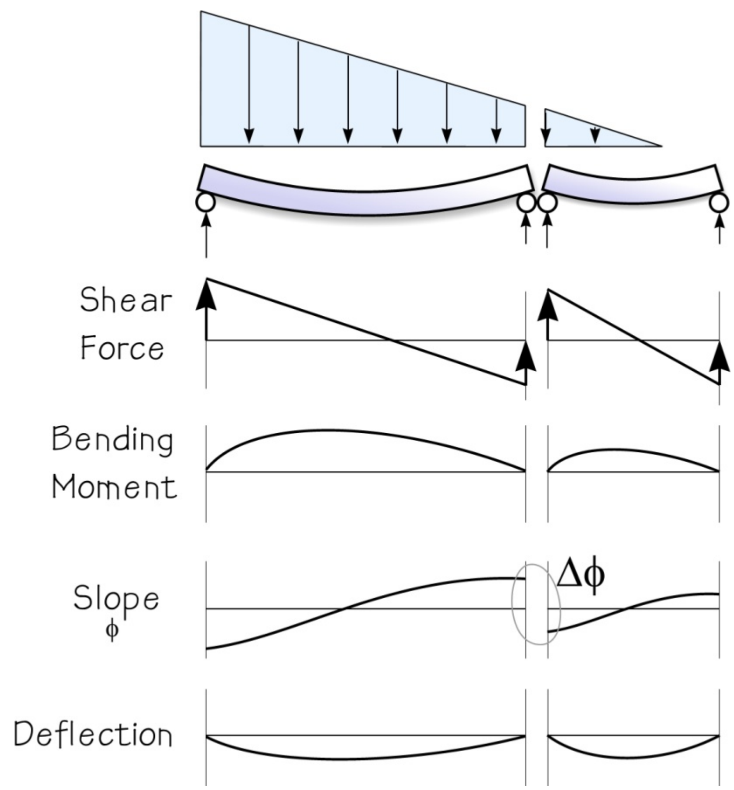
The structure: a beam over multiple supports:



step ① cut the structure to have one or more determinate systems

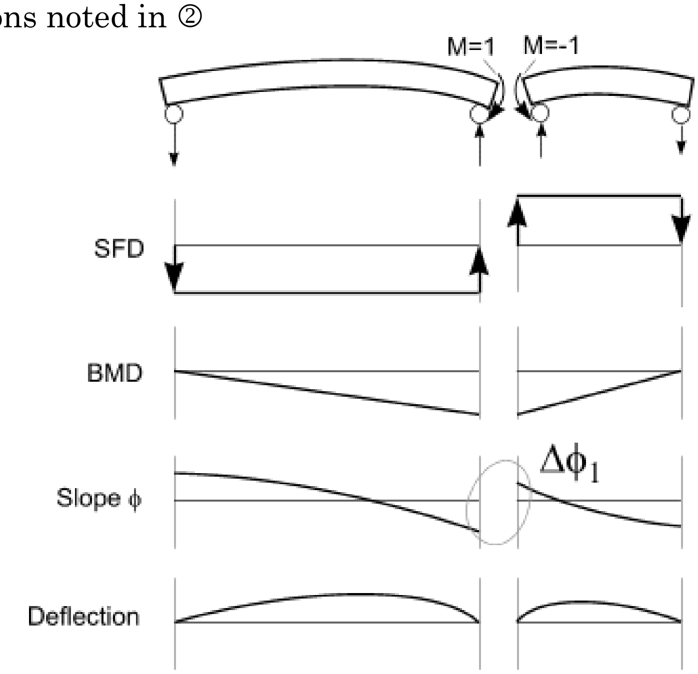


step ② solve each system. Note $\Delta\phi$ – an incompatible deflection.

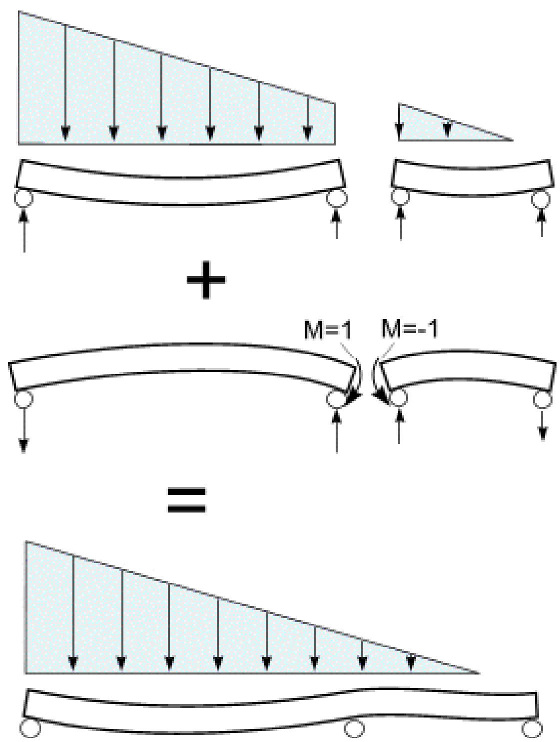


step ③ re-solve the cut structures with self-balancing internal unit forces*

step ③a scale these forces (moments) to cause the opposite of the incompatible deflections noted in ②

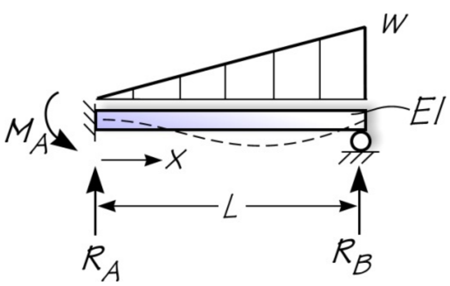


step ④ Add solutions (everything: loads, reactions, deflections…) from ② and ③a. Note that this will result in no incompatible deflections.



Example of the FORCE Method:

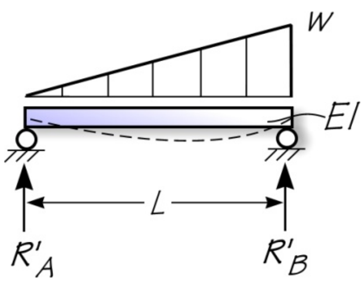
Manual Solution



Problem:

1 - Find M_A, R_A, R_B in terms of *w, EI, L*

2 – Find maximum displacement



Solution:

<u>Part 1</u> – solve with M_A released (denoted '). The reason we do this is because the structure is statically determinate.

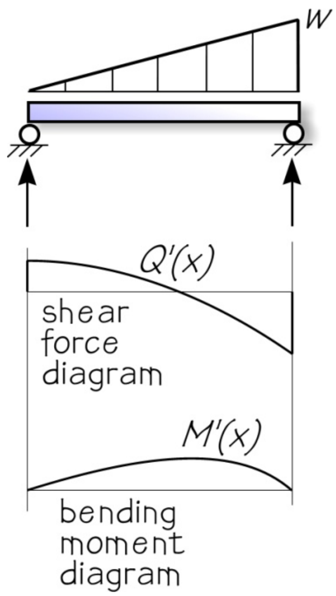
The line load function is:

$$w(x) = \frac{w\,x}{L}$$

Reactions are found from static equilibrium:

$$R'_A = \frac{1}{3}\frac{w\,L}{2} = \frac{w\,L}{6}$$

$$R'_B = \frac{2}{3}\frac{w\,L}{2} = \frac{w\,L}{3}$$



The shear $Q'(x)$ is found by integrating the line load:

$$Q'(x) = R'_A + \int_0^L -w(x)\,dx$$
$$= \frac{wL}{6} - \frac{w}{L}\int_0^L x\,dx = \frac{wL}{6} - \frac{w\,x^2}{2L}$$

The moment $M'(x)$ is found by integrating the shear:

$$M'(x) = M'_A + \int_0^L Q'(x)\,dx$$
$$= \frac{wLx}{6} - \frac{wx^3}{6L}$$

The slope $\phi'(x)$ is found by integrating the moment:

$$\phi'(x) = \phi'_A + \frac{1}{EI}\int_0^L M'(x)\,dx$$
$$= \phi'_A + \frac{1}{EI}\left(\frac{wLx^2}{12} - \frac{wx^4}{24L}\right)$$

And finally the deflection $y'(x)$ is found by integrating the slope:

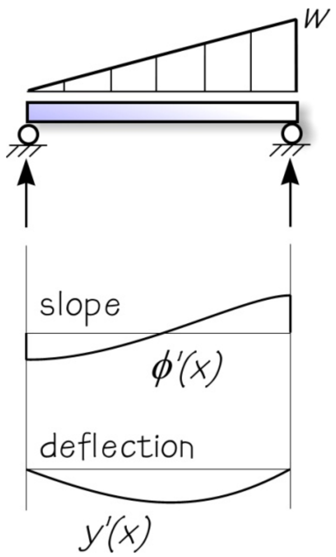
$$y'(x) = y'_A + \int_0^L \phi'(x)\,dx$$
$$= \phi'_A x + \frac{1}{EI}\left(\frac{wLx^3}{36} - \frac{wx^5}{120L}\right)$$

This leaves us with one left unknown to find, ϕ'_A which is the slope at A . We use the boundary condition:

$$y'(L) = 0 = \phi'_A L + \frac{1}{EI}\left(\frac{wL^4}{36} - \frac{wL^4}{120}\right)$$

which is solved to give;

$$\phi'_A = -\frac{7}{360}\frac{wL^3}{EI}$$



Substituting back gives;

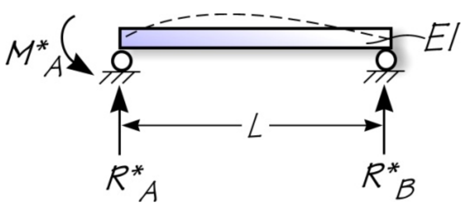
Slope:

$$\phi'(x) = \frac{1}{EI}\left(\frac{-7wL^3}{360} + \frac{wLx^2}{12} - \frac{wx^4}{24L}\right)$$

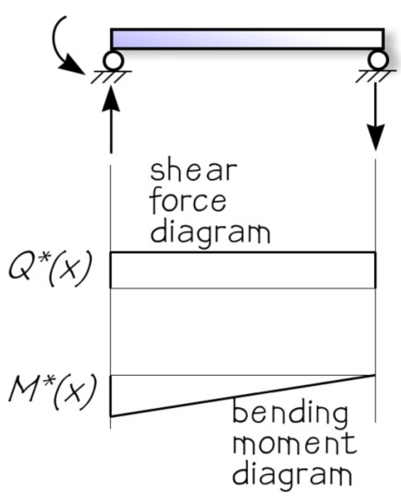
Deflection:

$$y'(x) = \frac{1}{EI}\left(\frac{-7wL^3x}{360} + \frac{wLx^3}{36} - \frac{wx^5}{120L}\right)$$

The gives us the first half of the solution. Now we need to 'correct' the solution, by removing the rotation at A (in Part 2). This is done by applying a moment at A, of just sufficient magnitude to cause $-\phi'_A$. This moment will be the true reaction moment at A. All other responses in Part 2 are added to the Part 1 responses (deflections, shear, moments, etc). Responses can be added because the systems are linear (superposition holds).



<u>Part 2</u> – solve with just M^*_A (the * denotes the corrective solution). M^*_A will cause a rotation opposite to ϕ'_A , which when added to the results of Part 1 will create a 'fixed' condition (no rotation) at A. Initially M^*_A is unknown.



Reactions are found from static equilibrium:

$$\Sigma M_A = 0 \qquad R^*_B L - M^*_A = 0$$
$$R^*_B = \frac{M^*_A}{L}$$

$$\Sigma F_V = 0 \qquad R^*_A + R^*_B = 0$$
$$R^*_A = -\frac{M^*_A}{L}$$

M^*_A is negative, so R^*_B is negative. R^*_A is positive.

The shear $Q^*(x)$ is found by:

$$Q^*(x) = R^*_A + \int_0^L -w(x)\,dx = -\frac{M^*_A}{L}$$

The moment $M^*(x)$ is found by integrating the shear:

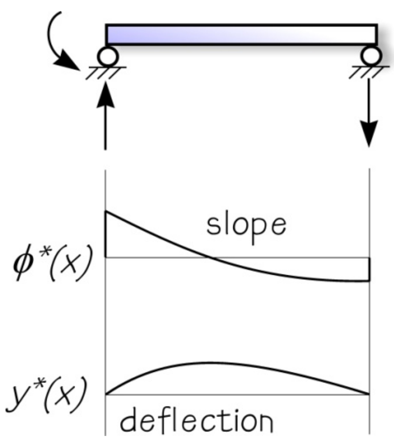
$$M^*(x) = M_A^* + \int_0^L Q^*(x)\,dx$$

$$= M_A^* - \frac{M_A^*\,x}{L}$$

The slope $\phi^*(x)$ is found by integrating the moment:

$$\phi^*(x) = \phi_A^* + \frac{1}{EI}\int_0^L M^*(x)\,dx$$

$$= \phi_A^* + \frac{M_A^*}{EI}\left(x - \frac{x^2}{2L}\right)$$



And finally the deflection y'(x) is found by integrating the slope:

$$y^*(x) = y_A^* + \int_0^L \phi^*(x)\,dx$$

$$= \phi_A^* x + \frac{M_A^*}{EI}\left(\frac{x^2}{2} - \frac{x^3}{6L}\right)$$

To fine ϕ^*_A and M^*_A , we use:

$$\phi_A^* = -\phi_A' = \frac{7}{360}\frac{wL^3}{EI}$$

$$y^*(L) = 0 = \phi_A^* L + \frac{M_A^*}{EI}\left(\frac{L^2}{2} - \frac{L^2}{6}\right)$$

$$0 = \frac{7}{360}\frac{wL^4}{EI} + \frac{M_A^*}{EI}\frac{L^2}{3}$$

$$M_A^* = \frac{-7}{120}wL^2$$

Substituting back gives;

Reactions:

$$R_A^* = \frac{-M_A^*}{L} = \frac{7}{120}wL \qquad (pushes\ up)$$

$$R_B^* = \frac{M_A^*}{L} = \frac{-7}{120}wL \qquad (pulls\ down)$$

Shear:

$$Q^*(x) = \frac{7}{120}wL$$

Moment:

$$M^*(x) = \frac{7}{120} wL(x-L)$$

Slope:

$$\phi^*(x) = \frac{1}{EI}\left(\frac{7}{360} wL^3 + \frac{-7}{120} wL^2 x + \frac{7}{240} wLx^2\right)$$

Deflection:

$$y^*(x) = \frac{1}{EI}\left(\frac{7}{360} wL^3 x - \frac{7}{240}\ wL^2 x^2 + \frac{7}{720} wLx^3\right)$$

This gives us the second half of the solution.

Now we sum the two parts together for the complete solution:

$$R_A = \frac{1}{6} wL + \frac{7}{120} wL = \frac{27}{120} wL$$

$$R_B = \frac{1}{3} wL - \frac{7}{120} wL = \frac{33}{120} wL$$

$$M_A = M_A^* = \frac{-7}{120} wL^2$$

$$\text{check forces} \qquad R_A + R_B = \frac{1}{2} wL \qquad OK$$

$\text{check moment about } A$

$$R_B L - \frac{wL}{2}\frac{2}{3}L \ = M_A$$

$$\frac{33}{120} wL^2 - \frac{40}{120} wL^2 = \ -\frac{7}{120} wL^2 \qquad OK$$

This is the answer to the first question. The maximum deflection is found where the slope is zero. The full expression for the slope is:

$$\phi(x) = \phi'(x) + \phi^*(x)$$

$$\phi'(x) = \frac{1}{EI}\left(\frac{27}{240} wLx^2 - \frac{7}{120} wL^2 x - \frac{wx^4}{24L}\right)$$

We can create a new normalized variable z, which ranges between 0 and 1. This gives us slope in a simpler form:

$$\phi'(x) = \frac{wL^3}{240\,EI}\,(27z^2 - 14z - 10z^4)$$

where

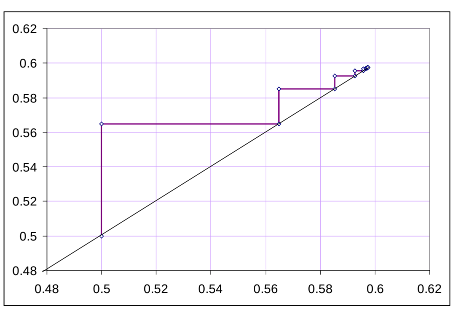
$$z = \frac{x}{L}$$

To find the location of zero slope we set the term inside the brackets above to zero, which can be simplified to:

$$27z - 14 - 10z^3 = 0$$

The solution of this equation will be the location of maximum deflection. One way to solve this (which can be done without derivatives or computers) is to solve the equation for z iteratively. This can be done on any hand-held calculator. We pick one of the z terms (the first term here), and express z as a function of z:

$$z = \frac{14 + 10\,z^3}{27}$$



This iterative equation might be expressed as:

$$z_{i+1} = \frac{14 + 10\,z_i^3}{27}$$

Recall, z ranges from 0 to 1. So any value between 0 and 1 is a possible starting value. We can guess that the maximum deflection will be at z >.5, so we could start with a guess of 0.6. It doesn't really matter, except that the better the initial guess, the quicker the solution will converge. Starting with z =0.6, we iterate to 0.5975 in 7 iterations.

Note: there is another possible iterative version of the z equation;

$$z_{i+1} = \sqrt[3]{\frac{27z_i - 14}{10}}$$

Unfortunately, it won't converge to an answer in the 0-1 range.

The equation for deflection is:

$$y(x) = y'(x) + y^*(x)$$

$$= \frac{w\,L^4}{EI}\left(\frac{27}{720}z^3 - \frac{7}{240}\,z^2 - \frac{z^5}{120}\right)$$

The final step in the solution, is to find y_{max} , which is at $z = 0.5975$:

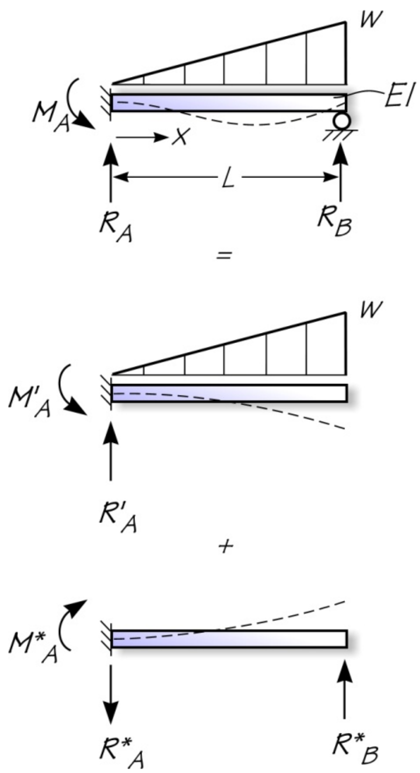
$$y_{max} = \frac{w\,L^4}{EI}\left(\frac{27}{720}0.5975^3 - \frac{7}{240}\,0.5975^2 - \frac{0.5975^5}{120}\right)$$

$$y_{max} = -.00305\,\frac{w\,L^4}{EI}$$

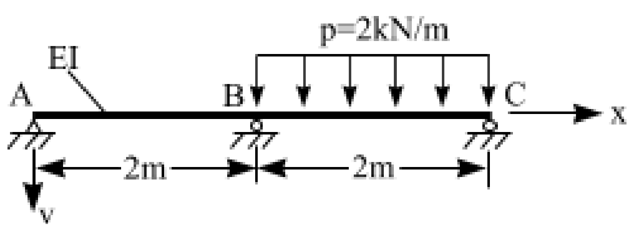
This answer can be checked in Roark, which gives the same answer. This completes the problem.

Topic 10: Problems

10.1. Solve the below by removing the reaction RB (as shown). This creates 'cut' problem that is a cantilever beam.



10.2. Force Method.



a) Sketch 3 alternative approaches to solving this indeterminate problem using the force method. For each approach, you will need two sketches of the auxiliary systems.

b) Using one of the approaches sketched in a) , solve the system to find the reaction at B (in kN)

Topic 11: Indeterminate Beams – Displacement Method

Cruise Ship in Adriatic

Introduction

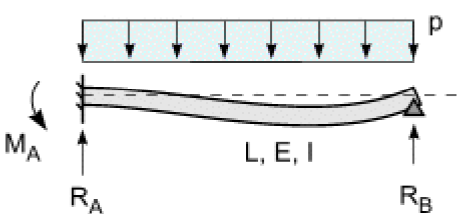
In this chapter we will

- introduce the displacement method used to solve structural problems
- introduce the standard stiffness components for a beam in 2D and 3D

~~~~~~~~~~~~~~~~~~~~

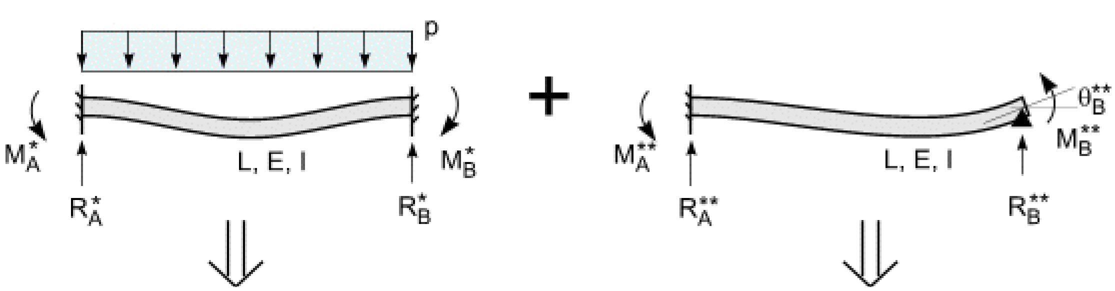
Indeterminate Problem

We start by considering the indeterminate beam as shown below. This could be described as a fixed-pinned beam or a cantilever with a pinned end.



Find M_A R_A and R_B

To solve this problem with the displacement (stiffness) method we create two sub-problems, each simpler than the whole problem. Rather than removing a support (removing a force or moment), we remove a movement (i.e we completely fix the structure). This becomes the problem marked * below. To the * problem, we add a second problem, the ** problem, that fixes any errors that we created with the * problem. In this case we have a moment M_B* that should not exist, while we have a θ_B* that should not be zero. So, in the ** problem, we impose θ_B**, (and <u>only</u> a θ_B**) sufficiently large to cause a moment M_B** that is equal and opposite to M_B*.


~~~~~~~~~~~~~~~~~~~~

- fixed-fixed beam
- known solution
- $M_A{}^* = -M_B{}^* = pL^2/12$
- $R_A{}^* = R_B{}^* = pL/2$

- applied moment at pin
- the moments and forces can be found from the "stiffness" terms, as shown below:
- $M_B{}^{**} = \theta_B{}^{**}\ 4EI/L$
- $M_A{}^{**} = \theta_B{}^{**}\ 2EI/L$
- $R_B{}^{**} = -\ \theta_B{}^{**}\ 6EI/L^2$
- $R_A{}^{**} = \theta_B{}^{**}\ 6EI/L^2$

To solve the problem we use;

$$M_B{}^{**} + M_B{}^{*} = 0$$

which gives;

$$\theta_B{}^{**}\ 4EI/L - pL^2/12 = 0$$

from this we can solve for $\theta_B{}^{**}$;

$$\theta_B{}^{**} = pL^3/(48\ EI) = 0$$

from this we can find all other ** terms;

$$M_A{}^{**} = pL^3/(48\ EI)\ 2EI/L = 1/24\ pL^2$$
$$R_B{}^{**} = -\ pL^3/(48\ EI)\ 6EI/L^2 = -\ 1/8\ pL$$
$$R_A{}^{**} = pL^3/(48\ EI)\ 6EI/L^2 = 1/8\ pL$$

from this we can find the reactions;

$$M_A = M_A{}^* + M_A{}^{**} = pL^2/12 + pL^2/24 = 1/8\ pL^2$$
$$R_B = R_B{}^* + R_B{}^{**} = -\ pL/8 + pL/2 = 3/8\ pL$$
$$R_A = R_A{}^* + R_A{}^{**} = pL/8 + pL/2 = 5/8\ pL$$

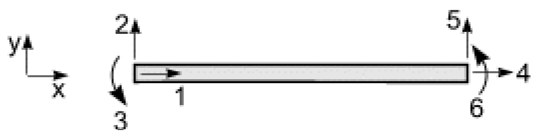
The terms used to find $M_B{}^{**}$, $M_A{}^{**}$, $R_B{}^{**}$ and $R_A{}^{**}$ are called stiffness terms because the are an 'action per unit movement', such as a force per unit displacement or moment per unit rotation. They can also be a kind of 'cross stiffness' such as a force per unit rotation or a moment per unit displacement. In the case of the example above, with the equations;

$$M_B{}^{**} = \theta_B{}^{**}\ 4EI/L$$
$$M_A{}^{**} = \theta_B{}^{**}\ 2EI/L$$
$$R_B{}^{**} = -\ \theta_B{}^{**}\ 6EI/L^2$$
$$R_A{}^{**} = \theta_B{}^{**}\ 6EI/L^2$$

The stiffness terms $4EI/L$, $2EI/L$, $-6EI/L^2$ and $6EI/L^2$ are forces and moment 'per unit rotation'. We will define these stiffness terms in the next section.

Stiffness Terms

When using the stiffness method, we always need to find a set of forces and moments that occur when we impose a movement at a support. The movement will correct a situation that involved the suppression of a movement at a support. In our case here, the structure is a beam, and the supports are at the ends of the beam. The supports prevent the ends of the beam from moving. There are 3 possible movements at a support for a 2D problem, and 6 for a 3D problem. Because of this we will define a standard set of 'degrees of freedom' for a beam. A 'degree of freedom' can have either a force or displacement, or a rotation or moment. The standard 2D degrees of freedom for a beam are shown below;



2D beam = 6 degrees of freedom

The degrees of freedom follow the Cartesian system, with the right-hand rule. These are essentially x, y, rotation (called rz). In general, to impose a unit movement in one (and only one) of these degrees of freedom, we need to also impose a set of forces/moments, The forces/moments must be in equilibrium. These forces/moments will be 'stiffnesses'.

The mechanics are linear. This means that the set of forces/moments corresponding to each movement can be added to those of any other movement. A general solution for any set of movements of the degrees of freedom can be found by superposition.

For now we will just consider the 2D case and derive the stiffness terms. There are 6 degrees of freedom. For each degree of freedom, there are potentially 6 forces or moments that develop. This means that there are a total of 36 stiffness terms. Any single term would be labeled k_{ij}, meaning the force/moment at i due to a displacement/rotation at j. For example;

k_{11} = force at 1 due to unit displacement at 1
k_{41} = moment at 4 due to unit displacement at 1
k_{26} = force at 2 due to unit rotation at 6

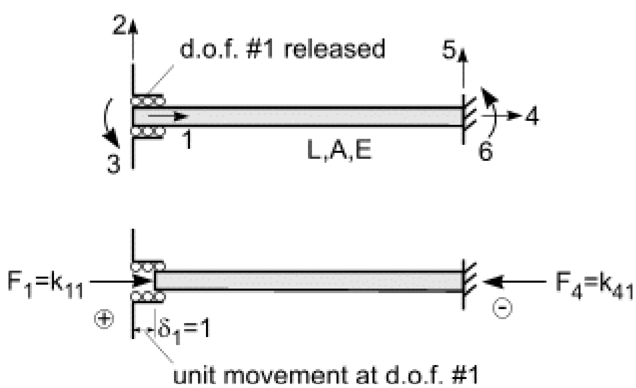
All the terms can be written in matrix form as;

$$K = \begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} & k_{15} & k_{16} \\ k_{21} & k_{22} & k_{23} & k_{24} & k_{25} & k_{26} \\ k_{31} & k_{32} & k_{33} & k_{34} & k_{35} & k_{36} \\ k_{41} & k_{42} & k_{43} & k_{44} & k_{45} & k_{46} \\ k_{51} & k_{52} & k_{53} & k_{54} & k_{55} & k_{56} \\ k_{61} & k_{62} & k_{63} & k_{64} & k_{65} & k_{66} \end{bmatrix}$$

We will now derive these 36 terms. Luckily they are not all unique.

Axial Terms

The axial terms are found by asking what set of forces is required to create a unit displacement at d.o.f. #1 (and only #1);



For axial compression, the deflection under load is;

$$\delta_1 = \frac{F_1 L}{AE} = 1 \Rightarrow \frac{F_1}{\delta_1} = k_{11} = \frac{AE}{L}$$

the force at d.o.f. #4 is equal and opposite to the force at #1;

$$F_4 = -F_1 \Rightarrow \frac{F_4}{\delta_1} = k_{41} = \frac{-AE}{L}$$

There are no other forces (at #2, 3, 5, 6), so we have;

$$\frac{F_2}{\delta_1} = k_{21} = 0 \text{ and } k_{31} = k_{51} = k_{61} = 0$$

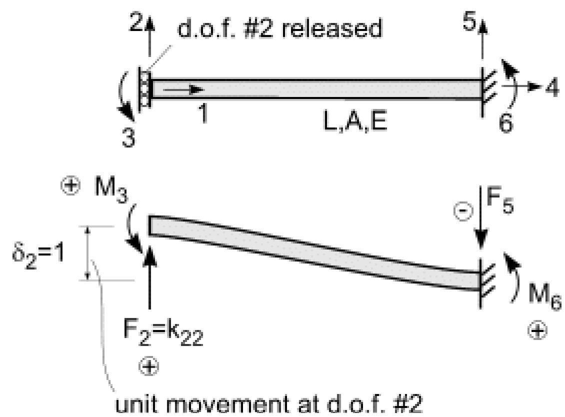
A displacement at 4 would require a similar set of forces, so that we can also write;

$$k_{44} = \frac{AE}{L},\ k_{14} = \frac{-AE}{L},\ k_{24} = k_{34} = k_{54} = k_{64} = 0$$

This has given us 12 terms, 1/3 of all the terms we need. Next we will find the terms for the #2 and #5 direction.

Shear Terms

The shear terms are found from the set of forces is required to create a unit displacement at d.o.f. #2 (and only #2);



For shear of this type, the deflection is;

$$\delta_2 = \frac{F_2 L^3}{12EI} = 1 \Rightarrow \frac{F_2}{\delta_2} = k_{22} = \frac{12EI}{L^3}$$

Note: to derive this easily, think of the beam as two cantilevers, each L/2 long, with a point load at the end, equal to F_2.

The force at d.o.f. #5 is equal and opposite to the force at #2;

$$F_5 = -F_2 \Rightarrow \frac{F_5}{\delta_2} = k_{52} = \frac{-12EI}{L^3}$$

Following from the double cantilever notion, the end moments (M_3, M_6) are ;

$$M_3 = M_6 = F_2 \frac{L}{2} \Rightarrow k_{32} = k_{62} = \frac{6EI}{L^2}$$

There are no axial forces, so;

$$k_{12} = k_{42} = 0$$

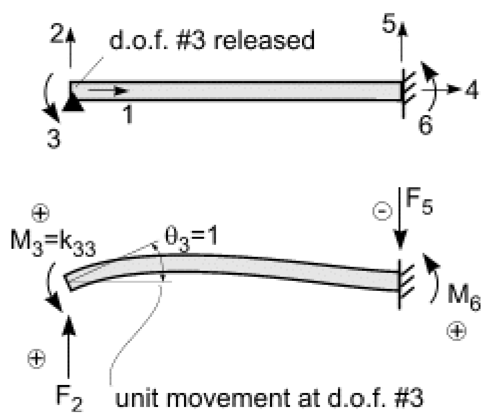
A displacement at #5 require a similar set of forces, so that we can also write;

$$k_{55} = \frac{12EI}{L^3}, \quad k_{25} = \frac{-12EI}{L^3}, \quad k_{35} = k_{65} = \frac{-6EI}{L^2}, \quad k_{15} = k_{45} = 0$$

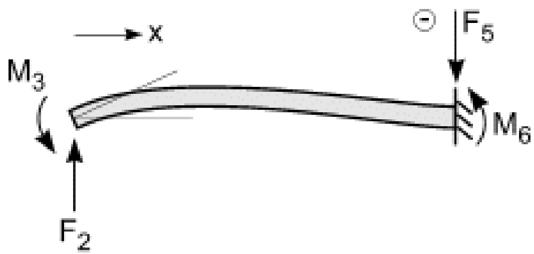
This has given us 12 more terms, for 2/3 of all the terms we need. Next we will find the terms for the #3 and #6 direction.

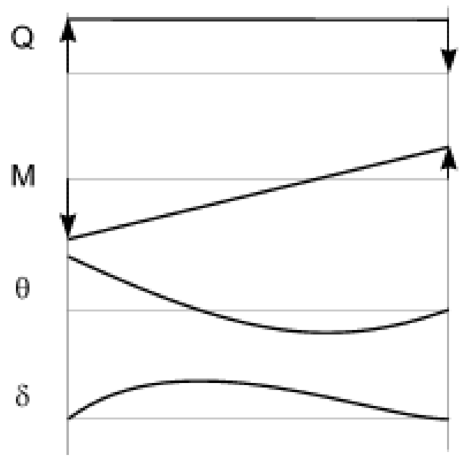
Rotary Terms

The rotary terms are found from the set of forces/moments required to create a unit rotation at d.o.f. #3 (and only #3);



For illustration and to find these stiffness terms we will solve the system. We can draw the shear force, moment, slope and deflection diagrams as below;





$$Q(x) = F_2$$

$$M(x) = -M_3 + F_2 \cdot x$$

$$\theta(x) = \theta_3 + \frac{1}{EI}\left(-M_3 x + F_2 \cdot \frac{x^2}{2}\right)$$

$$\delta(x) = \delta_2 + \theta_3 x + \frac{1}{EI}\left(-M_3 \frac{x^2}{2} + F_2 \cdot \frac{x^3}{6}\right)$$

We can use the boundary conditions (θ_3=1, δ_2=0, δ(L)=0, θ(L)=0) to find M_3 and F_2.

$$\theta(L) = 0 = 1 + \frac{1}{EI}\left(-M_3 L + F_2 \cdot \frac{L^2}{2}\right)$$

$$\delta(L) = 0 = 0 + L + \frac{1}{EI}\left(-M_3 \frac{L^2}{2} + F_2 \cdot \frac{L^3}{6}\right)$$

These two equations can be solved to get;

$$M_3 = \frac{4EI}{L},\ F_2 = \frac{6EI}{L^2}$$

from these we can find;

$$M_6 = \frac{2EI}{L},\ F_5 = \frac{-6EI}{L^2}$$

This allows to find the stiffness terms;

$$k_{33} = \frac{4EI}{L},\ k_{63} = \frac{2EI}{L}\ k_{23} = \frac{6EI}{L^2},\ k_{53} = \frac{-6EI}{L^2}\ ,\ k_{13} = k_{43} = 0$$

A rotation at #6 require a similar set of forces, so that we can also write;

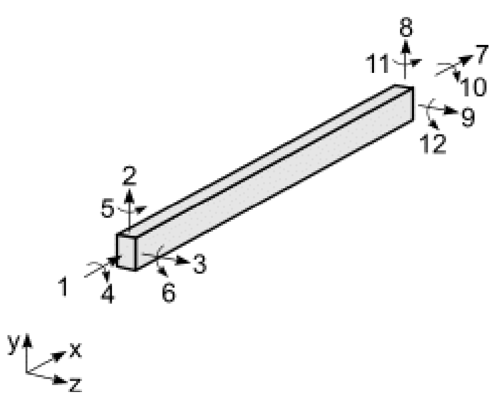
$$k_{66} = \frac{4EI}{L}, \quad k_{36} = \frac{2EI}{L} \quad k_{26} = \frac{6EI}{L^2}, \quad k_{56} = \frac{-6EI}{L^2} \quad , \quad k_{16} = k_{46} = 0$$

We can collect all these terms in the matrix;

$$K = \begin{bmatrix} \frac{AE}{L} & 0 & 0 & \frac{-AE}{L} & 0 & 0 \\ 0 & \frac{12EI}{L^3} & \frac{6EI}{L^2} & 0 & \frac{-12EI}{L^3} & \frac{6EI}{L^2} \\ 0 & \frac{6EI}{L^2} & \frac{4EI}{L} & 0 & \frac{-6EI}{L^2} & \frac{2EI}{L} \\ \frac{-AE}{L} & 0 & 0 & \frac{AE}{L} & 0 & 0 \\ 0 & \frac{-12EI}{L^3} & \frac{-6EI}{L^2} & 0 & \frac{12EI}{L^3} & \frac{-6EI}{L^2} \\ 0 & \frac{6EI}{L^2} & \frac{2EI}{L} & 0 & \frac{-6EI}{L^2} & \frac{4EI}{L} \end{bmatrix}$$

Note that the matrix is symmetrical. This means that terms such as k_{35} (moment at #3 due to displacement at #5) is equal to k_{35} (force at #5 due to rotation at #3). This may seem quite odd that these two items would be equal. We will examine this in the next Chapter.

The standard 3D degrees of freedom for a beam are shown below;

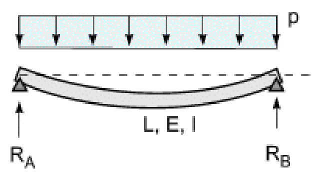


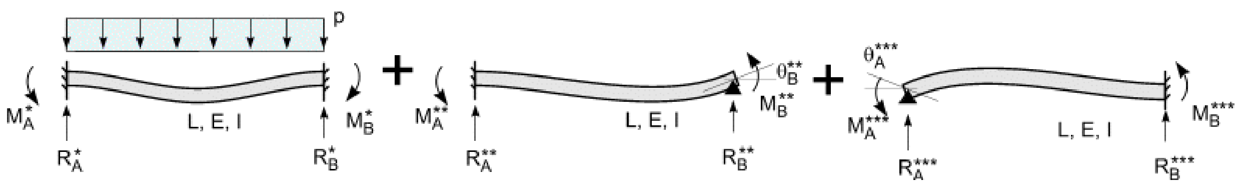
3D beam = 12 degrees of freedom

The K matrix for a 3D beam is a 12x12 (144 terms).

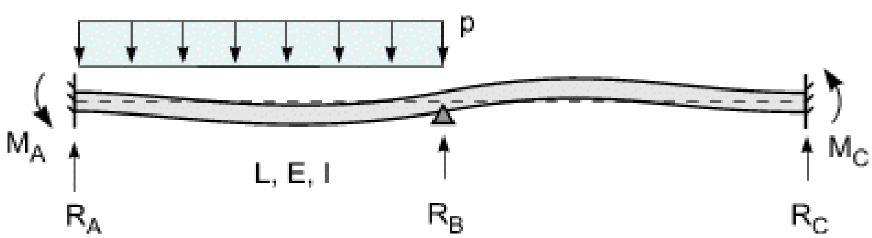
Topic 11: Problems

11.1. Solve the pinned-pinned beam by using the displacement method as sketched below. The solution for the fixed-fixed beam is the same as above. Then it is necessary to show that $M_B{}^{*}+M_B{}^{**}+M_B{}^{***}=0$ and $M_A{}^{*}+M_A{}^{**}+M_A{}^{***}=0$. <u>Note</u>: $M_A{}^{**} = ½\ M_B{}^{**}$, and $M_B{}^{***} = ½\ M_A{}^{***}$.

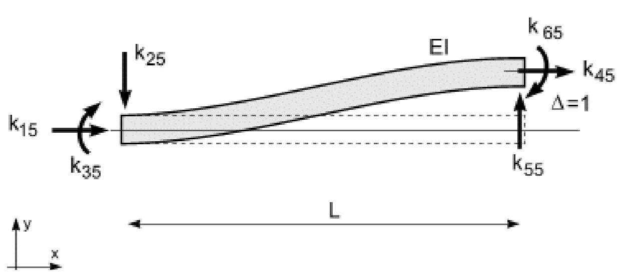


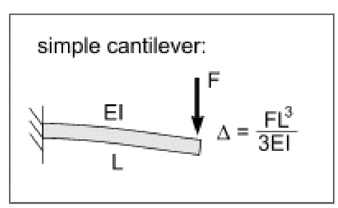


11.2. Describe how you would solve the beam shown below by using the displacement method.

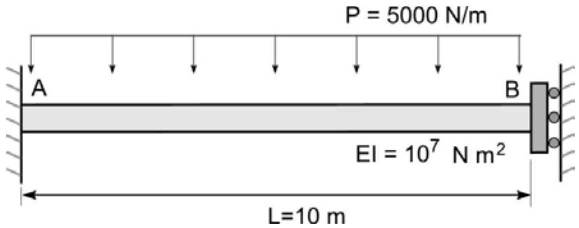


11.3. For the simple beam shown below, derive the shear stiffness terms (i.e k_{15} to k_{65})





11.4. Solve the beam shown below using the stiffness method. Find the reactions at A and B, and the deflection at B.



ANS: MA= 166667 N-m, MB = 83333N-m ΔB = -.2082m

11.5. Stiffness method .

sketch a 2D beam and show the degrees of freedom.

Describe the meaning of the terms (any, all) in the 6x6 stiffness matrix for a 2D beam, and give 2 examples.

11.6. Explain the difference between the "Force" method, and the "Displacement" method.

11.7. In the stiffness method for a 2D beam, the standard value for the k22 stiffness term is;

$$\boldsymbol{k_{22} = 12\frac{EI}{L^3}}$$

Derive this equation (Table 1 in appendix may be useful).

Topic 12: Energy Methods in Structural Analysis

Coliseum

Introduction

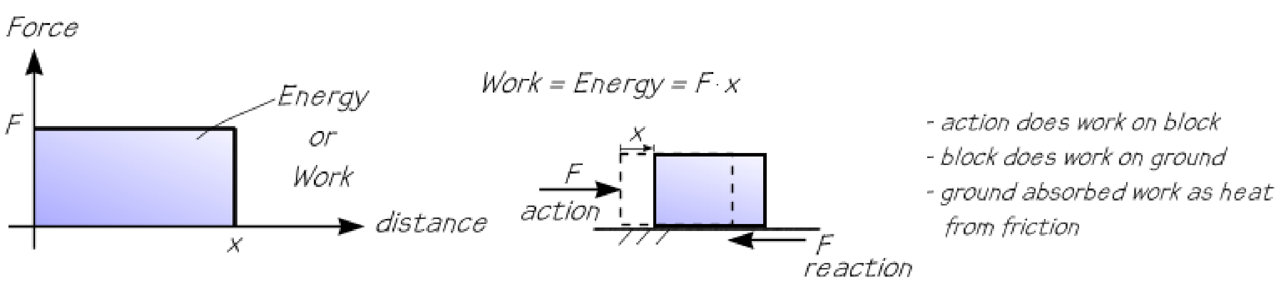
In this chapter we will

- Discuss application of energy methods in structural analysis
- Show how conservation of energy conservation to the symmetry of structural stiffness terms

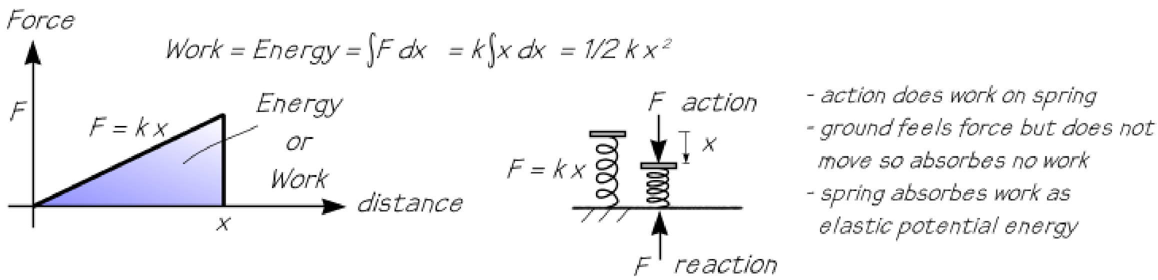
~~~~~~~~~~~~~~~~~~~~~~

Energy Methods

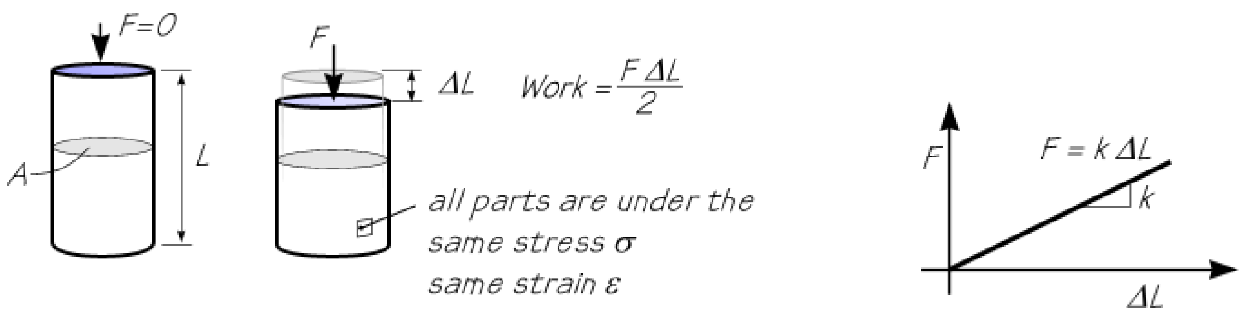
Structural analysis is concerned with forces, deflections, stresses and strains. All these involve energy. An analysis of energy can be a way to simplify structural analysis. Energy is a scalar, and must be conserved, somehow. In some cases the mechanical work done by a force is converted to heat by friction:


~~~~~~~~~~~~~~~~~~~~~~

In some cases the mechanical work done by a force is converted to elastic potential energy in a spring. Potential energy (in a spring or in a gravitational field) can later be recovered:



Consider a body subject to a simple axial load:



with

$$\sigma = \text{stress} = F / A \qquad F = \sigma A$$

$$\varepsilon = \text{strain} = \Delta L / L \qquad \Delta L = \varepsilon L$$

we can write:

$$\text{Work} = \text{Energy} = \frac{\sigma A \varepsilon L}{2} = \frac{\sigma \varepsilon \, Vol}{2}$$

with $\sigma = \varepsilon E$

$\varepsilon = \sigma / E$

we can write:

$$\text{Work} = \text{Energy} = \frac{1}{2}\int_{Vol} \sigma \varepsilon \, dvol$$

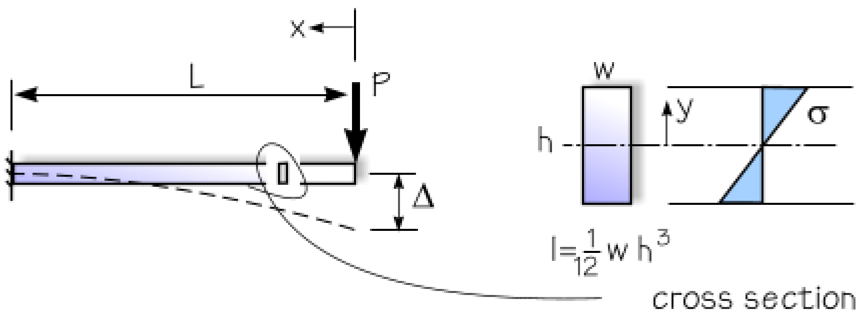
$$= \frac{1}{2E}\int_{Vol} \sigma^2 dvol$$

$$= \frac{E}{2}\int_{Vol} \varepsilon^2 dvol$$

The above is correct for situations where axial stresses dominate, as in column compression or simple beam bending. This does not take shear strain energy into account.

Example: derive formula for Cantilever beam deflection using energy methods.

Consider a simple cantilever with rectangular cross section.



Start with Energy Balance equation:

External Work (EW) done by the applied load P is balanced by the elastic potential energy (EPE) stored in the beam;

$$EW = EPE$$

$$1/2\, P\Delta = \frac{1}{2E}\int_{Vol} \sigma^2\, dvol$$

In this case we assume that the stress is the result of bending and we find the stress from;

$$\sigma = \frac{My}{I}$$

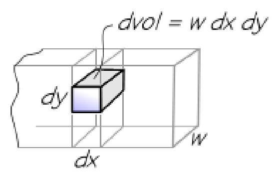
and

$$M = Px$$

which lets us write;

$$\sigma^2 = \frac{P^2x^2y^2}{I^2}$$

$$\Delta = \frac{P}{EI^2}\int_{Vol} x^2y^2\, dvol$$



We can re-write *dvol* as *w dx dy and use* :

$$= \frac{P}{EI^2} \int_L x^2 dx \int_h w\, y^2 dy$$

The last part of the above equation is the moment of inertia:

$$\int_h w\, y^2 dy = I$$

This simplifies the problem to:

$$\Delta = \frac{P}{EI} \int_L x^2\, dx$$

$$\Delta = \frac{P}{EI} \frac{x^3}{3} \Bigg]_0^L$$

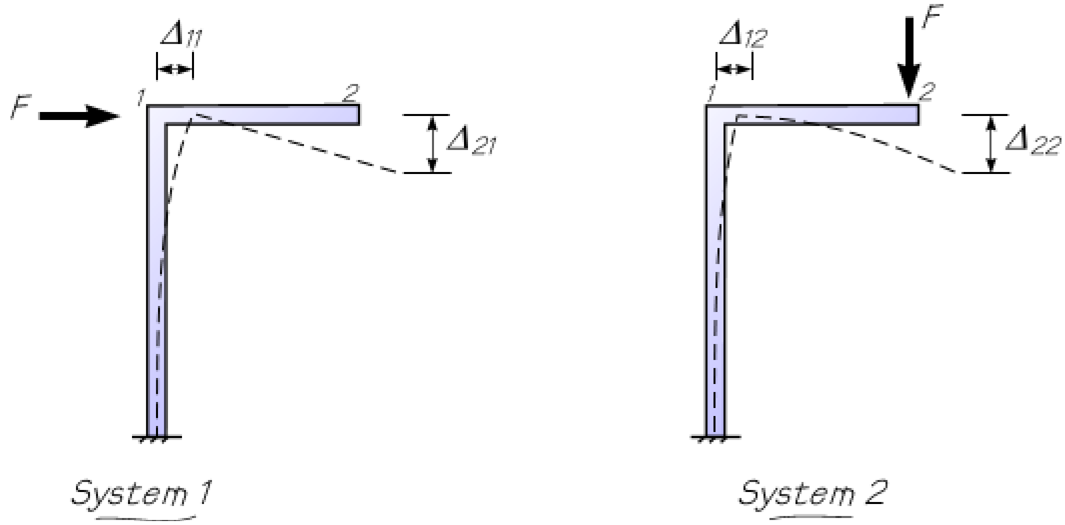
Which gives the final and correct answer:

$$\Delta = \frac{PL^3}{3EI}$$

Betti-Maxwell Reciprocal Theorem

The Betti-Maxwell theorem states that for any linear elastic body (also called a Hookean body), that the movement at a d.o.f. A, caused by the application of a force/moment F at a d.o.f. B, is exactly the same as the movement at a d.o.f. B, caused by the application of a force/moment F at a d.o.f. A. In the sketch below, Δ_{ij} refers to the movement at i due to the application of a force at . So we can write the Betti-Maxwell theorm as;

$$F\, \Delta_{12} = F\, \Delta_{21}$$



Proof:
As a linear system, superposition will hold. The structure will assume the same final position regardless of the order of application of the forces. This means that the same stored elastic energy will exist in either case. These are 'conservative' systems, meaning that all work done by the loads is converted to elastic potential energy (and is 'conserved' to be recovered later). We will apply F to the structure in two places, and compare the work done when we change the order in which we apply the forces.

When F is applied at both *1* and *2,* the total deflection at 1 and 2 will be;

$$\Delta_1 = \Delta_{11} + \Delta_{12}$$

$$\Delta_2 = \Delta_{21} + \Delta_{22}$$

If we imagine applying F at 1 first, and then at 2, the work done will be;

$$Work\ Done = \frac{F\ \Delta_{11}}{2} + \frac{F\ \Delta_{22}}{2} + F\ \Delta_{12}$$

If we imagine applying F at 2 first, and then at 1, the work done will be;

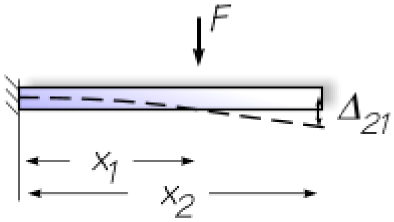
$$Work\ Done = \frac{F\ \Delta_{22}}{2} + \frac{F\ \Delta_{11}}{2} + F\ \Delta_{21}$$

The work done will be the same, so;

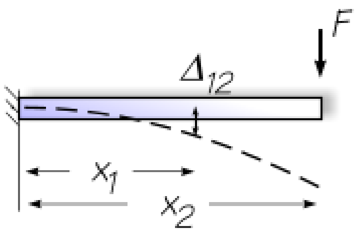
$$F\ \Delta_{12} = F\ \Delta_{21}$$

Hence Betti-Maxwell is proven.

Example 1 of Betti-Maxwell

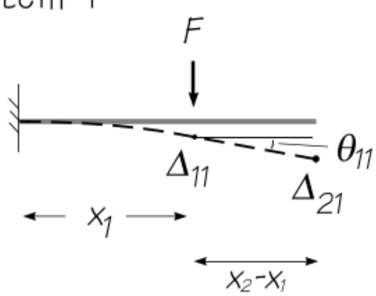


For a simple cantilever, the deflection at x_2 caused by a force F at x_1 should be the same as the deflection at x_1 when F is applied at x_2:



Solution:
The beam deflection tables (see Appendix) can be used to find Δ_{12} and Δ_{21}.
To find Δ_{21} we first find the deflection at x_1. The beam to the right of x_1 has no shear or bending. Consequently it is perfectly straight. It slopes downward at the same angle as the slop at x_1, which is θ_{11}. The addition deflection past is just equal to the slope angle times the distance. The total deflection at x_2 found as follows:

System 1



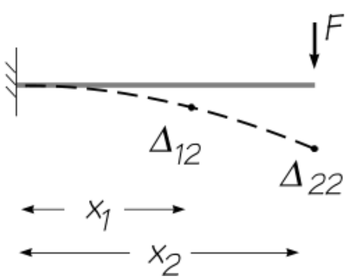
$$\Delta_{11} = \frac{F\,x_1^3}{3\,EI} \qquad \theta_{11} = \frac{F\,x_1^2}{2\,EI}$$

$$\Delta_{21} = \Delta_{11} + \theta_{11} \cdot (x_2 - x_1)$$

$$= \frac{F\,x_1^3}{3\,EI} + \frac{F\,x_1^2 x_2}{2\,EI} - \frac{F\,x_1^3}{2\,EI}$$

$$= \frac{F\,x_1^2}{6\,EI}(3\,x_2 - x_1)$$

System 2

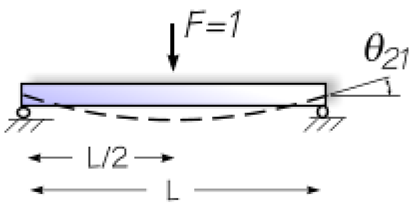


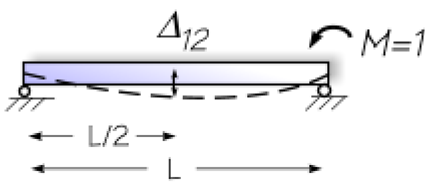
To find Δ_{12} we use the general equation for the deflections in a cantilever of length x_2 and solve for the deflection at x_1.

$$\Delta_{12} = \frac{F\,x_1^2}{6\,EI}(3\,x_2 - x_1)$$

The two results are identical, as Betti-Maxwell predicted.

Example 2 of Betti-Maxwell





For a simply supported beam, the rotation at the right hand end caused by a unit vertical force F in the center should be the same as the vertical deflection at the center caused by a unit moment at the right hand end :

Solution:
The beam deflection tables (see Appendix) can be used to find Δ_{12} and θ_{21}.
The rotation θ_{21} *is* as follows:

$$\theta_{21} = \frac{F\,L^2}{16\,EI}$$

$$= \frac{L^2}{16\,EI}$$

To find Δ_{12} we use the general equation for the deflections in a simply supported beam with an end moment and solve for the deflection at $L/2$.

$$\Delta_{12} = \frac{M\,x}{6\,EI\,L}\,(L^2 - x^2)$$

$$= \frac{L/2}{6\,EI\,L}\,(L^2 - L^2/4)$$

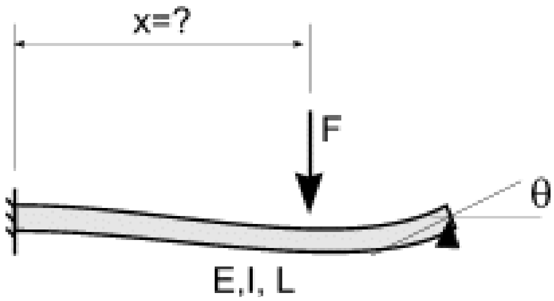
$$= \frac{L^2}{12\,EI}\,(1 - 1/4)$$

$$= \frac{L^2}{16\,EI}$$

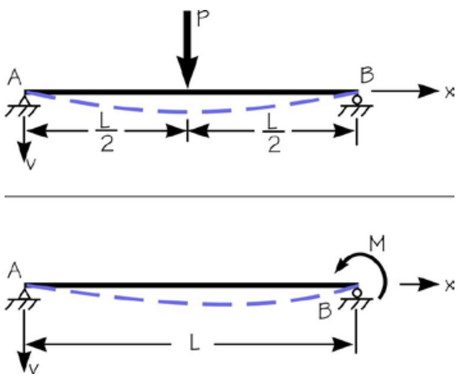
The two results are identical, as Betti-Maxwell predicted.

Topic 12: Problems

12.1. Find the location of the force F so that ☐ is a maximu of Betti-Maxwell.



12.2. Illustrate the Betti-Maxwell theorem using the beam load cases shown below. Use the deflection table on pg 8 at the end of the paper.



Topic 13: The Moment Distribution Method

Venice

Introduction

In this chapter we will describe the moment distribution method for solving indeterminate beams

~~~~~~

Overview

The moment distribution method is a type of displacement (stiffness) method because it makes use of the stiffness terms we derived earlier. It is particularly useful for solving problems involving beams over multiple supports, and frames with moment connections. It is what can be termed a 'relaxation' method. This refers to the iterative way that errors are 'relaxed'. The method can be solved manually on paper with a simple calculator, and was once the dominant method used in professional practice. These days it can easily be solved with a spreadsheet, but is seldom used professionally. Its current value is in helping students develop an understanding of structural behavior. The essence of structures is the interconnected behavior of structural elements. The moment distribution method is all about the way neighboring elements interact.

Hardy Cross (Wikipedia)

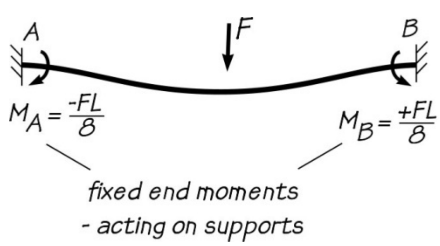
The method was developed by Prof. Hardy Cross in the 1920s and 30s. Cross studied at MIT and Harvard, taught at Brown, Illinois and Yale and consulted extensively.

Prof. Hardy Cross described his procedure as follows:
~~~~~~

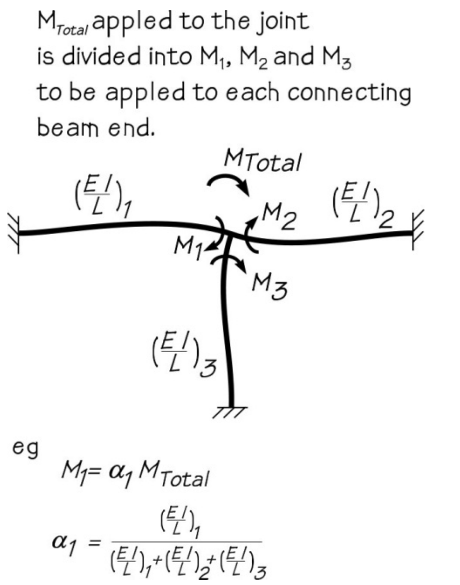
" The method of moment distribution is this:
1. *Imagine all joints in the structure held so that they cannot rotate and compute the moments at the ends of the members for this condition;*
2. *At each joint distribute the unbalanced fixed-end moment among the connecting members in proportion to the constant for each member defined as "stiffness";*
3. *Multiply the moment distributed to each member at a joint by the carry-over factor at the end of the member and set this product at the other end of the member;*
4. *Distribute these moments just "carried over";*
5. *Repeat the process until the moments to be carried over are small enough to be neglected; and*
6. *Add all moments - fixed-end moments, distributed moments, moments carried over - at each end of each member to obtain the true moment at the end."*

Description of Method

The moment distribution method is a way to solve indeterminate structures comprised of beams. The method works for continuous beams over multiple supports and for frames. In its basic form it does not consider joint translation. All joints are only assumed to rotate, as would occur at a pin or roller support, or at a frame connection (beams to column) where sway is prevented. Subsidence of a support can easily be handled. An extended version can treat sway of a frame system.



Fixed End Moments – FEM : To start the procedure, all joint are considered fixed and all fixed-end moments are calculated. One example of fixed end moments is shown below for a beam with a central point force. The moments are expressed as <u>true moments</u> acting <u>on the supports</u>. This is an <u>important point</u>. Note that both end moments in the sketch cause concave downward bending, and would this have the same sign in a bending moment diagram. But here they have opposite true senses (clockwise on left and counterclockwise on right) and so have opposite signs. And we keep tract of the moments acting from the beam, not the reactions by the support.



Moment Distribution factors - α: At each joint where two or more beams connect, each beam provides part of the rotary stiffness. When an external moment is applied to the joint, it rotates as a unit, with each of the connecting beams resisting part of the total moment. The portion of the total is called the moment distribution factor - α. For each beam the moment will be :

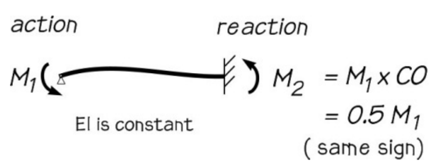
$$M_i = k_{33_i}\,\theta_{joint}$$

where k_{33} is beam end rotation stiffness (see Ch10);

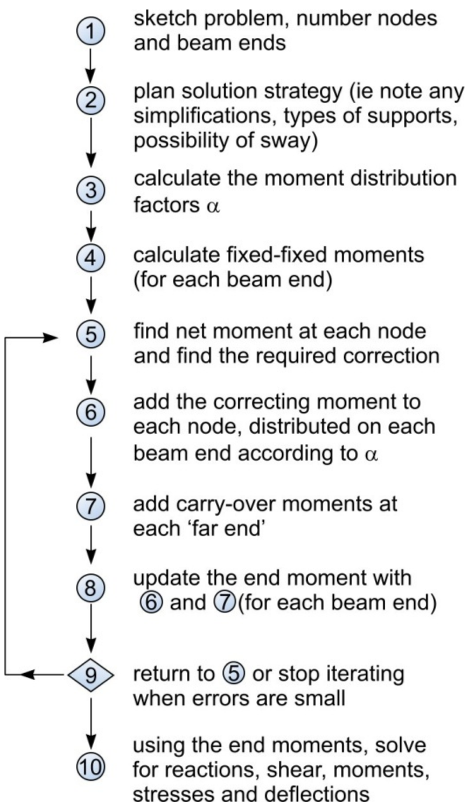
$$k_{33_i} = \frac{4\,EI_i}{L_i}\ \textit{for beam } i$$

The moment distribution factor is;

$$\alpha_i = \frac{M_i}{M_{total}} = \frac{k_{33_i}\,\theta_{joint}}{\theta_{joint}\sum_{all} k_{33_j}} = \frac{(EI/L)_i}{\sum_{all}(EI/L)}$$

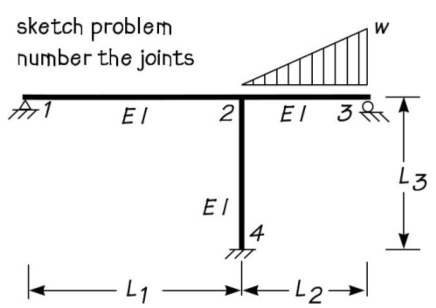


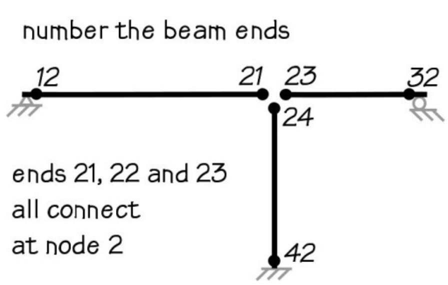
Carry-Over factors - CO: As we saw earlier, when one end of a bean is rotated, the other end of the beam experiences a moment as well. This is the k_{63} moment. In other words, when a moment is applied to one end of a beam, and the far end is fixed, that other end experiences a moment. Because k_{63} is half of k_{33}, the far end moment is always half of the near end moment. Therefore the carry over factor is always 0.5.



Steps in the Moment-Distribution Method

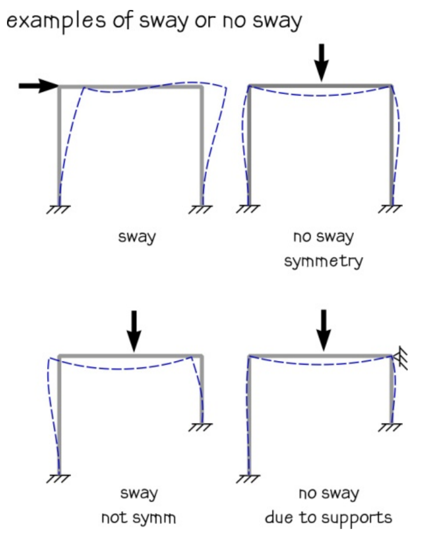
The steps in the MDM are shown on the left. The steps are discussed in more detail below.





<u>Step 1</u>: sketch the structure:

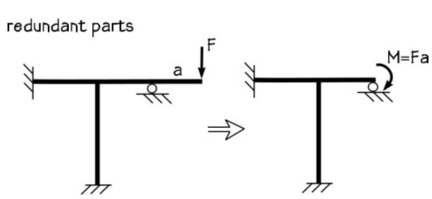
Sketch the structure, show the loads and number the joints. In the case of two or more members connected at a joint, there is one 'end' for each beam. Any correcting moment applied to the joint is divided among the ends according to the moment distribution factor.



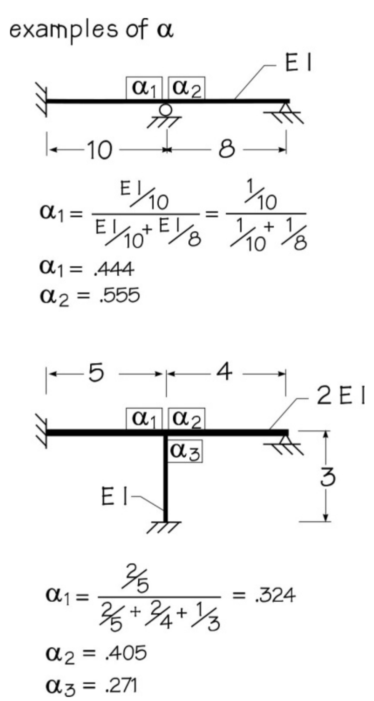
Step 2: plan the solution strategy and determine if the structure will sway

In the standard type of problem the joints do not translate, they only rotate. Axial and shear deformations are ignored. Only bending deformations are considered. If the model supports permit one or more joints to translate, and the load is such that it will cause such a movement, we need to consider sway. The example structures at the left show both types (no-sway and sway).

Note: And 'imposed' joint movement, as would occur when a support 'settles' a fixed amount, is not a sway problem. Imposed movements are just as easy to solve as are applied loads.



In cases where there are redundant parts of the structure (a determinant sub-structure), such as cantilever portions as shown at left, these should be removed and replaced with the moments or forces that they cause on the remaining structure.



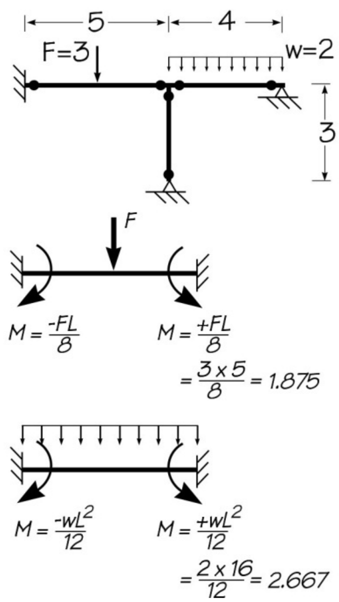
Step 3: Find moment distribution factors α :

For each joint we find the set of moment distribution factors. In general;

$$\alpha_i = \frac{(EI/L)_i}{\sum_{all}(EI/L)}$$

The moments will tend to be larger in the stiffer members, where rotary stiffness is EI/L. Thus the shorter members will tend to have the higher α factors.

examples of fixed-end moments



<u>Step 4</u>: Find fixed-end moments:

In this step, we find the fixed end moments for each beam end. In the example at left, we have 3 beams connected in a frame. The top two have loads and so have fixed-end moments. The vertical beam (the column) is unloaded so its FEM are zero.

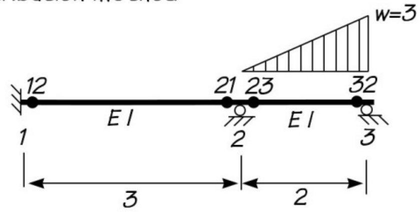
<u>Steps 5, 6, 7, 8, 9</u>: Perform iterative calculation to correct end moments. The fixed-end moments found in step 4 are the first estimate of the solution. The moments are in equilibrium with the external loads, with the only problem being that some of the joints are incorrectly fixed, when they should be free to rotate. We will set up a calculation table that will allow us to add a correcting moment to each joint. We will perform the corrections iteratively and the solution will converge to the correct answer.

The table with the solution is shown on the next page. With two beam, there are 4 end and so there are 4 columns in the table. The first row contains the moment distribution factors. The second contains a note describing the target moment (this is an extra feature normally not included). The third row contains the fixed end moments. The fourth row shows the total correction (later ignored), with the fifth row dividing the correction among the beam ends. The sixth row adds the carry-over moments from the neighboring ends. And then the seventh row add the third, fifth and sixth row terms to get a new estimate for the end moments.

The whole process is repeated until the solution is sufficiently converged.

Example showing the Moment Distribution method

① sketch problem
label the joints

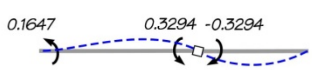


③ moment distribution factors $\alpha_{21} = \dfrac{\frac{1}{3}}{\frac{1}{3}+\frac{1}{2}} = \dfrac{2}{5}$ $\quad \alpha_{23} = \dfrac{3}{5}$

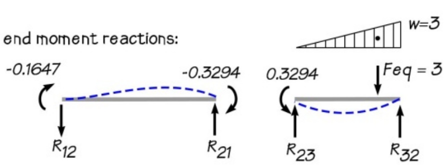
④ fixed end moments $M_{23} = \dfrac{-wL^2}{30} = -0.40$ $\quad M_{32} = \dfrac{wL^2}{20} = 0.60$

	location	12		21	23	32
moment distribution factors	α	0	③	0.4	0.6	1
target moment	TM	any	②	net 0	net 0	0
fixed end moments	FEM	0	④	0	-0.40	0.60
correction needed	cn	0	⑤	+0.40		-0.60
correction applied	corr	0	⑥	+0.16	+0.24	-0.60
carry over moment	co	+0.08	⑦	0	-0.30	+0.12
estimated end moment	eEM	+0.08	⑧	+0.16	-0.46	+0.12
	cn	0		+0.30		-0.12
repeat	corr	0		+0.12	+0.18	-0.12
	co	+0.06		0	-0.06	+0.09
	eEM	+0.14		0.28	-0.34	+0.09
repeat	⑨	⋮		⋮	⋮	⋮
		0.1647		0.3294	-0.3294	0.0000

⑩ final end moments:



end moment reactions:



find reactions on beam 1:

ΣM_1 (+) $\quad R_{21} \times 3 + (-.1647 - .3294) = 0$

$R_{21} = \dfrac{.4941}{3} = 0.1647$

ΣF_v (+) $\quad R_{12} = -0.1647$

find reactions on beam 2:

ΣM_2 (+) $\quad R_{32} \times 2 + -3 \times 1.33 + .3294 = 0$

$R_{32} = \dfrac{4 - .3294}{2} = 1.8353$

ΣF_v (+) $\quad R_{23} + R_{32} - 3 = 0$

$R_{23} = 1.1647$

continue solution ...

Steps 10: Solve for the other reactions and beam responses.

Once the end moments on a beam are known, the vertical reactions can be found from static equilibrium.

Remember that the end moments found in the MDM are moments acting "on" the supports. Moment reactions "from" the supports are opposite to these.

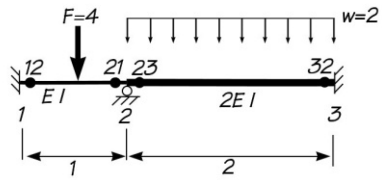
Once the vertical reactions are found, all other responses (distribution of shear, bending, slope deflection, stress) can be found using normal beam theory.

<u>Example 2</u>: Here is a simple case that solves fully in 1 iteration. This will happen when there is only one joint that needs to rotate to bring the problem into equilibrium.

Also note that this example shows a case of different EI values.

Example #2

① problem



③ α factors $\alpha_{21} = \frac{1/1}{1/1 + 2/2} = 1/2 \quad \alpha_{23} = 1/2$

④ fem $M_{23} = \frac{-FL}{8} = -0.5 \qquad M_{23} = \frac{-wL^2}{12} = -0.667$

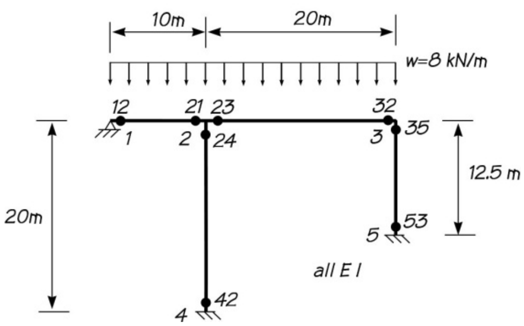
$M_{32} = 0.5 \qquad M_{32} = 0.667$

location	12	21	23	32
③ α	0	0.5	0.5	0
② TM	any	net 0	net 0	any
④ FEM	-0.5	0.5	-0.667	0.667
⑤ cn	0	+0.167		-0.667
⑥ corr	0	0.083	0.083	0
⑦ co	0.042	0	0	0.042
⑧ eEM	-0.458	0.583	-0.583	0.708
cn	+0.458	+0.0		-0.708
corr	0	0	0	0
co	0	0	0	0
eEM	-0.458	0.583	-0.583	0.708

errors are zero, so the solution is converged

Example #3

1 problem *all E I, no sway*



<u>Example 3</u>: Here is a case that shows a frame with two columns. This is a relatively complex case, though without sway.

3 joint 2 $\alpha_{21} = \frac{1/10}{1/10 + 1/20 + 1/20} = 0.5$ $\alpha_{23} = \frac{1/20}{1/10 + 1/20 + 1/20} = 0.25$ $\alpha_{24} = 0.25$

joint 3 $\alpha_{32} = \frac{1/20}{1/12.5 + 1/20} = 0.3846$ $\alpha_{35} = (1 - .3846) = .6154$

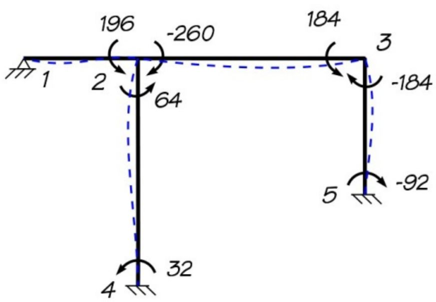
4 fem $M_{12} = \frac{-wL^2}{12} = -66.7$ $M_{23} = \frac{-wL^2}{12} = -267$

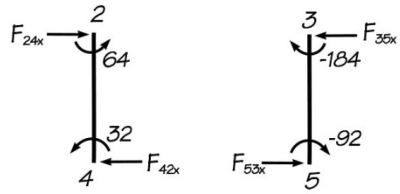
$M_{21} = 66.7$ $M_{32} = 267$

location	12	21	23	24	42	32	35	53
3 α	1	0.5	0.25	0.25	0	0.385	0.615	0
2 TM	0	net 0	net 0	net 0	any	net 0	net 0	any
4 FEM	-66.7	66.7	-266.7	0	0	266.7	0	0
5 cn	66.7	+200			0	-266.7		0
6 corr	66.7	100	50	50	0	-102.6	-164.1	0
7 co	50	33.3	-51.3	0	25	25	0	-82.1
8 eEM	50	200	-267.9	50	25	189.1	-164.1	-82.1
cn	-50	+17.9			-25	-25		82.1
corr	-50	9	4.5	4.5	0	-9.6	-15.4	0
co	4.5	-25	-4.8	0	2.2	2.2	0	-7.7
eEM	4.5	184.0	-268.3	54.5	27.2	181.7	-179.5	-89.7
	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
9	0	196	-260	64	32	184	-184	-92

with the end moments solved, the full set of horizontal and vertical reactions can be found using force and moment equilibrium.

solution for end moments



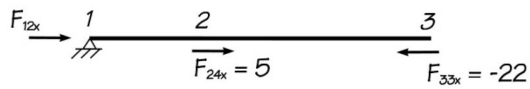


$\Sigma M = 0$ (+ counterclockwise)

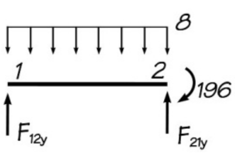
$F_{35x} \times 12.5 = (-184-93)$
$= -22$

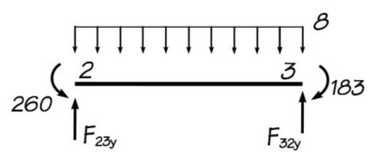
$F_{24x} \times 20 = (64+32)$
$= 5$

$F_{42x} = -5 \quad F_{53x} = 22$



$\Sigma F_x = 0$ (→ +) $\quad F_{12x} = 17$





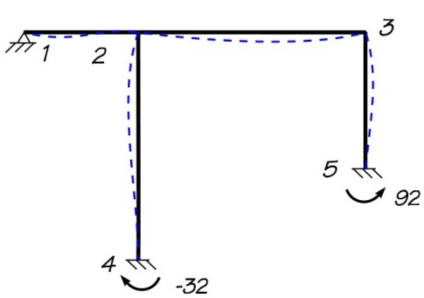
$\Sigma M_1 = 0$ (+ counterclockwise)
$F_{21y} \times 10 - 8 \times 10 \times 5 - 196 = 0$
$F_{21y} = 59.6$

$\Sigma M_2 = 0$ (+ counterclockwise)
$F_{32y} \times 20 - 8 \times 20 \times 10 - 183 + 260 = 0$
$F_{32y} = 76.2$

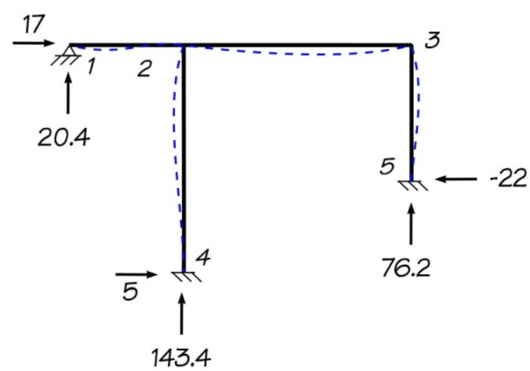
$\Sigma F_y = 0$ (↑+) $\quad F_{12y} = 20.4$

$\Sigma F_y = 0$ (↑+) $\quad F_{23y} = 83.8$
$F_{53y} = 76.2 \quad F_{42y} = 83.8+59.6=143.4$

solution for moment reactions



solution for force reactions

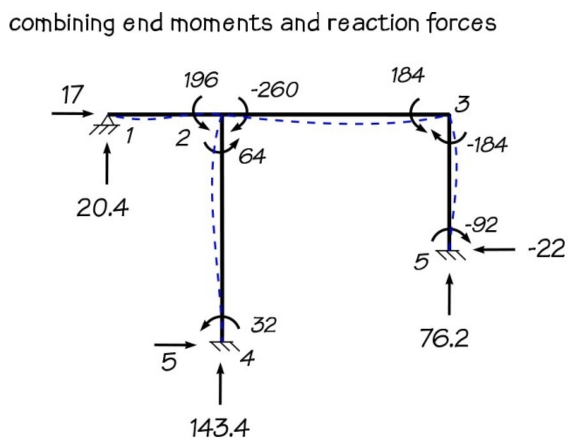


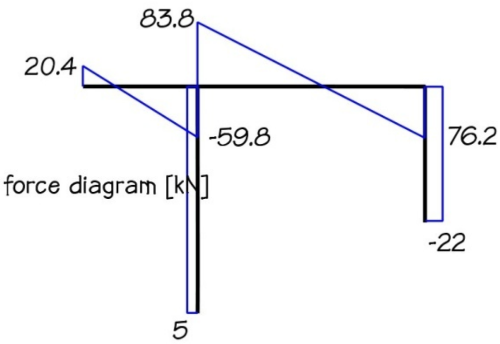
check

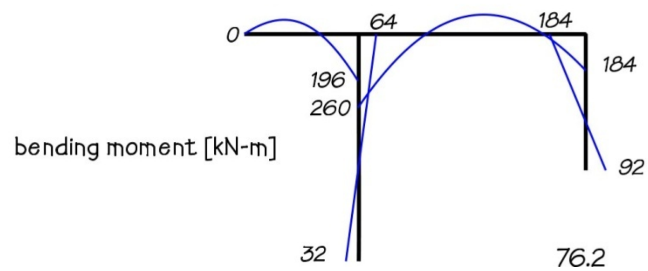
$\Sigma F_y = 20.4 + 143.4 + 76.2 = 240$ OK

$\Sigma F_x = 17 + 5 - 22 = 0$ OK

With the reactions found, the shear force and bending moment diagrams can be sketched as follows:





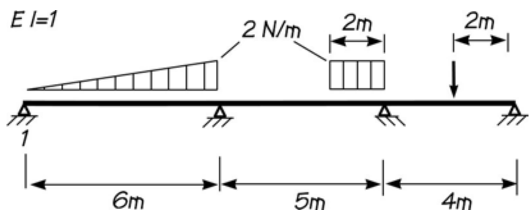


The bending moments above are drawn on the compression side of the beam. Deflections can be found by double integration of the moment diagram.

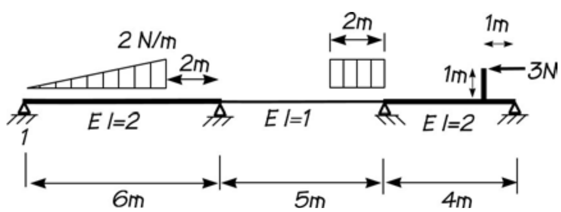
Exercise: What is the slope at joint #3?

Topic 13: Problems

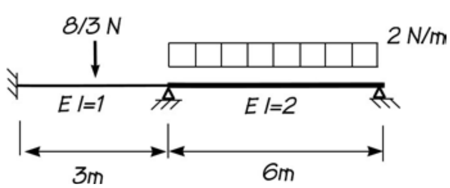
13.1. Moment distribution method



13.2. Moment distribution method



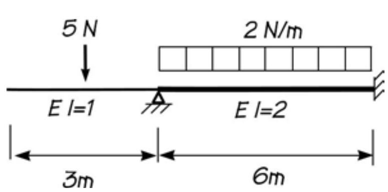
13.3. **Moment distribution method**. For the case shown on the attached page (Figure 1), fill in the first two cycles of the MD calculations.



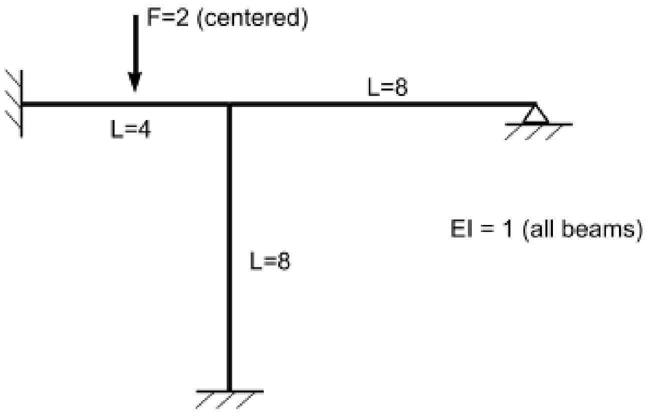
13.4. For the statically indeterminate beam shown below, with the loads, properties and end conditions as given,

a) Solve using the moment distribution method.

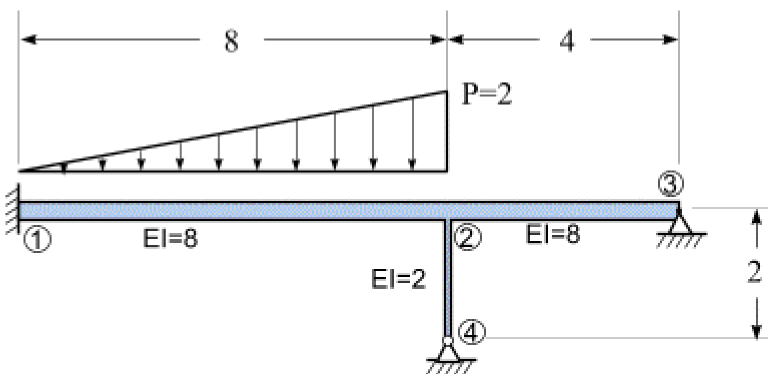
b) What is the vertical reaction at the middle support



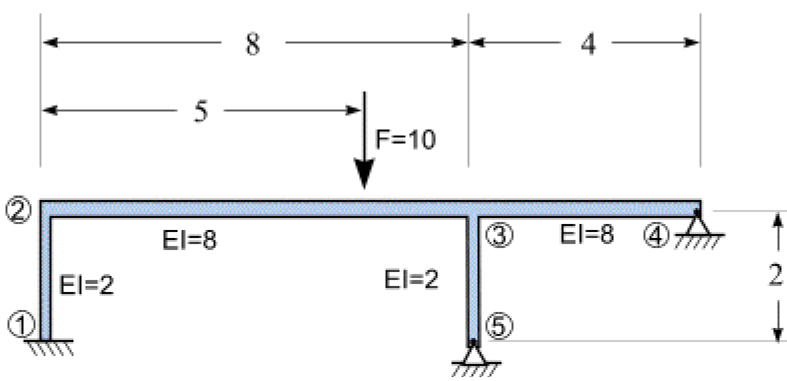
13.5. A 3 bar frame is shown below.
Solve for the moments using the moment distribution method.
Sketch the deformed shape.
Find the vertical reaction at the pin (the right hand end).



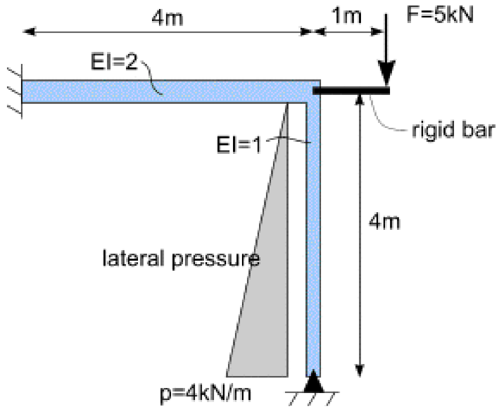
13.6. Solve the frame using the MDM method (suggest you use a spreadsheet).



13.7. Solve the frame using the MDM method (suggest you use a spreadsheet).

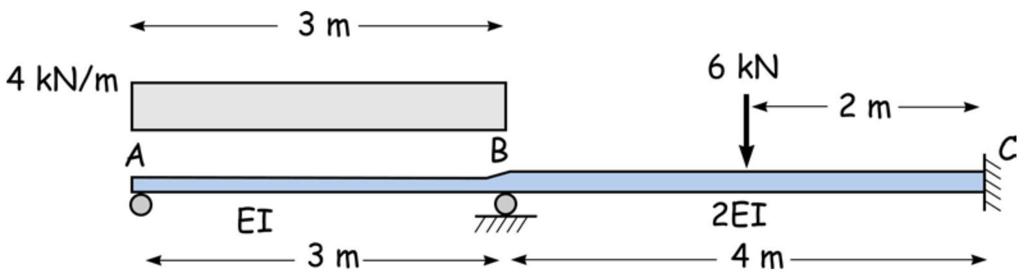


13.8. For the case shown below, set up and fill in the first two cycles of the Moment Distribution calculations.



13.9. A 2 bar structure is shown below.

a) Solve for the moments using the moment distribution method.
b) Find the vertical reaction at the pin A (the left).



Topic 14: The Moment Distribution Method with Sway

a Quadrant

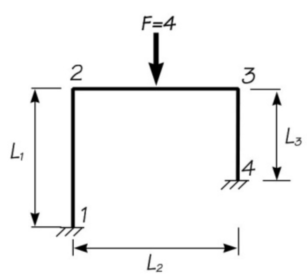
Introduction

In this chapter we will
extend the application of the moment distribution method for solving frames with sway

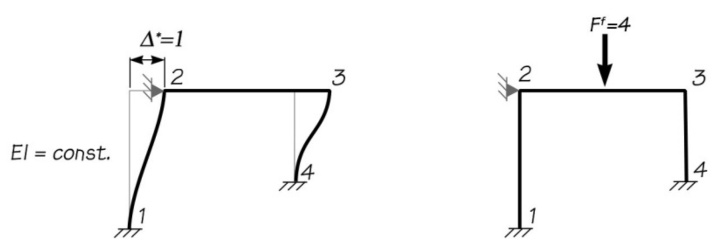
~~~~~~

In the previous chapter we dealt with beams and frames in which joints could not translate due to bending. In this chapter we all add the possibility of sway motion. For simplicity we will only consider one sway motion.

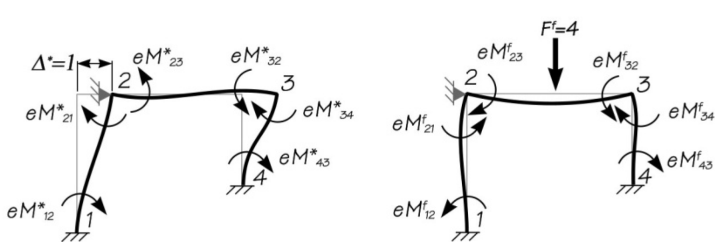
MDM problem with sway



Step 1: set up two problems, one with imposed unit sway (*) and one with sway fixed (f). Apply the general loads to the 'f' system.



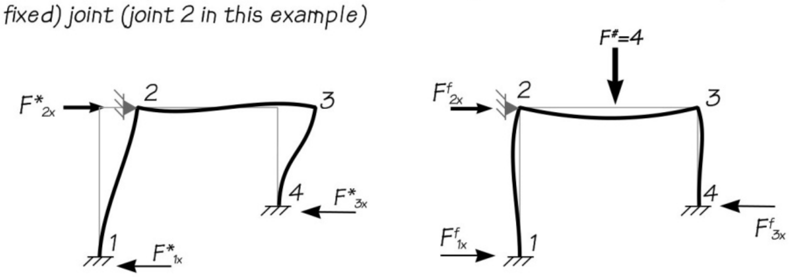
Step 2: solve both problems for moments using MDM



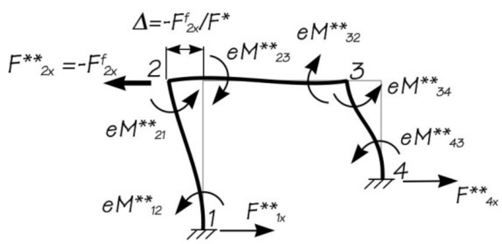
The solution of a sway problem takes two parts. In the first part a unit sway sway is imposed on the structure (call this the * problem). The imposed motion causes initial fixed end moments, which relax as the solution progresses, just as happens with applied forces. The force required to impose the unit sway can be found once the solution is found, just like the other reactions. In the example at left this is F^*_{2x} .

In the second problem (the 'f' problem) the sway is prevented, and the problem solved.
~~~~~~

Step 3: solve both problems for reactions including the reaction at the imposed (and fixed) joint (joint 2 in this example)



*Step 4: The next step is to scale the * problem so that the force at the sway joint (joint 2 in this example) is corrected (to zero in this case, or possibly to the applied load at 2 if there were an applied load - see {note a}). Call this the ** solution.*



{note a}
in the case of no load at joint 2:
$F^{**}_{2x} + F^f_{2x} = 0 \quad => \quad F^{**}_{2x} = -F^f_{2x}$
in the case of a load at joint 2:
$F^{**}_{2x} + F^f_{2x} = F_{2x} \quad => \quad F^{**}_{2x} = F_{2x} - F^f_{2x}$

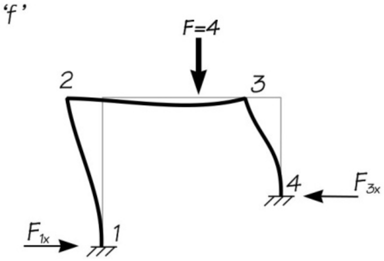
*Scale all moments and reactions from the * problem by the scale factor* $\lambda = F^{**}_{2x}/F^*_{2x}$
eg.

$$\lambda = -F^f_{2x}/F^*_{2x}$$
$$eM^{**}_{43} = \lambda\, eM^*_{43}$$
$$F^{**}_{4x} = \lambda\, F^*_{4x}$$

*Step 5: Add the ** problem with the 'f' problem, to get the complete solution.*

*add all values from ** and 'f' to get complete solution:*
eg.

$$eM_{12} = eM^{**}_{12} + eM^f_{12}$$



Step 6: Check that the solution makes sense
eg. in this case $F_{1x} + F_{3x} = 0$, $eM_{43} = 1/2\, eM_{34}$ *etc.*

To get the total solution we need to scale the * problem by λ (we call this the ** problem) and add it to the 'f ' problem.

How large is λ ?

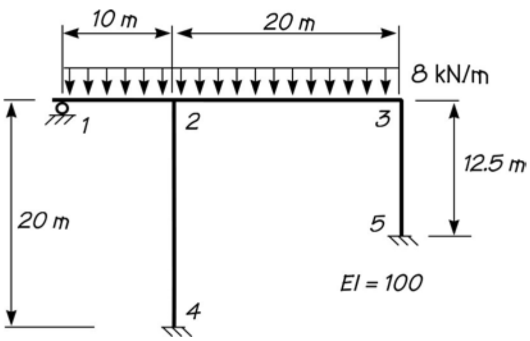
λ is chosen so that the conditions at the "false" sway support are corrected.

If there is no direct force at the false support, (as in the example at left), we want:

$$\lambda F^*_{2x} = -F^f_{2x}$$

If there is a direct force at the false support, we would want:

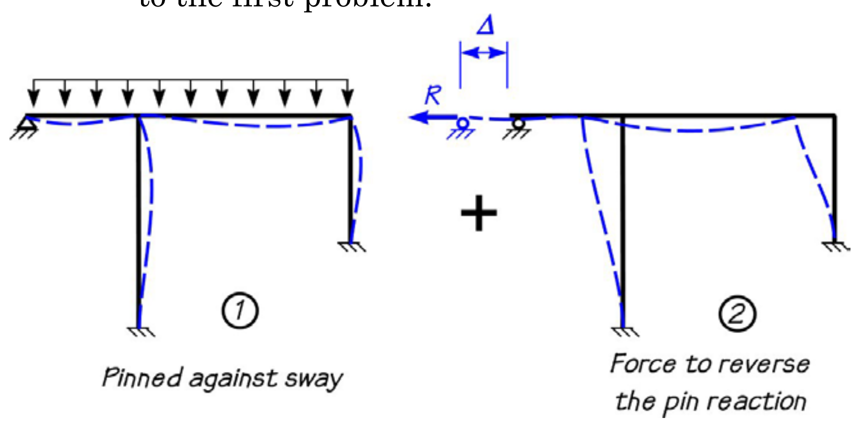
$$\lambda F^*_{2x} = F_{2x} - F^f_{2x}$$



Example of MDM with Sway

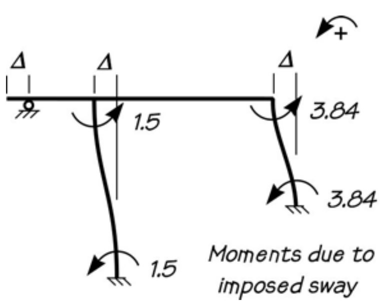
To illustrate the moment distribution method with sway, we will use a problem similar to Example 3 in Topic 13. In this case the problem has a roller on the left, instead of a pin. As a result the frame can sway.

To solve the problem we need to split the problem into two component problems. The first problem has sway prevented (by a pin on the left support). The complementary problem has an imposed sway which will create a reaction of opposite magnitude to the first problem.



The first problem was solved in Example 3 above. The reaction at the left hand pin was (see pg. 130);

$$F_{1x} = 17.1\ kN$$



Now we solve the second problem with a unit displacement Δ applied to the roller. For the imposed unit displacement, we have the initial fixed end moments as shown at the left. For example the moments in the right column are;

$$M = \frac{6\ EI\ \Delta}{L^2} = \frac{6 \cdot 100\ \cdot 1}{12.5^2} = 3.84\ kNm$$

Once we have solved the second problem, and found the reaction at the roller, we scale the whole solution to match the reaction with the 17.1 kN we need. The final answer is the sum of the scaled solution of second problem and the solution of the first problem. All the solutions needed are presented below in the form of spreadsheets.

The solution of Problem #1:

Problem 1

Joint

	1	2			3		4	5
	A	B	C	D	E	F	G	H
L		10	20	20	20	12.5		
afla	1	0.5	0.25	0.25	0.384615	0.615385	1	1
FEM	-66.66667	66.66667	-266.6667		266.6667		0	0
err	66.67		200.00		-266.67		0.00	0.00
corr	66.67	100.00	50.00	50.00	-102.56	-164.10	0.00	0.00
CO	50.00	33.33	-51.28		25.00		25.00	-82.05
EST	50.00	200.00	-267.95	50.00	189.10	-164.10	25.00	-82.05
err	-50.00		17.95		-25.00		0.00	0.00
corr	-50.00	8.97	4.49	4.49	-9.62	-15.38	0.00	0.00
CO	4.49	-25.00	-4.81		2.24		2.24	-7.69
EST	4.49	183.97	-268.27	54.49	181.73	-179.49	27.24	-89.74
err	-4.49		29.81		-2.24		0.00	0.00
corr	-4.49	14.90	7.45	7.45	-0.86	-1.38	0.00	0.00
CO	7.45	-2.24	-0.43		3.73		3.73	-0.69
EST	7.45	196.63	-261.25	61.94	184.59	-180.87	30.97	-90.43
[illegible]	-7.45		2.68		-3.73		0.00	0.00
[illegible]	-7.45	1.34	0.67	[illegible]	[illegible]	-2.29	0.00	[illegible]
⋮								
[illegible]	[illegible]	-0.01	0.00	[illegible]	[illegible]	[illegible]	[illegible]	0.00
EST	0.00	196.04	-260.08	64.03	183.80	-183.80	32.01	-91.90
err	0.00		0.01		0.00		0.00	0.00
corr	0.00	0.01	0.00	0.00	0.00	0.00	0.00	0.00
CO	0.00	0.00	0.00		0.00		0.00	0.00
EST	0.00	196.05	-260.08	64.03	183.80	-183.80	32.01	-91.90

This is the solution of Problem #2:

Problem **2**

Joint	1	2			3		4	5
	A	B	C	D	E	F	G	H
L		10.0	20.0	20.0	20.0	12.5		
afla	1.000	0.500	0.250	0.250	0.385	0.615	1.000	1.000
FEM	-	-	-	1.500	-	3.840	1.500	3.840
err	-		-1.500		-3.840		-	-
corr	-	-0.750	-0.375	-0.375	-1.477	-2.363	-	-
CO	-0.375	-	-0.738		-0.188		-0.188	-1.182
EST	-0.375	-0.750	-1.113	1.125	-1.664	1.477	1.313	2.658
err	0.375		0.738		0.188		-	-
corr	0.375	0.369	0.185	0.185	0.072	0.115	-	-
CO	0.185	0.188	0.036		0.092		0.092	0.058
EST	0.185	-0.193	-0.893	1.310	-1.500	1.592	1.405	2.716
err	-0.185		-0.224		-0.092		-	-
corr	-0.185	-0.112	-0.056	-0.056	-0.036	-0.057	-	-
CO	-0.056	-0.092	-0.018		-0.028		-0.028	-0.028
EST	-0.056	-0.397	-0.966	1.254	-1.563	1.536	1.377	2.688
err	0.056		0.110		0.028		-	-
	0.056	0.055	0.028	0.028	0.011	0.017	-	-
							0.014	
⋮								
EST	0.000	-0.336	-0.941	1.276	-1.546	1.546	1.388	2.693
err	-0.000		-0.000		-0.000		-	-
corr	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-	-
CO	-0.000	-0.000	-0.000		-0.000		-0.000	-0.000
EST	-0.000	-0.336	-0.941	1.276	-1.546	1.546	1.388	2.693

This is the solution of Problem #2, scaled to counteract the pin force from problem #1 (call this #2a):

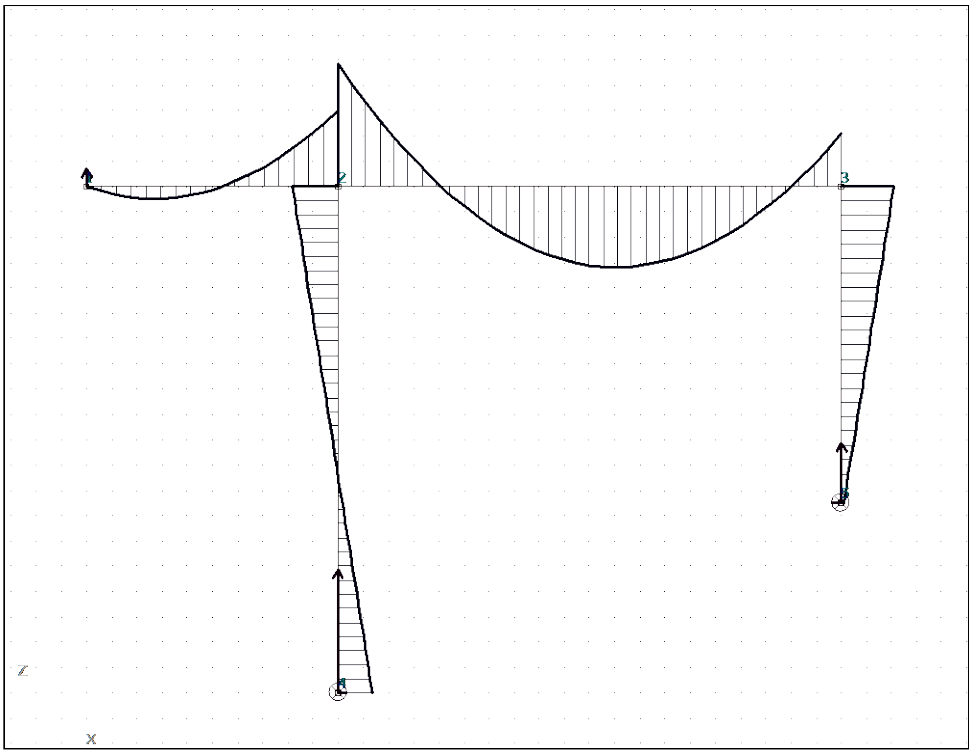
Problem **2** Scaled

Joint	1	2			3		4	5
	A	B	C	D	E	F	G	H
L		10.0	20.0	20.0	20.0	12.5		
afla	1.000	0.500	0.250	0.250	0.385	0.615	1.000	1.000
FEM	-	-	-	54.307	-	139.027	54.307	139.027
err	0.00	0.00	-54.31	0.00	-139.03	0.00	0.00	0.00
corr	0.00	-27.15	-13.58	-13.58	-53.47	-85.55	0.00	0.00
CO	-13.58	0.00	-26.74	0.00	-6.79	0.00	-6.79	-42.78
	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
EST	-13.58	-27.15	-40.31	40.73	-60.26	53.47	47.52	96.25
err	13.58	0.00	26.74	0.00	6.79	0.00	0.00	0.00
corr	13.58	13.37	6.68	6.68	2.61	4.18	0.00	0.00
CO	6.68	6.79	1.31	0.00	3.34	0.00	3.34	2.09
	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
EST	6.68	-7.00	-32.32	47.41	-54.31	57.65	50.86	98.34
err	-6.68	0.00	-8.09	0.00	-3.34	0.00	0.00	0.00
corr	-6.68	-4.05	-2.02	-2.02	-1.29	-2.06	0.00	0.00
CO	-2.02	-3.34	-0.64	0.00	-1.01	0.00	-1.01	-1.03
	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
EST	-2.02	-14.39	-34.99	45.39	-56.60	55.59	49.85	97.31
	2.02	0.00	3.98	0.00	1.01	0.00	0.00	0.00
	2.02	1.99	1.00					
⋮								
CO	0.00	0.00						
	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
EST	0.00	-12.15	-34.06	46.21	-55.96	55.97	50.26	97.50
err	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
corr	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
CO	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
EST	0.00	-12.15	-34.06	46.21	-55.96	55.96	50.26	97.50

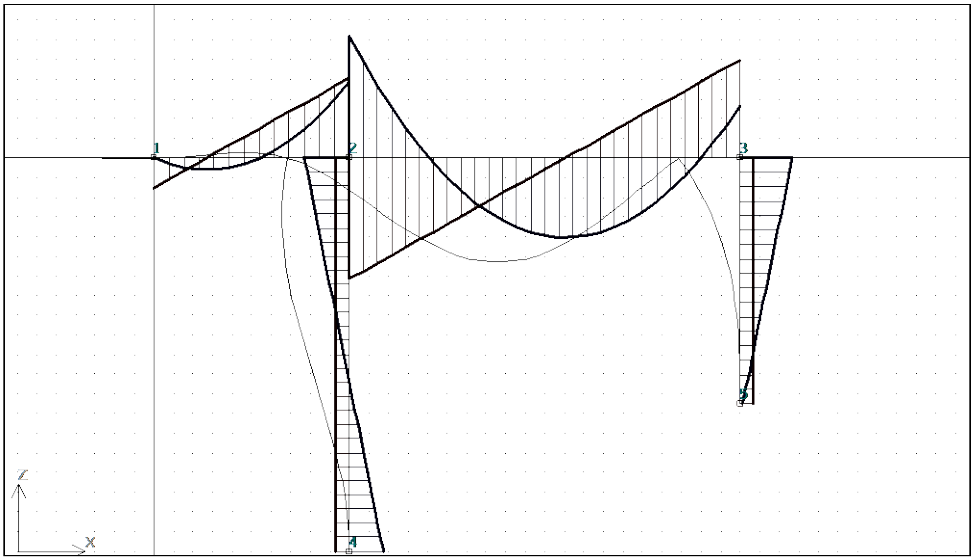
This is the sum of Problem #1 + #2a, which is the frame with roller solution. The values are moments at the locations indicated.

	1	2		3		4	5
	A	B	C	D	E	F	G	H
MDM	0.0	183.89	-294.13	110.24	127.84	-127.84	82.27	5.59
BEAM3D	0.0	183.8	-294.4	110.6	127.3	-127.3	83.9	6.47

To confirm these values independently, the same problem was analyzed in the DnV program BEAM3D. The values shown above correspond very well with the MDM results. The plots from BEAM3D are shown below;



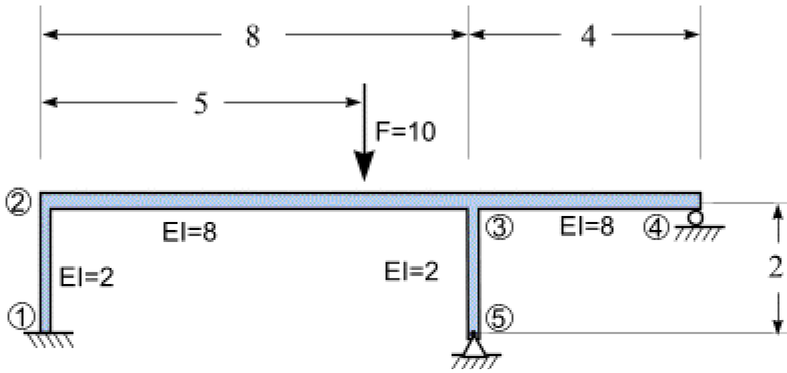
bending moment diagram, and reactions.



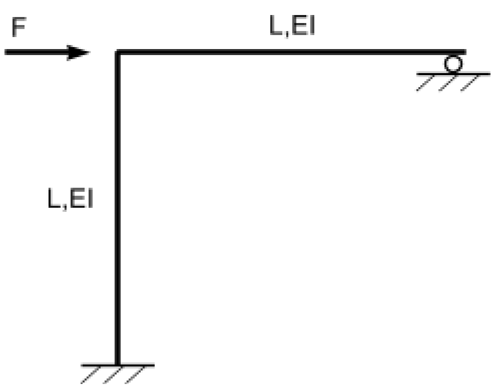
shear force (red), bending moment and deflections (exaggerated)

Topic 14: Problems

14.1. Solve the frame using the MDM method (suggest you use a spreadsheet).



14.2. A 3 bar frame is shown below.
Solve for the moments using the moment distribution method.
Sketch the deformed shape.
Find the vertical reaction at the pin (the right hand end).



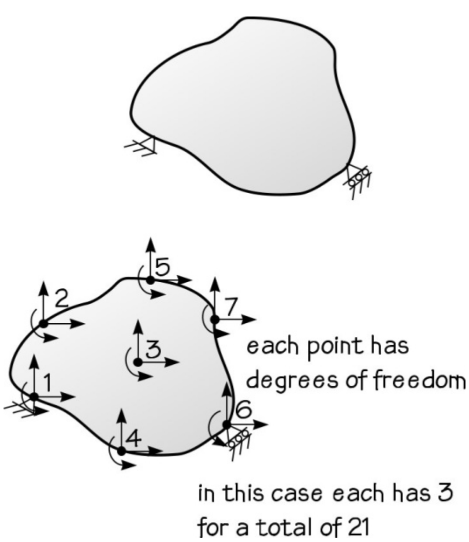
Topic 15: Matrix Structural Analysis

Introduction

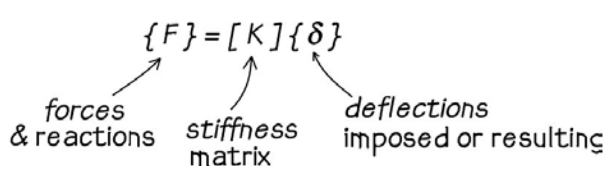
In this chapter we will

- Discuss a very general method to analyze structures, to give bending moments and axial forces in general frame structures.

~~~~~~~~~~~~~~~~~~~~



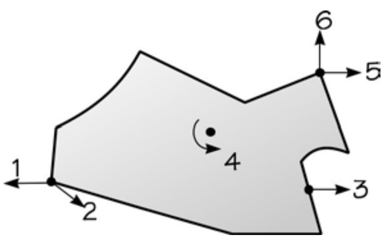
The behavior of a structure can be expressed in matrix form as;



This type of equation is 'discrete'. It represents a set of relationships among a finite set of degrees of freedom (dof).

For a general structure or arbitrary shape, the behavior can be adequately described by
~~~~~~~~~~~~~~~~~~~~

describing the behavior of a set of points. In such a case, all forces would have to be applied at the points and all responses would be determined at the points. Actions and responses at other points can be considered, as long as there is a way to gather actions to points and to interpolate response to locations between points.



We might define arbitrary degrees of freedom, for which we could write;

$$\begin{Bmatrix} F_1 \\ F_2 \\ \vdots \\ F_6 \end{Bmatrix} = \begin{bmatrix} k_{11} & k_{12} & \cdots & \\ k_{21} & \ddots & & \\ & & \ddots & \\ & & & k_{66} \end{bmatrix} \begin{Bmatrix} \delta_1 \\ \delta_2 \\ \vdots \\ \delta_6 \end{Bmatrix}$$

But how would we find the *kij* terms? For an arbitrary body (a violin, a rock, a teapot ...) the *kij* terms would be hard to find. There would be no table of standard values.

The *kij* terms could be found by experiment.
- apply a test force at dof "i", measure all displacements at dofs "j":

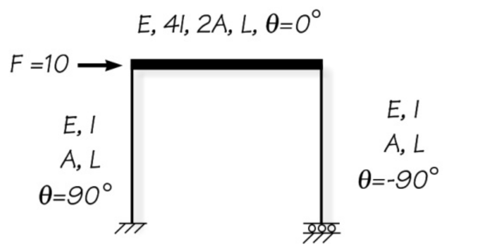
$$k_{ij} = {F_i}/{\delta_j}$$

But is it even possible to apply a force at "i" and only "i" ? Remember that F_i includes reactions as well as applied forces (there is no difference as far as the structure is concerned!)

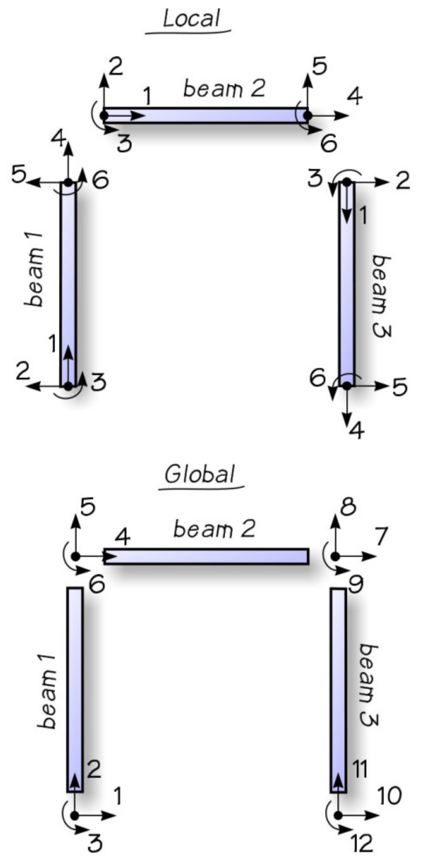
Determining k_{ij} experimentally is not practical. The best one can do is to attempt to validate the k_{ij} matrix experimentally by measuring responses and comparing to predictions.

To make the determination of a structural stiffness matrix practical, we normally describe a structure using regularly shaped parts, with standard degrees of freedom.

Example: a 3 bar frame with a lateral load



Degrees of Freedom



In this problem the force is applied at dof #4
and movement is prevented at dofs # 1,2,3,10,11,12

For the 3 bar frame at left, we can define the define local degrees of freedom for each member using the same standard approach that was described in Chapter 11. We will start from the local element stiffness matrices and assemble the full structural global stiffness matrix, just to illustrate the process.

The local degrees of freedom follow the individual members, while the global degrees of freedom are all aligned to the Cartesian (x-y) system. The other aspect is that global degrees of freedom refer to nodes of a structure, rather than to ends of members. This means that several member ends can share a single set of degrees of freedom.

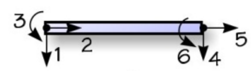
The matrices below show the local and global versions of the stiffness matrix for beam 1. The difference is the way the degrees of freedom are defined. In this case the global degrees of freedom are just versions of the local dofs.

Beam 1

local k matrix

$$K_{local} = \begin{bmatrix} \frac{AE}{L} & 0 & 0 & \frac{-AE}{L} & 0 & 0 \\ 0 & \frac{12\,EI}{L^3} & \frac{6\,EI}{L^2} & 0 & \frac{-12\,EI}{L^3} & \frac{6\,EI}{L^2} \\ 0 & \frac{-6\,EI}{L^2} & \frac{4\,EI}{L} & 0 & \frac{-6\,EI}{L^2} & \frac{2\,EI}{L} \\ \frac{-AE}{L} & 0 & 0 & \frac{AE}{L} & 0 & 0 \\ 0 & \frac{-12\,EI}{L^3} & \frac{-6\,EI}{L^2} & 0 & \frac{12\,EI}{L^3} & \frac{-6\,EI}{L^2} \\ 0 & \frac{6\,EI}{L^2} & \frac{2\,EI}{L} & 0 & \frac{-6\,EI}{L^2} & \frac{4\,EI}{L} \end{bmatrix}$$

global k matrix



local	2	1	3	5	4	6
global	1	2	3	4	5	6

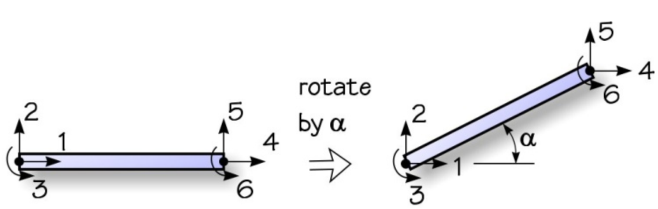
$$K_{global} = \begin{array}{c c} \begin{bmatrix} \frac{12\,EI}{L^3} & 0 & \frac{6\,EI}{L^2} & \frac{-12\,EI}{L^3} & 0 & \frac{6\,EI}{L^2} \\ 0 & \frac{AE}{L} & 0 & 0 & \frac{-AE}{L} & 0 \\ \frac{-6\,EI}{L^2} & 0 & \frac{4\,EI}{L} & \frac{-6\,EI}{L^2} & 0 & \frac{2\,EI}{L} \\ \frac{-12\,EI}{L^3} & 0 & \frac{-6\,EI}{L^2} & \frac{12\,EI}{L^3} & 0 & \frac{-6\,EI}{L^2} \\ 0 & \frac{-AE}{L} & 0 & 0 & \frac{AE}{L} & 0 \\ \frac{6\,EI}{L^2} & 0 & \frac{2\,EI}{L} & \frac{-6\,EI}{L^2} & 0 & \frac{4\,EI}{L} \end{bmatrix} & \begin{array}{cc} g & l \\ 1 & 2 \\ 2 & 1 \\ 3 & 3 \\ 4 & 5 \\ 5 & 4 \\ 6 & 6 \end{array} \end{array}$$

ie. $\underbrace{k_{11}}_{local} \Rightarrow \underbrace{k_{22}}_{global}$ $\quad \underbrace{k_{22}}_{local} \Rightarrow \underbrace{k_{11}}_{global}$

$\underbrace{k_{23}}_{local} \Rightarrow \underbrace{-k_{12}}_{global}$

diagonal terms - k_{ii} - are always positive: + force always causes a + displacement, directly but may cause a - displacement, indirectly

<u>Aside</u>: There is a general way to find the global stiffnesses for a rotated bar. The rotation matrix can be used to find the stiffness terms for a rotated beam. In a rotated beam dof 1 is partly axial and partly shear, as is dof 2. But as superposition holds, any movement along dof 1 can be expresses as some axial and some shear, and the resulting axial and shear forces can be resolved back into the 1 and 2 directions.



The matrix below and the matrix operation expresses the mix of effects in a concise way.

(suggestion: derive the rotation matrix using vector algebra).

the rotation matrix is:

$$\lambda = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 & 0 & 0 & 0 \\ -\sin\alpha & \cos\alpha & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos\alpha & \sin\alpha & 0 \\ 0 & 0 & 0 & -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\underset{global}{[K]} = \lambda^T \underset{local}{[K]} \lambda$$

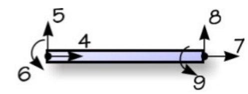
In the case of a 90 degree rotation, the rotation matrix has the effect of doing row-column swaps. For other angles the effect is more complicated.

$$\lambda(90°) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\lambda(45°) = \begin{bmatrix} .707 & .707 & 0 & 0 & 0 & 0 \\ -.707 & .707 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & .707 & .707 & 0 \\ 0 & 0 & 0 & -.707 & .707 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Beam 2 has a local [k] that is similar to beam 1 except that area is 2A and modulus is 4I. The global [k] looks similar to the local [k], except that the numbering is shifted.

Beam 2

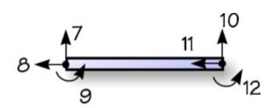


global k matrix

$$K_{global} = \begin{array}{c} \begin{array}{ccccccc} \text{global} & 4 & 5 & 6 & 7 & 8 & 9 \end{array} \\ \left[\begin{array}{cccccc} \frac{2AE}{L} & 0 & 0 & \frac{-2AE}{L} & 0 & 0 \\ & \frac{48\,EI}{L^3} & \frac{24\,EI}{L^2} & 0 & \frac{-48\,EI}{L^3} & \frac{24\,EI}{L^2} \\ & & \frac{16\,EI}{L} & 0 & \frac{-24\,EI}{L^2} & \frac{8\,EI}{L} \\ & & & \frac{2AE}{L} & 0 & 0 \\ & \text{symm} & & & \frac{48\,EI}{L^3} & \frac{-24\,EI}{L^2} \\ & & & & & \frac{16\,EI}{L} \end{array}\right] \end{array} \begin{array}{c} g \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \end{array}$$

Beam 3 has a local [k] that is the same as beam 1. The global [k] also looks similar to the global [k], of beam 1 because a rotation of +90 produces a similar effect to -90. The only change is that the numbering is shifted.

Beam 3



global k matrix

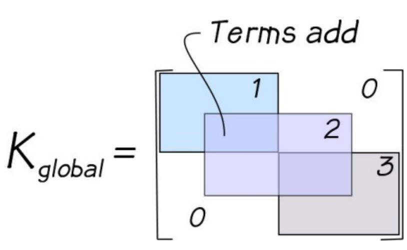
$$K_{global} = \begin{array}{c} \begin{array}{ccccccc} \text{global} & 7 & 8 & 9 & 10 & 11 & 12 \end{array} \\ \left[\begin{array}{cccccc} \frac{12\,EI}{L^3} & 0 & \frac{6\,EI}{L^2} & \frac{-12\,EI}{L^3} & 0 & \frac{6\,EI}{L^2} \\ & \frac{AE}{L} & 0 & 0 & \frac{-AE}{L} & 0 \\ & & \frac{4\,EI}{L} & \frac{-6\,EI}{L^2} & 0 & \frac{2\,EI}{L} \\ & & & \frac{12\,EI}{L^3} & 0 & \frac{-6\,EI}{L^2} \\ & \text{symm} & & & \frac{AE}{L} & 0 \\ & & & & & \frac{4\,EI}{L} \end{array}\right] \end{array} \begin{array}{c} g \\ 7 \\ 8 \\ 9 \\ 10 \\ 11 \\ 12 \end{array}$$

The structural stiffness matrix is just the sum of the global versions of the member stiffness matrices. Where two terms share a dof, the two values are added. This is again reflecting the simple idea of superposition in linear systems that Hooke first saw.

structural K matrix

$$
K_{global} = \begin{array}{c}
\begin{array}{cccccccccccc} \text{global} & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \end{array} \\
\left[\begin{array}{cccccccccccc}
\frac{12EI}{L^3} & 0 & \frac{6EI}{L^2} & \frac{-12EI}{L^3} & 0 & \frac{6EI}{L^2} & & & & & & \\
 & \frac{AE}{L} & 0 & 0 & \frac{-AE}{L} & 0 & & & & & 0 & \\
 & & \frac{4EI}{L} & \frac{-6EI}{L^2} & 0 & \frac{2EI}{L} & & & & & & \\
 & & & \frac{12EI}{L^3}+\frac{2AE}{L} & 0 & \frac{-6EI}{L^2} & \frac{-2AE}{L} & 0 & 0 & & & \\
 & \text{symm} & & & \frac{AE}{L}+\frac{48EI}{L^3} & \frac{24EI}{L^2} & 0 & \frac{-48EI}{L^3} & \frac{24EI}{L^2} & & & \\
 & & & & & \frac{20EI}{L} & 0 & \frac{-24EI}{L^2} & \frac{8EI}{L} & & & \\
 & & & & & & \frac{12EI}{L^3}+\frac{2AE}{L} & 0 & \frac{6EI}{L^2} & \frac{-12EI}{L^3} & 0 & \frac{6EI}{L^2} \\
 & & & & \text{symm} & & & \frac{AE}{L}+\frac{48EI}{L^3} & \frac{-24EI}{L^2} & 0 & \frac{-AE}{L} & 0 \\
 & & & & & & & & \frac{20EI}{L} & \frac{-6EI}{L^2} & 0 & \frac{2EI}{L} \\
 & & & & & & & & & \frac{12EI}{L^3} & 0 & \frac{-6EI}{L^2} \\
 & 0 & & & & & & \text{symm} & & & \frac{AE}{L} & 0 \\
 & & & & & & & & & & & \frac{4EI}{L}
\end{array}\right]
\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \\ 11 \\ 12 \end{array}
\end{array}
$$

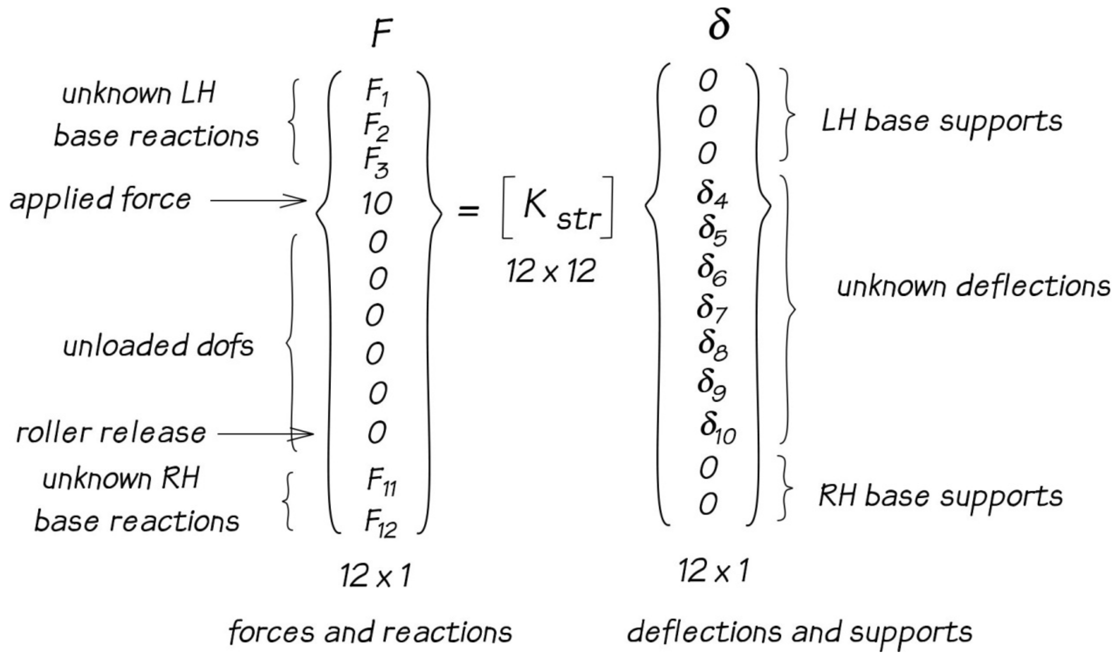
Stiffness matrices are symmetrical. This is a curious property, especially when you think about the off-diagonal terms. Some of the terms refer to forces per unit rotation and moments per unit translation.



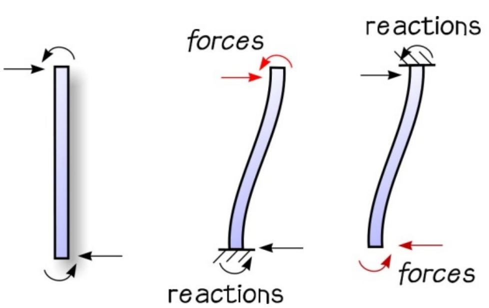
With the whole stiffness matrix assembled, we have a single equation that relates all actions (forces and moments) with all movements (translation and rotations):

$$
\underset{12 \times 1}{\{F\}} = \underset{12 \times 12}{\left[K_{str}\right]} \underset{12 \times 1}{\{\delta\}}
$$

To solve the system of twelve by twelve equations we need to identify the twelve unknowns. It is (almost) never the case that we would know twelve deflections and want to know twelve forces. Nor would we know twelve forces and look for the deflections. Typically we know some forces (mostly zero) and some
deflections (zero at supports):



We should have some combination of unknown loads and deflections that adds up to twelve. If we don't, we can't solve the system.



Note that the structure does not know what is an applied force and what is a reaction. All the structure know is whether it is in equilibrium.

There are a variety of ways of solving matrix equations like:

$$\underset{12\times 1}{\{F\}} = \underset{12\times 12}{[K_{str}]}\ \underset{12\times 1}{\{\delta\}}$$

There are various numerical strategies used in linear algebra that are used to solve such systems.

Gaussian elimination is one common method. We can assume that if we have N equations in N unknowns that we can solve it.

To solve these in Maple (see 3bar_frame.pdf or 3bar_frame.mw), we would just expand the matrix expression into a set of 12 simultaneous equations;

$$eqn1 \quad F_1 = k_{11}\,\delta_1 + k_{12}\,\delta_2 + \ldots$$

$$eqn2 \quad F_2 = k_{21}\,\delta_1 + k_{22}\,\delta_2 + \ldots$$

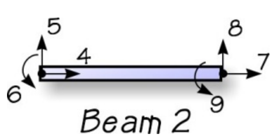
$$eqns = \{eqn1,\ eqn2,\ \ldots\ eqn12\}$$

Maple will solve these equations in either numerical or algebraic form, giving expressions for all results in terms of the variable. For example, for this problem, Maple will give;

$$\delta_{10} = \frac{10\,L^3}{9\,EI} \qquad F_{12} = \frac{40\,L^2(A\,L^2 - 48\,I)}{9\,(25\,A\,L^2 + 96\,I)\,EI}$$

Q1: With the above solution for force and deflections at the nodes (the dofs), how would we find the stresses in each member?

A1: To find the stresses we have to return to the individual beams. We use the global stiffness matrix of a single member. For example, for the cross beam in the previous example (beam 2), we find the member forces as follows;

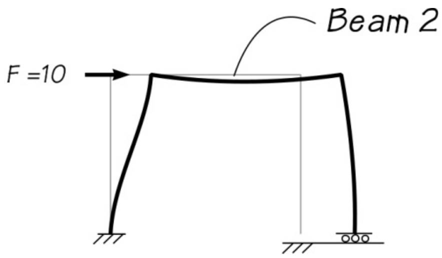


these are not from the solution of the structure

these are from the solution of the structure

$$\underset{6\times 1}{\begin{Bmatrix} F_4 \\ F_5 \\ F_6 \\ F_7 \\ F_8 \\ F_9 \end{Bmatrix}} = \underset{6\times 6}{\overset{\text{Beam 2}}{\begin{bmatrix} k_{global} \end{bmatrix}}} \underset{6\times 1}{\begin{Bmatrix} \delta_4 \\ \delta_5 \\ \delta_6 \\ \delta_7 \\ \delta_8 \\ \delta_9 \end{Bmatrix}}$$

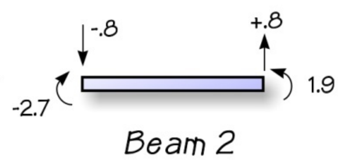
The forces are not the same as found above. They are only the forces that act on the individual member.
The beam forces are found as follows:



from solution of the whole structure

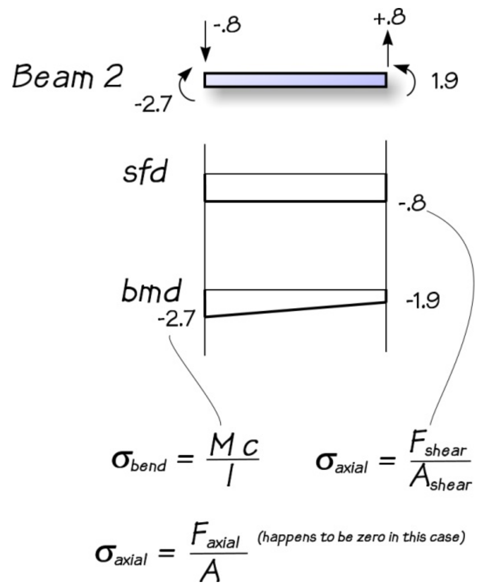
$$\underset{6\times1}{\begin{Bmatrix} F_4 \\ F_5 \\ F_6 \\ F_7 \\ F_8 \\ F_9 \end{Bmatrix}} = \overset{\text{Beam 2}}{\underset{6\times6}{\left[k_{global}\right]}} \underset{6\times1}{\begin{Bmatrix} .198 \\ .099 \\ -.228 \\ .198 \\ -.099 \\ -.172 \end{Bmatrix}}$$

$$\begin{Bmatrix} F_4 \\ F_5 \\ F_6 \\ F_7 \\ F_8 \\ F_9 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -.80 \\ -2.70 \\ 0 \\ .80 \\ 1.9 \end{Bmatrix} \Rightarrow$$



solution of just Beam 2

Note that there is no axial force (would be F_4, F_7) in Beam 2. This is because the roller at bottom of beam 3 releases all horizontal force. The applied load of 10 must all be transmitted to ground through Beam1. With these forces and moments we can find the shear force and bending moment diagrams, along with the axial, shear and bending stresses:

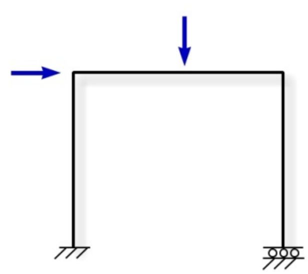


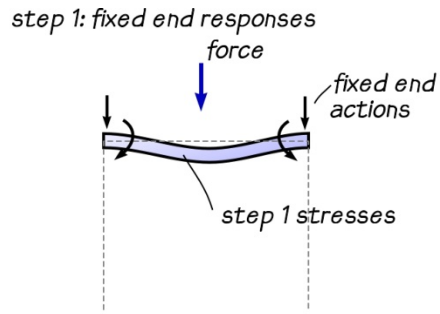
Because there was no load along the member, the maximum stresses in the above case occurred at the ends of the beam.

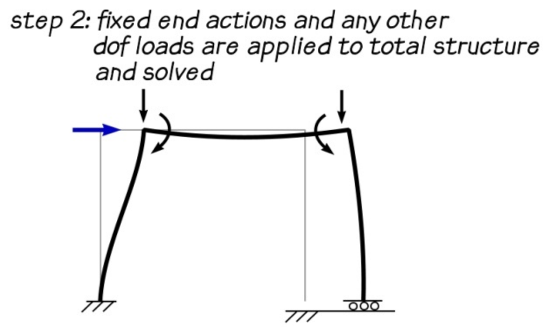
Q2: How are loads along a beam dealt with?

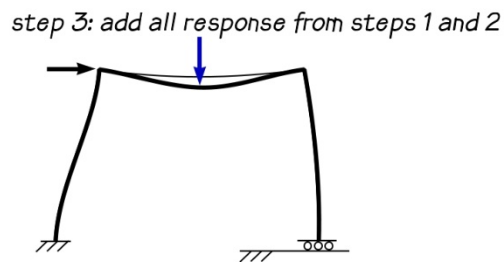
A2: Loads that are act between dofs are dealt with in three steps. In step 1, the fixed end forces and stresses that the loads cause are found. In step 2, the fixed end actions are placed on a full structural model and solved. All responses, including deflection, stresses, strains, for the full structure (including the beam where the loads acted) can be found for the whole structure.
The complete solution comes from adding the two solutions (step1 + step2):

problem with loads along members




step 1: fixed end responses
force
fixed end
actions
step 1 stresses


step 2: fixed end actions and any other
dof loads are applied to total structure
and solved


step 3: add all response from steps 1 and 2

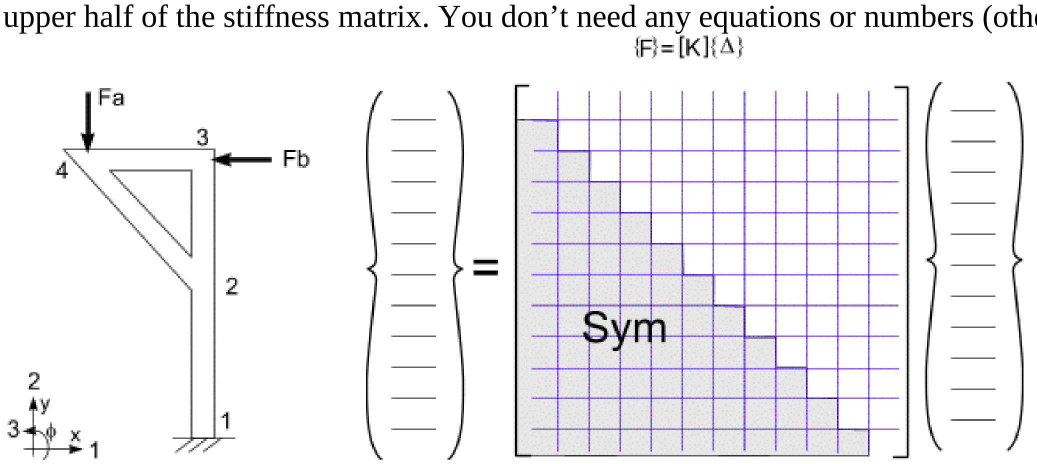
Topic 15: Problems

15.1 Frame Structures can be analyzed by "Matrix Structural Analysis" or by solution of sets of continuous differential equations. Compare and contrast these two approaches.

15.2 The stiffness matrix for a 2D beam is said to have axial, shear and rotary terms. Give examples of each of the 3 types of stiffness (i.e. 3 examples of the individual k_{ij} terms), with a sketch of the terms.

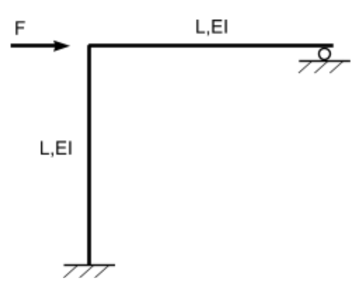
15.3 Describe what is meant by the "rotary stiffness terms" in the stiffness matrix of a beam. Explain which terms in the matrix are rotary terms and how they are derived.

15.4 For the 4-bar frame shown below, the 2D solution is found by solving 12 equations in matrix form shown beneath. For the case of the loads and boundary conditions as shown, fill in the 14 columns (there is 1 column for forces, 1 for displacements and 12 in the stiffness matrix), with any known values. In the force and displacement vectors, write in a zero (*0*) for known zero values and the letter *X* or variable name for other unknown values. In the stiffness matrix write a 0 for the zero terms and the letter *K* for a non-zero stiffness terms. You only need to fill in the upper half of the stiffness matrix. You don't need any equations or numbers (other than 0).

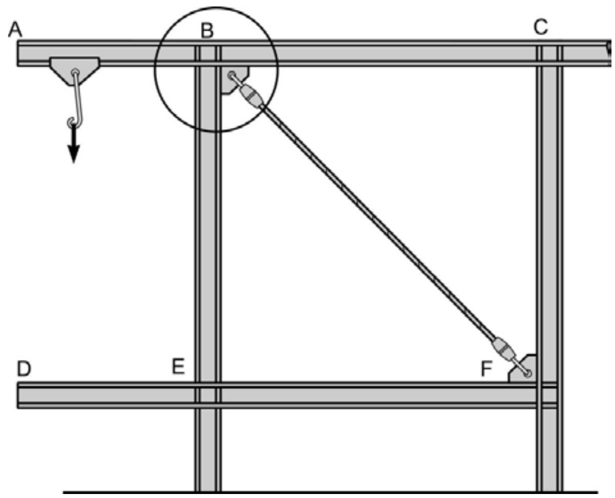


15.5 A 2 part frame is shown below.

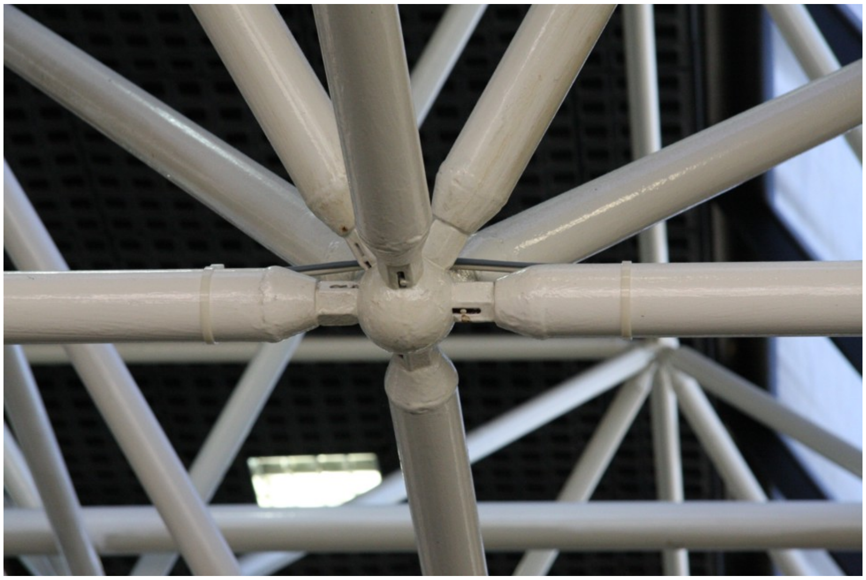
a) Construct the full structural stiffness matrix for the structure. Describe the steps you take to do so.
b) Write the force-deflection equation for the structure in matrix format, showing all terms (ie include all terms in the matrices or vectors). Explain which, if any, terms are unknown.



15.6 Assuming that you are using a program that performs matrix structural analysis, explain concisely how the global stiffness terms for the joint circled in the sketch below are determined. You don't have to solve this frame.



Topic 16 Overview of Finite Element Theory

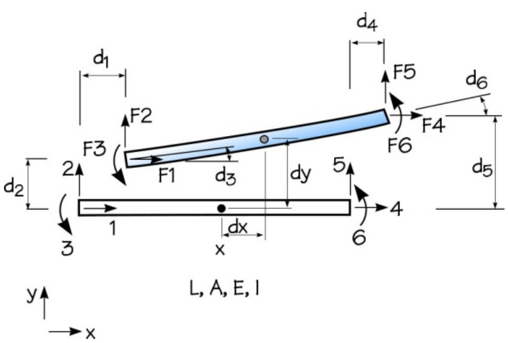


Introduction

In this chapter we will

- introduce the 2D finite element called the constant stress triangle (cst)
- show how to derive the element stiffness and all output values from energy considerations

~~~~~~~~~~~~~~~~~~~~~~



Finite element method

Recall that for a beam, we can relate the end loads by a stiffness equation in matrix form;

$$\{F\} = [K]\{x\}$$

We can find the $\boldsymbol{K}$ terms for a beam by solving the beam bending equation for various end movements. To find the displacement of some point along the beam (at x) we could solve the system for the displaced shape. We would find that the displacements would be;

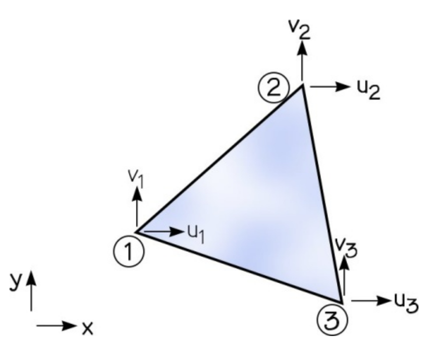
$$dx = d_1 + x\,(d_4 - d_1)$$ (why so simple?)

and
~~~~~~~~~~~~~~~~~~~~~~

$$dy = d_2 + f(x, d_3, d_5, d_6)$$ (why is this more complex?)

For this beam element, we made use of what is called 'beam theory', to solve for the loads and deflections under certain loading conditions.

However, in the case of most finite elements, such as 2D planar elements, plate elements, and solid elements, we will not start from some general analytical solution of a loaded membrane, plate or solid. These solutions are too complex and will not give practical results. Instead, we assume some very simple behaviors, highly idealized, but which satisfy the basic requirements for equilibrium (i.e. forces balance, energy is conserved). With this approach, the single element does not really model the behavior or a comparable real solid object of the same shape. This is ok, because the aggregate behavior of a set of these simple elements will model the behavior quite well. This is something like modeling a smooth curve as a series of straight lines (even horizontal steps). This is locally wrong, but overall quite accurate.



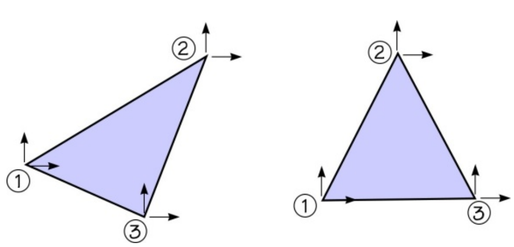
Constant Stress Triangle

To illustrate the way that finite elements are formulated, we will derive the full description of an element called the constant stress triangle (cst). This is a standard 2D element that is available in most finite element models.

Consider a 2D element which is only able to take in-plane stress. The three corners of the triangle can only move in the plane.

For this element the force balance is;

$$\underset{\{6x1\}}{F} = \underset{[6x6]}{K^e}\underset{\{6x1\}}{\delta}$$



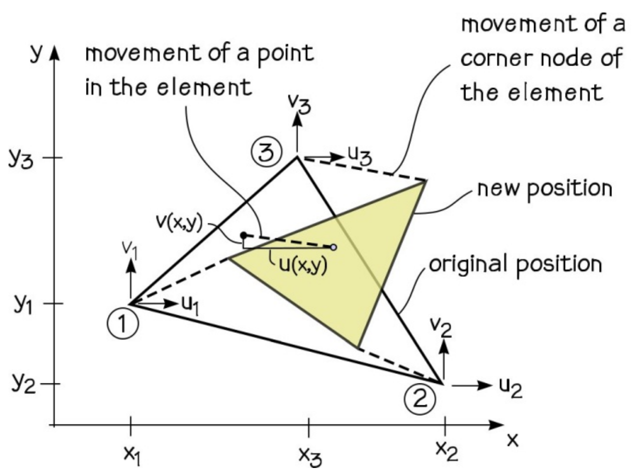
We want to determine the element stiffness matrix K^e, and we want it to be valid for any triangle;

So, while we have six degrees of freedom, as we did in the beam case, we don't have any hand analytical solutions. To create a general solution that will apply to all triangles we will make some very simple assumptions which will allow us to model 2D stress problems (such as a web in shear, or stresses in plane around a cutout in a web. .

We will follow the outline in Hughes (p. 245-253).

Step 1 - select a suitable displacement function.

Consider the movement of a general triangle. Each corner moves differently, and every point inside moves.



The movement in x is defined as u and the movement is y is defined a v. Both u and v are functions of x and y;

$$\Delta x = u$$
$$\Delta y = v$$

$$u = f_u(x, y)$$
$$v = f_v(x, y)$$

Assuming that the material in the triangle is isotropic (no preferred direction), then we would expect the two displacement functions $f_u(x, y)$ and $f_v(x, y)$ to look similar.

The functions for u and v can only depend on the 6 nodal displacements (that all the info that we have to define movement), so we can have no more than 6 unknown coefficients for both functions.

A trial function;
a) lets try: $u = c_1 x + c_2 y + c_3(x + y)$

is this ok? No! Why? Because it means that at (0,0) (the origin) there is no movement. It would be as if all elements are pinned to the origin.

b) lets try: $u = c_1 x + c_2 y + c_3(xy)$

is this ok? No! Why? same problem.

The simplest viable functions for u and v that has 6 coefficients is;

$$u = c_1 + c_2 x + c_3 y$$
$$v = c_4 + c_5 x + c_6 y$$

Occam's razor, in latin: "***lex parsimoniae***" (the law of simple), is a principle that says: from among alternative explanations, the one that works, but makes the fewest new assumptions is usually correct. The concept is central to rational thought. William Occam was a 14th century English Friar and writer.

This provides a very simple but viable general description of the displacement field. We can re-write the displacement function in matrix form;

$$\delta(x,y) = \begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} 1 & x & y & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & x & y \end{bmatrix} \begin{Bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ c_6 \end{Bmatrix}$$

$$\delta(x,y) = \boldsymbol{H}\,\boldsymbol{C}$$

Now we have the displacement function.

<u>Step 2</u> - Find the constants in $\boldsymbol{C}$
at the corners we can write;

$$\delta_1 = \delta(x_1, y_1) = \begin{bmatrix} 1 & x_1 & y_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & x_1 & y_1 \end{bmatrix} \boldsymbol{C}$$

$$\delta_2 = \delta(x_2, y_2) = \begin{bmatrix} 1 & x_2 & y_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & x_2 & y_2 \end{bmatrix} \boldsymbol{C}$$

$$\delta_3 = \delta(x_3, y_3) = \begin{bmatrix} 1 & x_3 & y_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & x_3 & y_3 \end{bmatrix} \boldsymbol{C}$$

The total displacement of the corners can be written;

or

$$\boldsymbol{\delta} = \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix} = \underbrace{\begin{bmatrix} 1 & x_1 & y_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & x_1 & y_1 \\ 1 & x_2 & y_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & x_2 & y_2 \\ 1 & x_3 & y_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & x_3 & y_3 \end{bmatrix} \begin{Bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ c_6 \end{Bmatrix}}_{\boldsymbol{\delta} = \boldsymbol{A}\,\boldsymbol{C}}$$

$\boldsymbol{A}$ is called the connectivity matrix. It contains the geometric information, the coordinates of the nodes of the triangle. The terms in the C vector can be found;

$$\boldsymbol{C} = \boldsymbol{A}^{-1}\,\boldsymbol{\delta}$$

$\boldsymbol{A}^{-1}$ is a 6x6 matrix;

$$\boldsymbol{A}^{-1} = \frac{1}{2\,A_{123}} \begin{bmatrix} x_2y_3 - x_3y_2 & 0 & & \\ & y_2 - y_3 & & \\ & & & \\ & & & \cdots \\ & & & \end{bmatrix}$$

where $2\,A_{123}$ is the determinant of the 3x3 coordinate matrix;

$$2\,A_{123} = det \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix}$$

where:

$$2\,A_{123} = det \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix} = x_2y_3 - y_2x_3 + x_3y_1 - x_1y_3 + x_1y_2 - x_2y_1$$

which happens to be 2x the area of the triangle (ie A_{123} is the area of the triangle).

We can now go back to;

$$\delta(x, y) = \boldsymbol{H}(x, y)\,\boldsymbol{C}$$

which we can re-write as;

$$\delta(x, y) = \boldsymbol{H}(x, y)\,\boldsymbol{A}^{-1}\,\boldsymbol{\delta}$$

where $\delta(x,y)$ is the displacement of any point in the triangle, $\boldsymbol{A^{-1}}$ contains information on the geometry of the triangle and $\boldsymbol{\delta}$ contains the displacements of the corner nodes of the triangle. This lets of find the displacement anywhere by just tracking the displacements of the nodes. Remember that the finite element method lets us model a continuum by modeling a discrete system of connected nodes.

Step 3 - Find the strain in the element

We need to find the stress and strain in the element so that we can determine the stiffness of the element.
The (2D) strains at any point in the element have 3 components;

$$\varepsilon(x,y) = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}$$

where the strains are found from the partial derivatives of the displacement field:

$$\varepsilon_x = \frac{\partial u}{\partial x}$$

$$\varepsilon_y = \frac{\partial v}{\partial y}$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

recall that;

$$u = c_1 + c_2 x + c_3 y$$
$$v = c_4 + c_5 x + c_6 y$$

so that we have;

$$\varepsilon_x = \frac{\partial u}{\partial x} = c_2$$

$$\varepsilon_y = \frac{\partial v}{\partial y} = c_6$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = c_3 + c_5$$

which allows us to write;

$$\varepsilon(x,y) = \begin{Bmatrix} c_2 \\ c_6 \\ c_3 + c_5 \end{Bmatrix}$$

Note that the strains in the triangle are just constants, and do not vary with *x* and *y*. This is the reason that this element is called the *CST* or constant stress triangle.
We can write the strains in matrix form;

$$\varepsilon(x,y) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix} \begin{Bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ c_6 \end{Bmatrix}$$

and simplified to be;

$$\varepsilon(x,y) = \boldsymbol{G}\ \boldsymbol{C}$$

We can substitute for $\boldsymbol{C}$ to get;

$$\boldsymbol{\varepsilon} = \boldsymbol{G}\ \ \boldsymbol{A}^{-1}\ \boldsymbol{\delta}$$

This is the strain fully described in terms of nodal coordinates and nodal displacements. We can collect terms;

$$\boldsymbol{B} = \boldsymbol{G}\ \ \boldsymbol{A}^{-1}$$

where $\boldsymbol{B}$ is called the strain coefficient matrix, and so write;

$$\boldsymbol{\varepsilon} = \boldsymbol{B}\ \boldsymbol{\delta}$$

$\boldsymbol{G}$ is a 3x6 matrix. $\boldsymbol{A}^{-1}$ is a 6x6, so $\boldsymbol{B}$ is a 3x6 matrix that relates the 3 strains to the 6 nodal displacements.

<u>Step 4</u> - Find the element stresses (and forces)

Start by defining the stresses;

$$\boldsymbol{\sigma} = \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}$$

We can write Hooke's law in matrix form as;

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \frac{1}{E}\begin{bmatrix} 1 & -\nu & 0 \\ -\nu & 1 & 0 \\ 0 & 0 & 2(1+\nu) \end{bmatrix}\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}$$

or in terms of stress;

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \dfrac{1-\nu^2}{2} \end{bmatrix}\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}$$

In simpler form we write the stresses as;

$$\boldsymbol{\sigma} = \boldsymbol{D}\,\boldsymbol{\varepsilon}$$

where $\boldsymbol{D}$ is called the elasticity matrix. Now we can use $\boldsymbol{\varepsilon} = \boldsymbol{B}\,\boldsymbol{\delta}$ to let us write;

$$\boldsymbol{\sigma} = \boldsymbol{D}\,\boldsymbol{B}\,\boldsymbol{\delta}$$

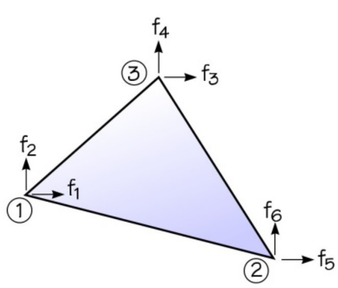
or

$$\boldsymbol{\sigma} = \boldsymbol{S}\,\boldsymbol{\delta}$$

where $\boldsymbol{S} = \boldsymbol{D}\,\boldsymbol{B}$ and is called the stress matrix.

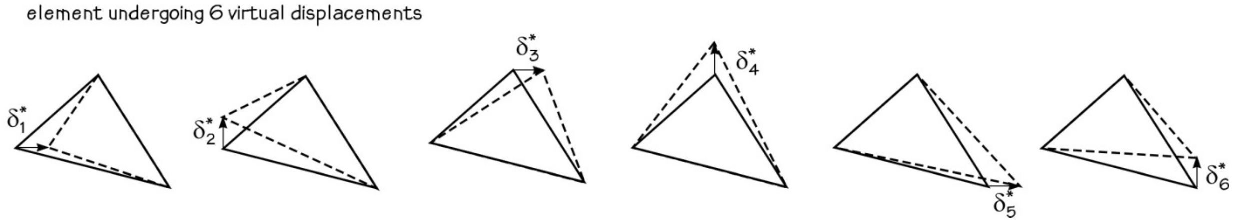
Step 5 - Obtain the Element Stiffness Matrix

element with 6 nodal forces



Idea: To obtain the element stiffness we will use the principle of virtual work. The principle of virtual work states that for a body in equilibrium, the virtual work done by real forces f_i acting through any viable pattern of virtual displacements δ^* will be zero. In our case we wish to equate the work done by the real nodal forces with the work done to distort the element.

element undergoing 6 virtual displacements



The external work done for a set of 6 virtual displacements will be;

$$\boldsymbol{W_{ext}} = \boldsymbol{\delta}^{*\boldsymbol{T}}\, \boldsymbol{f}$$

or

$$\boldsymbol{W_{ext}} = \begin{bmatrix} \delta_1^* & \delta_2^* & \delta_3^* & \delta_4^* & \delta_5^* & \delta_6^* \end{bmatrix} \begin{Bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \end{Bmatrix}$$

Note that, for example, f_3 only does work when δ_3* moves. And the work is the full amount of, f_3 δ_3*, as f_3 is fully active during the whole of δ_3* . Remember that f_3 does not cause δ_3*. We just imagine that δ_3* occurs even as the nodal forces stay acting.

The internal work done is equal to the integral of the stress time the strain over the volume;

$$\boldsymbol{W_{int}} = \int_{Vol} \varepsilon\, \sigma\, dvol$$

which in the case of the virtual work done one element becomes;

$$\boldsymbol{W_{int}} = \int_{Vol} [\varepsilon^*(x,y)]^T\, \sigma(x,y)\, dvol$$

which when making use of the strain coefficient matrix and the elasticity matrix can be written as;

$$\boldsymbol{W_{int}} = \int_{Vol} [\boldsymbol{B}\, \boldsymbol{\delta}^*]^T\, \mathbf{D}\, \mathbf{B}\, \boldsymbol{\delta}\;\, dvol$$

In this equation $\boldsymbol{\delta}^*$ refers to virtual displacements , while $\boldsymbol{\delta}$ refers to real (existing) displacements.

$$\boldsymbol{W_{int}} = \int_{Vol} \boldsymbol{B^T}\, \boldsymbol{\delta}^{*\mathbf{T}}\, \mathbf{D}\, \mathbf{B}\, \boldsymbol{\delta}\;\, dvol$$

So if we say;

$$\mathbf{W_{ext}} = \boldsymbol{W_{int}}$$

we can obtain;

$$\boldsymbol{\delta}^{*\boldsymbol{T}}\,\boldsymbol{f} = \boldsymbol{\delta}^{*\mathbf{T}}\left(\int_{Vol} \boldsymbol{B^T}\ \mathbf{D}\ \mathbf{B}\ \ dvol\right)\ \ \boldsymbol{\delta}$$

which simplifies to;

$$\boldsymbol{f} = \left(\boldsymbol{B^T}\ \mathbf{D}\ \mathbf{B}\int_{Vol} dvol\right)\ \ \boldsymbol{\delta}$$

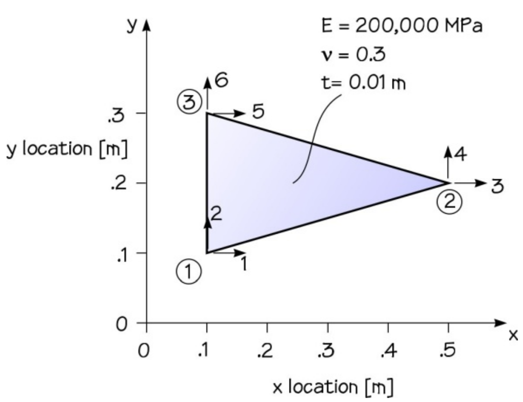
and;

$$\boldsymbol{f} = (\boldsymbol{B^T}\ \mathbf{D}\ \mathbf{B}(A_{123}t))\ \ \boldsymbol{\delta}$$

where t is the element thickness and A_{123} is the element area. The term in the brackets is the element stiffness;

$$\boldsymbol{K^e} = (\boldsymbol{B^T}\ \mathbf{D}\ \mathbf{B}(A_{123}t))$$

$\boldsymbol{K^e}$ is a 6x6 matrix ($\boldsymbol{B^T}$ **D B** is 6x3 x 3x3 x 3x6 = 6x6)



Numerical Example: Consider this triangular element with properties shown.

For this case the matrices are;

$$\boldsymbol{A} = \begin{bmatrix} 1 & .1 & .1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & .1 & .1 \\ 1 & .5 & .2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & .5 & .2 \\ 1 & .1 & .3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & .1 & .3 \end{bmatrix}$$

$$\boldsymbol{A^{-1}} = \begin{bmatrix} 1.62 & 0 & -.25 & 0 & -.375 & 0 \\ -1.25 & 0 & 2.5 & 0 & -1.25 & 0 \\ -5 & 0 & 0 & 0 & 5 & 0 \\ 0 & 1.62 & 0 & -.25 & 0 & -.375 \\ 0 & -1.25 & 0 & 2.5 & 0 & -1.25 \\ 0 & -5 & 0 & 0 & 0 & 5 \end{bmatrix}$$

$$A_{123} = .08$$

$$\boldsymbol{G} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}$$

$$\boldsymbol{B} = \boldsymbol{G} \quad \boldsymbol{A}^{-1}$$
$$= \begin{bmatrix} -1.25 & 0 & 2.5 & 0 & -1.25 & 0 \\ 0 & -5 & 0 & 0 & 0 & 5 \\ -5 & -1.25 & 0 & 2.5 & 5 & -1.25 \end{bmatrix}$$

$$\boldsymbol{D} = \begin{bmatrix} 220000 & 65900 & 0 \\ 65900 & 220000 & 0 \\ 0 & 0 & 100000 \end{bmatrix}$$

$$\boldsymbol{B^T} = \begin{bmatrix} -1.25 & 0 & -5 \\ 0 & -5 & -1.25 \\ 2.5 & 0 & 0 \\ 0 & 0 & 2.5 \\ -1.25 & 0 & 2 \\ 0 & 5 & -1.25 \end{bmatrix}$$

$$\boldsymbol{K^e} = (\boldsymbol{B^T}\ \mathbf{D}\ \mathbf{B}(A_{123}t))$$

$$\boldsymbol{K^e}$$
$$= \begin{bmatrix} 4540 & 1660 & -1100 & -2000 & -3460 & 341 \\ 1660 & 9060 & -1320 & -499 & -341 & -8540 \\ -1100 & -1320 & 2210 & 0 & -1100 & 1320 \\ -2000 & -499 & 0 & 1000 & 2000 & -499 \\ -3460 & -341 & -1100 & 2000 & 4540 & -1660 \\ 341 & -8540 & 1320 & -499 & -1660 & 9060 \end{bmatrix}$$

This is the stiffness matrix for a specific CST element.

Topic 16: Problems

16.1 The displacement functions of the constant stress triangular element are:

$u(x,y) = C1 + C2\ x + C3\ y$

$v(x,y) = C4 + C5\ x + C6\ y$

where u represents the x-translation of any point (x,y) and v represents the y-translation of the point.

16.2 A beam has only one coordinate (x). However, most beam models would allow a point on the beam to rotate as well as translate. So, construct 3 simple displacement functions;
u(x),
v(x),
θ(x),
of a 'beam element', using the same logic as was used to create the displacement functions of the constant stress triangular element.

Topic 17: Hull Girder Shear Stresses

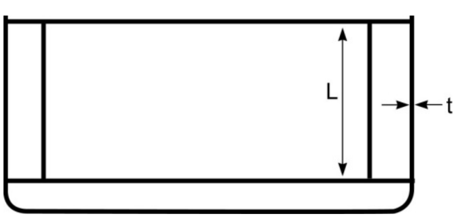
Italian Stone

Introduction

In this Chapter we will

- Examine vertical shear in a ship
- Describe the idea of shear flow.

~~~~~~~~~~~~~~~

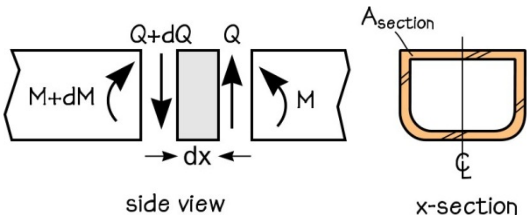


Ships are made of steel plate. This means that ships are thin walled shells. Even for the local components such as individual frames the width of a plate is much greater than its thickness;

$$L \gg t$$

Overall, the cross section of a ship contains long sections of connected plate. Such sections transfer shear very effectively. Ships are generally very stiff in shear, and need to be.

We wish to be able to determine the shear forces and stresses everywhere in the cross section of a ship. We will start by examining the shear that is associated with the vertical bending stress. In a later chapter we will examine torsion.
~~~~~~~~~~~~~~~



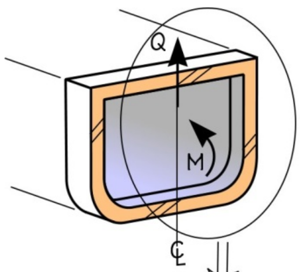
Recall from beam theory that shear is the slope of the bending moment:

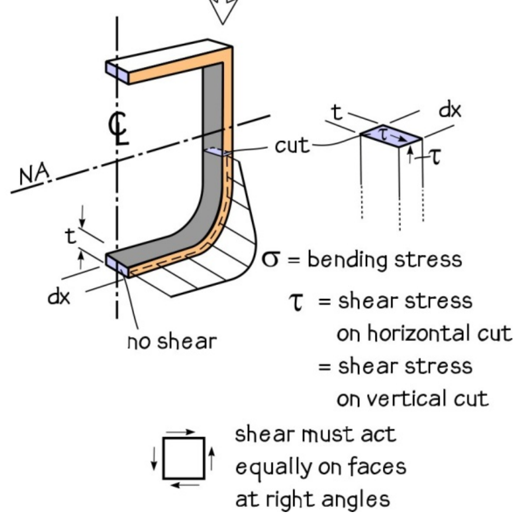
$$dM = Q\ dx$$

$$Q = dM/\ dx$$

There is significant shear is a ship. How is it distributed in the cross section? Shear is not just in the vertical plates. There is shear in all parts of the vessel. The average shear stress can be found by dividing the shear force by the cross-section area;

$$\tau_{avg} = Q/A_{shear}$$

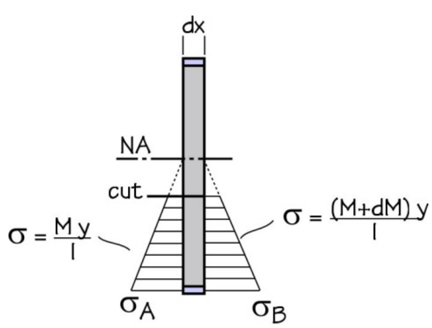




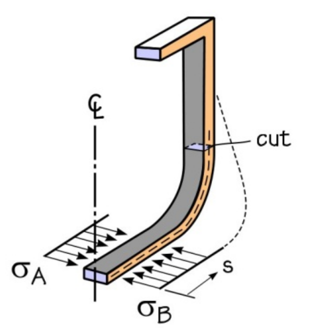
How is Q distributed around the x-section of the ship? Is the shear stress uniform? Is it only in vertically oriented members? To find the pattern of stress, we construct a free body diagram of a part of a slice of the ship's cross section.

To find the shear on the cross section, we cut the section longitudinally and note that the shear stress on the cut must be the same as the shear stress on the cross section at that point. We can assume;

- there is no shear on the centerline
- the shear force on the cut is $\tau\ t\ dx$



We find the force on the cut by integrating all horizontal forces on out slice atarting from the centerline (keel). We integrate along the shell plating, using the path variable 's'.



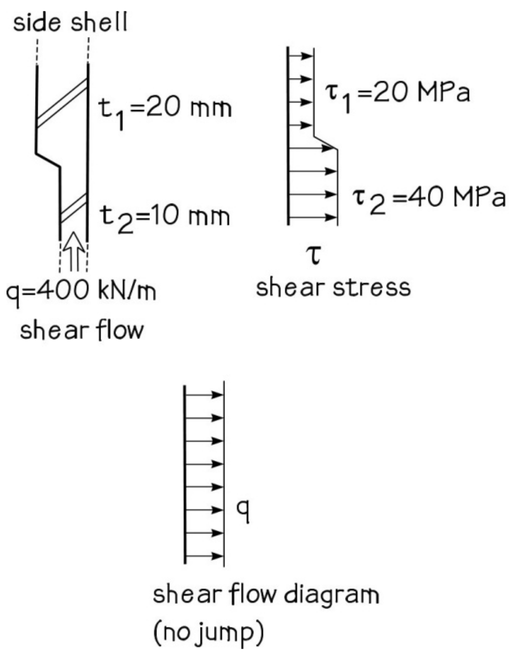
bending stress (acts in x direction)

$$\tau\, t\, dx = \int_0^s \sigma_A t\, ds - \int_0^s \sigma_B t\, ds$$
$$= \frac{M_A - M_B}{I}\int y\, t\, ds$$
$$= \frac{dM}{I}\int y\, t\, ds$$
$$\tau\, t = \frac{dM}{dx}\frac{1}{I}\int y\, t\, ds$$
$$= \frac{Q}{I}\int y\, t\, ds$$

<u>Define</u>:

$$m = \int_0^s y\, t\, ds \quad \text{(for 1/2 section)}$$

m : 1st moment of area, about the neutral axis, of all the material from the start to the cut at S (where τ is determined)



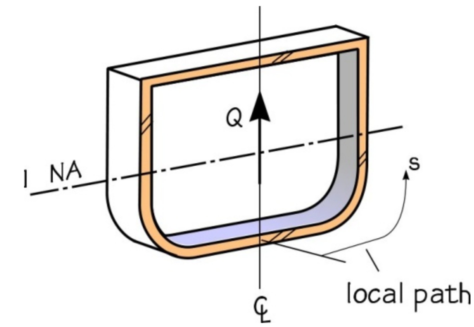
<u>Define</u>:

$$q \equiv \tau\, t \Leftarrow \text{shear flow}$$

The units of shear flow is N/m.

There is an analogy between shear flow and fluid flow. At an abrupt change in section, the shear flow remains constant, while the stress abruptly changes. This is analogous to water flow where at a change in pipe size the mass flow rate (kg/s) would stay constant while the velocity would abruptly change.

We can combine the above concepts into one equation;



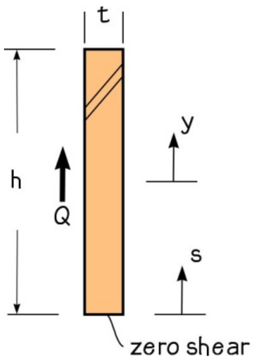
$$\tau t = \frac{Q}{I}\int_0^s y\,t\,ds$$

$$\Downarrow$$

$$\boxed{q = \frac{Q\,m}{I}}$$ shear flow equation

Q, I ← entire section
m ← local path
q ← local path

Shear Flow Example 1: a rectangular steel bar subject to a shear force Q.



Q = shear

$I = \frac{1}{12} t h^3$

$s = 0 \rightarrow h$

$y = s - h/2$

$s = y + h/2$

shear flow :

$$q = \frac{Q\,m}{I}$$

$$\tau t = \frac{Q}{I}\int_0^s y\,t\,ds$$

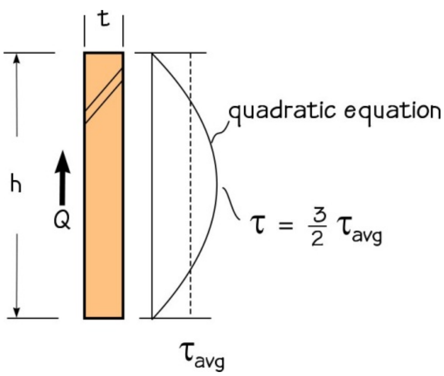
$$= \frac{12\,Q\,t}{t\,h^3}\int_0^s (s-h/2)\,ds$$

$$= \frac{12\,Q}{h^3}\left[\frac{s^2}{2} - \frac{h\,s}{2}\right]$$

$$= \frac{12\,Q}{h^3}\left[\frac{(y+h/2)^2}{2} - \frac{h\,(y+h/2)}{2}\right]$$

$$= \frac{6\,Q}{h^3}\left[y^2 + yh + \frac{h^2}{4} - yh - \frac{h^2}{2}\right]$$

$$= \frac{6\,Q}{h^3}\left[y^2 - \frac{h^2}{4}\right]$$



$$\tau t = -\frac{3}{2}\frac{Q}{h}\left[1 - \left(\frac{2y}{h}\right)^2\right]$$

$$\tau = -\frac{3}{2}\frac{Q}{th}\left[1 - \left(\frac{2y}{h}\right)^2\right]$$

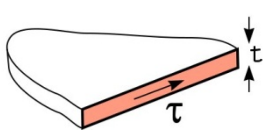
$$\tau = -\frac{3}{2}\tau_{avg}\left[1 - \left(\frac{2y}{h}\right)^2\right] \quad \text{quadratic equation}$$

@ NA, y=0 $\quad \underline{\tau = -\frac{3}{2}\tau_{avg}}$

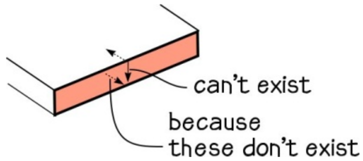
@ top, bot, y=+- h/2 $\quad \tau = 0$

Summary:

shear flow $\equiv q \equiv \tau\, t$



Shear flow acts along the cross section of a plate. There can be no significant shear across a thin plate, because there is no shear on the inner and outer surfaces. The shear flow is found by determining the value of 'm' (a path integral) along with Q (the total shear force) and I (the moment of inertia);

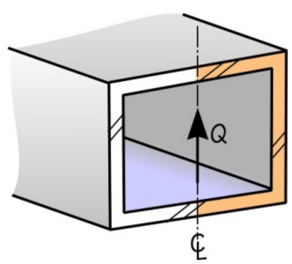


$$q = \frac{Q\,m}{I} \qquad m = \int_0^s y\,t\,ds$$

$s \equiv$ path coordinate

Example 2: Shear Flow in a simple box-girder.

Consider the simple box girder with the dimensions as show below. This is like a simple barge without the frames. The overall vertical shear Q is 20 MN. To find the pattern of shear flow and then the shear stresses we first calculate the location of the neutral axis, and I.

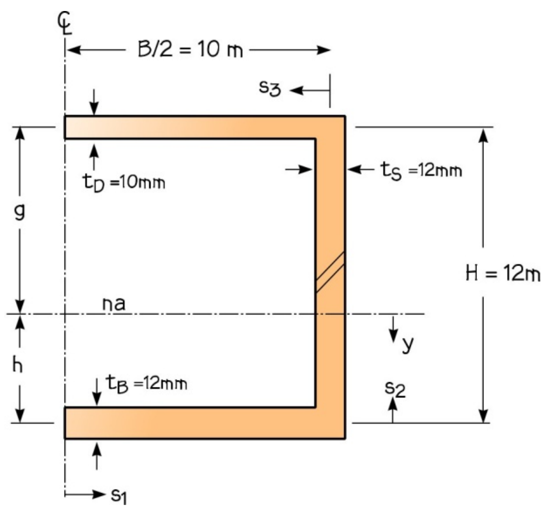


g and h are the distances from the deck and bottom to the neutral axis;

$$h = \frac{\sum ay}{\sum a} = \frac{.010 \cdot 10 \cdot 12 + .012 \cdot 12 \cdot 6 + 0}{.010 \cdot 10 + .012 \cdot 12 + .015 \cdot 10} = \frac{2.064}{.394}$$

$$= 5.24 \text{ m}$$

$$g = 12 - 5.24 = 6.76 \text{ m}$$



The moment of inertia about the base can be approximated by;

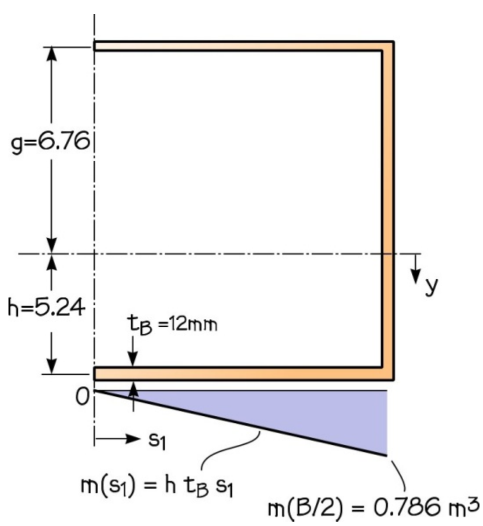
$$I_{base} \cong t_D \cdot \frac{B}{2} \cdot H^2 + \frac{1}{3}\, t_S \cdot H^3$$
$$= 21.31\ m^4 \qquad (half\ ship)$$

The moment of inertia about the neutral axis is;

$$I_{na} = 2 \cdot (I_{BASE} - A\,h^2) = 21\ m^4 \quad \text{(whole ship)}$$

Now we can determine m

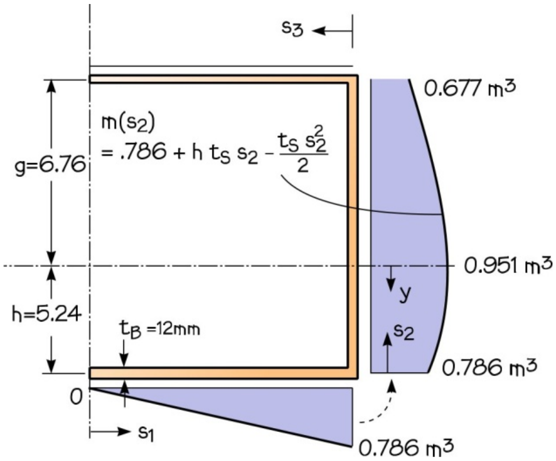
$$I_{na} = 2 \cdot (I_{BASE} - A\,h^2) = 21\ m^4$$



Next we find *m*. We will start at the centerline on the bottom, where s_1 starts;

$$m = \int_0^{s_1} y\, t_B\ ds = h\ t_B \int_0^{s_1} ds = y\, t_B\, s_1$$

$$@\, s_1 = \frac{B}{2} \qquad m = y\, t_B \frac{B}{2} = 0.786\ m^3$$



Next we find m on the side shell, The initial value for the side is the same as the final value for the bottom. The shear flow continues around the corner. We integrate along s_2 (note: $y = h - s_2$) ;

$$m(s_2) = m(s_1 = B/2) + \int_0^{s_2} y\, t_S \; ds$$

$$= 0.786 + \int_0^{s_2} (h - s_2)\, t_S \; ds$$

$$= 0.786 + h\, t_S\, s_2 - \frac{t_S\, s_2^2}{2}$$

This is a quadratic equation in s_2. To find the location of the maximum value, we set its derivative to zero;

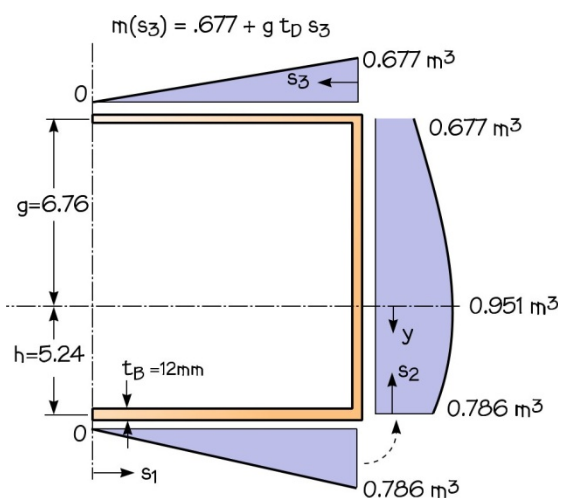
$$\frac{dm}{ds_2} = h\, t_S - \; t_S\, s_2 = 0$$

$$s_2 = h$$

This shows that the maximum shear flow is occurring at the neutral axis;

$$m(s_2 = h) = 0.786 + h^2\, t_S \; - \frac{t_S\, h^2}{2}$$

$$= 0.786 + \frac{.012\; 5.24^2}{2} = 0.951\; m^3$$



Continuing the integral to the deck gives;

$$m(s_2 = H) = 0.786 + \frac{t_S\, H^2}{2} = 0.677\; m^3$$

Next we continue the integral along the deck, along s_3, to the centerline;

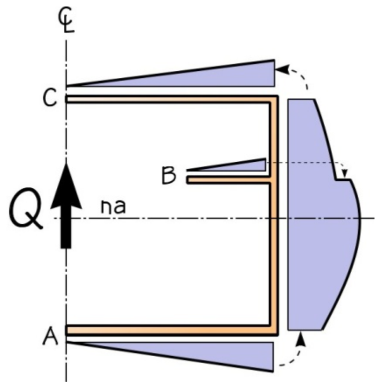
$$m(s_3) = 0.677 + \int_0^{s_3} y\, t_D \; ds = .677 + \; g\, t_D\, s_3$$

$$m(s_3) = 0.677 - 6.76 \; \cdot 0.01 \cdot s_3$$

$$@\; s_3 = \frac{B}{2} \quad m = 0.677 - 6.76 \; \cdot 0.01 \cdot 10 = 0\; m^3$$

With the shear force of 20 MN (about 2000 tonnes) The maximum shear stress is;

$$\tau_{max} = \frac{Q\,m}{I\,t} = \frac{20 \cdot 0.951}{21 \cdot 0.012} = 75.5\,MPa$$



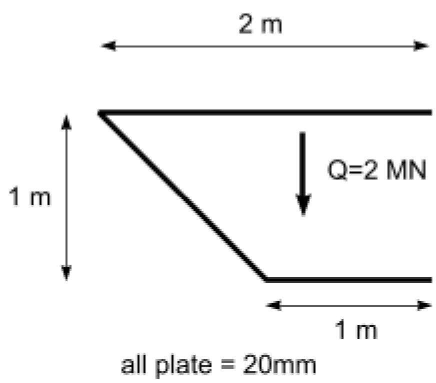
Branching Shear:

At a T junction, the shear flow branches. As long as there are no closed loops between the points of zero shear (ie. pts A, B and C in the sketch at left) the shear flow can be found easily. Such situations are statically determinate.

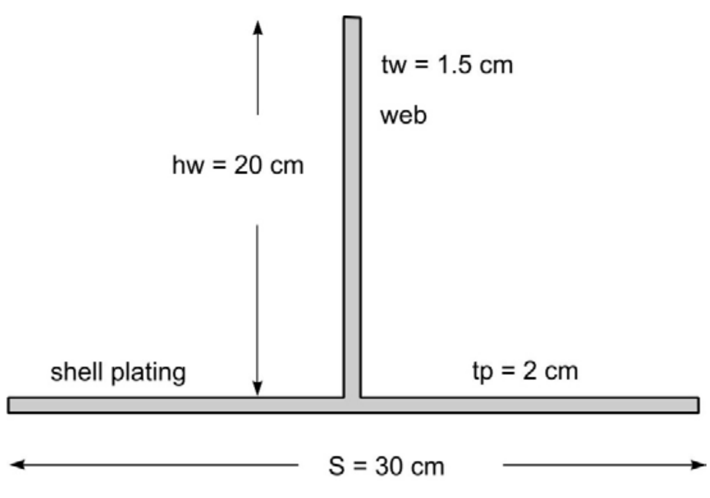
Topic 17: Problems

17.1 An open section is shown below. This is the cross section of a long folded steel plate. The cross section is subject to a shear force of 2 MN

Solve the shear flow, plot it and then also show the shear stress values.
If this is a section of a long cantilever (fixed at one end and free at the other) explain what types of deformations would you expect to see.



17.2 An open section is shown below. This is the cross section of transverse frame in a ship. The shear force of 200kN.
Solve the shear flow, plot it and then also show the shear stress values.
The web is welded to the shell plate. What shear force must be resisted at this joint?



transverse frame cross section

Topic 18: Shear Stresses in multi-cell sections

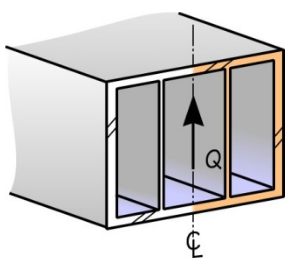
Croatian Coast

Introduction

In this Chapter we will

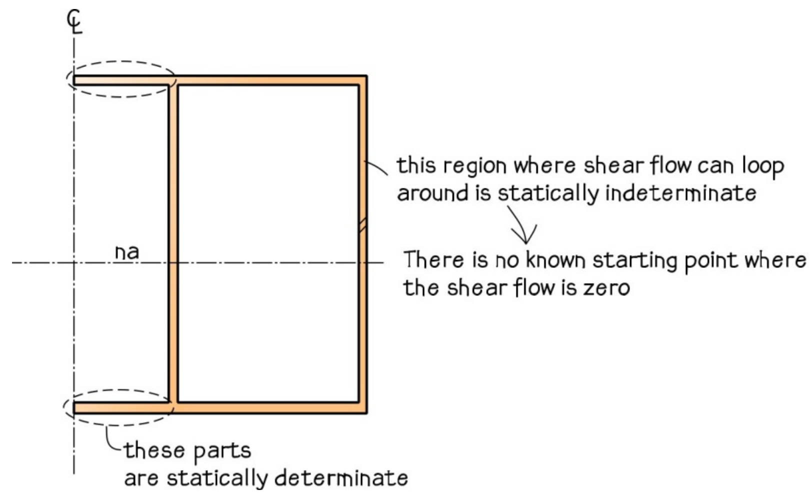
- Discuss indeterminate shear flow
- Calculate shear slip in a cut section.
- Do an example of shear flow in a ship

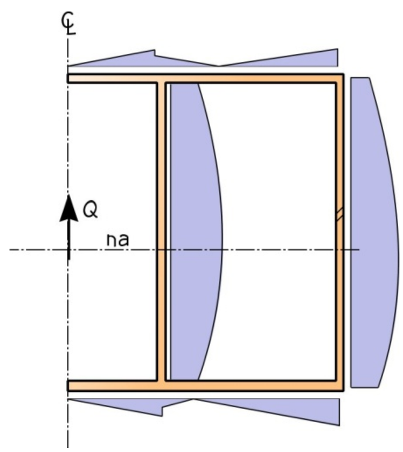
~~~~~~~~~~~~~~~~~~~~~
~~~~~~~~~~~~~~~~~~~~~



Shear in Multi-cell Sections

Consider a tanker with two longitudinal bulkheads;

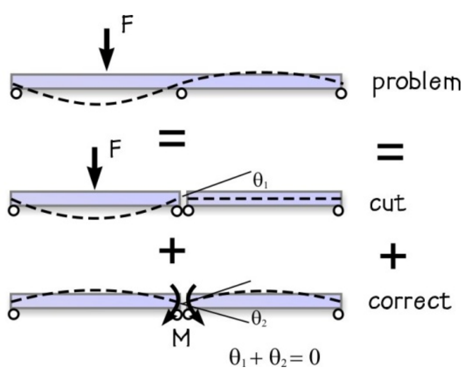




There will likely be two spots in the cell where m=0. The shear flow will look something like the sketch to the left.

To solve the statically indeterminate problem, we apply the same kind of technique that we used in the Force Method to solve indeterminate beams.

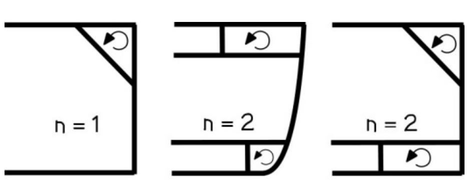
We will cut the structure, releasing the shear force and allowing shear deflection (called 'slip'). We will then determine how much shear we have to apply to the cell to remove the slip.



This is qualitatively similar to the correction of movements in the force method.

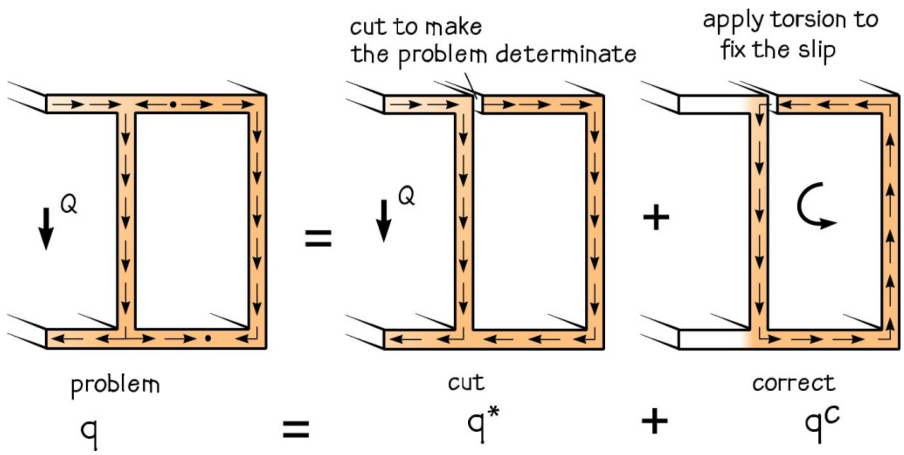
For any case where the loops are not adjacent, the <u>steps in the solution</u> process are;

1) Make n cuts to make the problem into a statically determinate problem.
2) Solve the statically determinate problem.
3) Find the N incompatible deflections (slips).

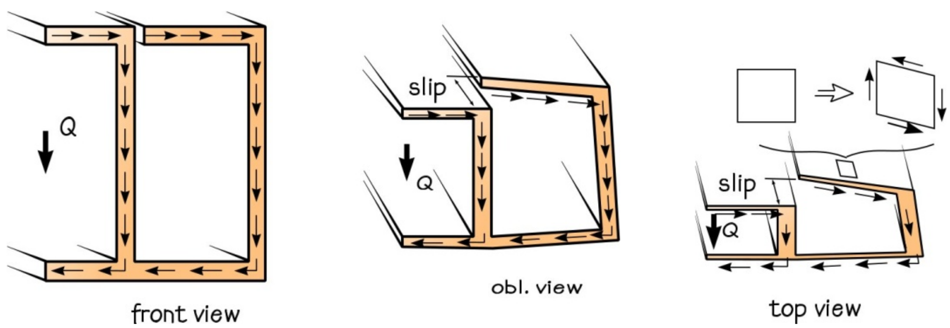


non-adjacent loops

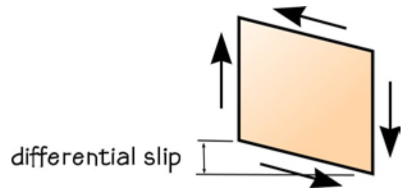
4) Apply N internal forces (actually torsions) to reverse the incompatible deflections
5) Add #2 and #4(s) to get the solution
The above steps are sketched below;



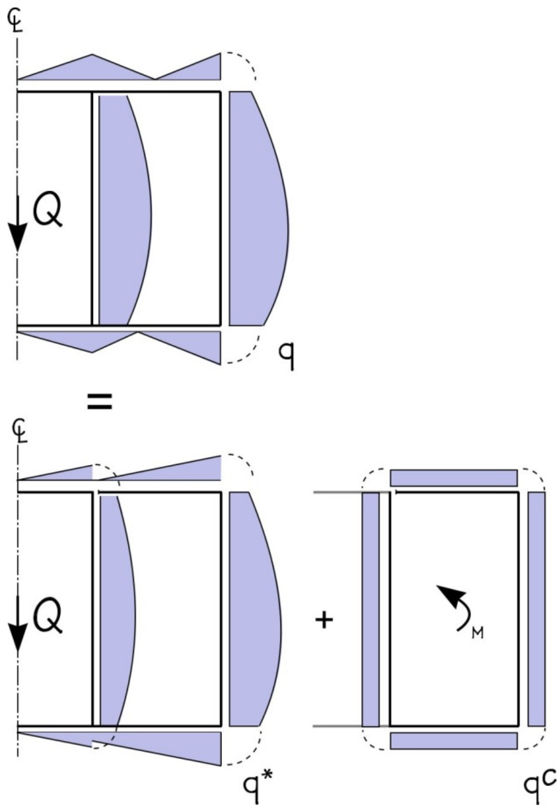
The cuts and the slip at the cuts are in the longitudinal direction;



The shear flow occurs on the cross section, which is a transverse vertical plane. The shear stresses on this plane will also occur on a longitudinal plane at right angles to the transverse plane. The longitudinal plane may be horizontal or vertical or inclined. The stressed plate will respond to the shear by distorting into a 'diamond' with relative movement in the longitudinal direction, which creates a differential slip over a small part of the cross section.



The total slip is found by integrating the slip over the whole loop from one side of a cut to the other. If the loop is symmetrical , the fore and aft slip will cancel out and result in no slip. In an unsymmetrical section there is a net slip.



$$slip = \oint \gamma \, ds$$

s = the path variable (length) around any loop
γ = shear strain $\gamma = \tau/G$
$\oint$ = a cyclic or loop integral

The slip can be found from the shear flow;

$$slip = \oint \tau/G \, ds = \frac{1}{G}\oint q/t \, ds$$

To correct the slip in a cut loop, we impose a correcting shear flow q^c , such that;

$$\frac{1}{G}\oint q^*/t \, ds \; + \frac{1}{G}\oint q^c/t \, ds \; = 0$$

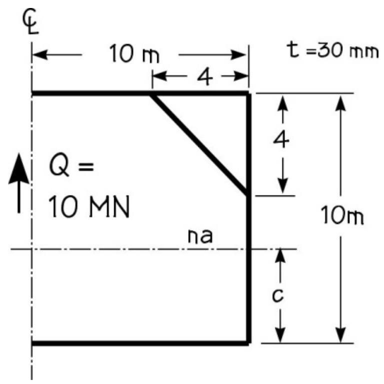
q^c is a constant so we can find it as;

$$q^C = \frac{-\oint \frac{q^*}{t} ds}{\oint \frac{1}{t} ds}$$

q^c is constant around the loop and zero elsewhere.
q^* is a determiate solution, found in the usual way.
The total solution is;

$$q \; = q^* + \; q^c$$

Shear Flow Example #2



Find the shear stresses in the section below. The total shear is 10MN (5 MN on the half section.

First we find the section properties:

Item	Desc.	w	h	Io	a	y	ay	a y^2
1	deck	10	.03	*	.3	10	3	30
2	w.t pl.		4	.23	.17	8	1.36	10.9
3	side		10	2.5	.3	5	1.5	7.5
4	bot.	10	.03	*	.3	0	0	0
Σ				2.73	1.07		5.86	48.38

The centroid and moment of inertias are (for half section) ;

$$c = \frac{\Sigma\ \mathrm{ay}}{\mathrm{A}} = \frac{5.86}{1.07} = 5.48\ \mathrm{m}$$

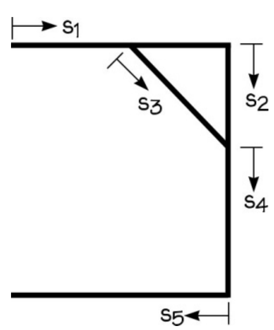
$$I_{base} = \Sigma\, Io + \ \Sigma\, a\, y^2 = 2.73 + 48.38 = 51.1\ \mathrm{m}^4$$

$$I_{na} = \mathrm{I_{base}} - \ A \cdot c^2 = 51.1 - 1.07\ \cdot 5.48^2 = 19.0\ \mathrm{m}^4$$

The shear flow and stress in the half section can be found from;

$$q = \frac{Q\, m}{I}$$
$$q = .2634\, m$$

$$\tau = \frac{Q\, m}{I\, t} = \frac{5}{19\ \cdot 0.03} \cdot m = 8.78\ \cdot m$$



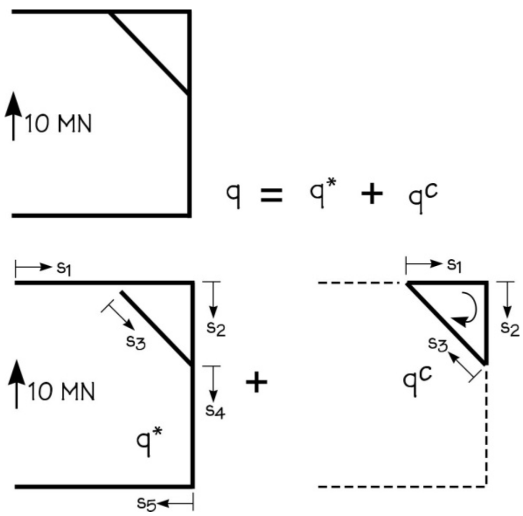
where

$$m = \int_0^s y\, t\ \ ds$$

So to find τ, we just need to find *m*. To find m we need to integrate along the 5 branches of the problem.

Because we have a loop, the problem is indeterminate and we need to cut the loop, find the slip and add a correcting shear flow.

The solution to the cut problem is called q*. The correcting flow is called q^c.



For s_1 (along deck);

$$m = 0 + \int_0^{s_1} y\,t\;ds$$

y = 10 - 5.48 = 4.52, y t = 0.1357
m = .1357 s_1
= 0.814 (@ s_1 = 6)
= 1.357 (@ s_1 = 10)

For s_2 (side shell above wing tank);

$$m = 1.357 + \int_0^{s_2} y\,t\;ds$$

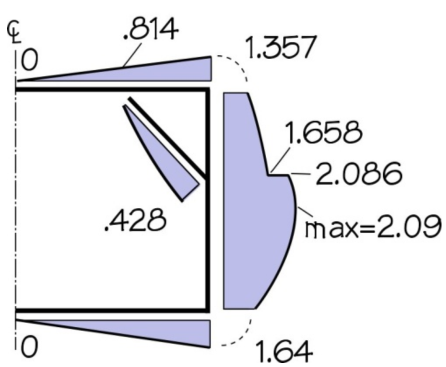
y = 4.52 - s_2 ,
m = 1.357 + .03 (4.52 s_2 - s_2^2 / 2)
= 1.357 + .1357 s_2 - .015 s_2^2
= 1.658 (@ s_2 = 4) (at wing tank plate)

For s_3 (inclined plate of wing tank);

$$m = 0 + \int_0^{s_3} y\,t\;ds$$

y = 4.52 - $s_3/\sqrt{2}$, s_3 = 0 -> $4\sqrt{2}$
m = .03 (4.52 s_3 - $s_3^2 / 2\sqrt{2}$)
= .1357 s_3 - .0106 s_3^2
= .428 (@ $s_3 = 4\sqrt{2}$) (at side shell)

plot of m* [m³]



For s_4 (side shell below wing tank);

$$m = .428 + 1.658 + \int_0^{s_4} y\,t\;ds$$

y = 0.52 - s_4 , s_4 = 0 -> 6
m = 2.086+ .03 (0.52 s_4 - s_4^2 / 2)
= 2.086 + .0156 s_4 - .015 s_4^2
= 1.64 (@ s_4 = 6) (at bottom)
= 2.09 (@ s_4 = .52) (max value at n.a.)

For s_5 (along bottom);

$$m = 1.64 + \int_0^{s_5} y\,t\;ds$$

y = -5.48 , s_5 = 0 -> 10
m = 1.64 - .164 s_5
= 0 (@ s_5 = 10) (at centerline) ok

Now we can calculate the corrective shear needed to close the slip that occurs at the wing tank cut;

$$slip^{*} + slip^{C} = 0$$

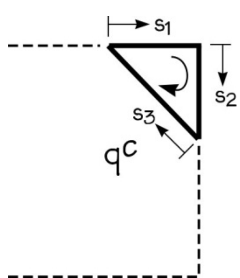
$$\frac{1}{G}\oint q^{*}/t\,ds \; + \; \frac{1}{G}\oint q^{c}/t\,ds \; = 0$$

q^{c} is a constant so we can find it as;

$$q^{C} = \frac{-\oint \frac{q^{*}}{t}\,ds}{\oint \frac{1}{t}\,ds}$$

In this case t is a constant so;

$$q^{C} = \frac{-\oint q^{*}\,ds}{\mathrm{S}}$$



where S is the length around the loop. S = 8 + 4√2. We can use the definition of shear flow to get;

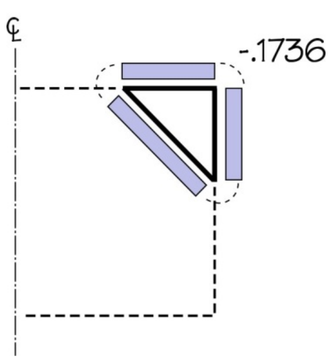
$$q^{C} = -\frac{\mathrm{Q}_{\frac{1}{2}}}{\mathrm{I}_{\frac{1}{2}}\,\mathrm{S}}\oint m^{*}\,ds$$

$$= -.01929\oint m^{*}\,ds$$

m^{*}_{deck} = .814 +.1357 s
m^{*}_{side} = 1.357 + .03 (4.52 s_2 - s_2^2 / 2)
m^{*}_{wt} = .03 (4.52 s_3 - s_3^2 / 2√2)

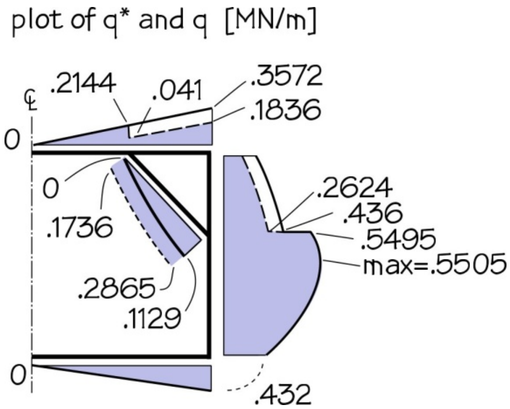
$$\oint m^{*}\,ds = \int_0^4 (.814 + .1357\,\mathrm{s})\,ds$$
$$+ \int_0^4 (1.357 + .03\,(4.52\,\mathrm{s} - \mathrm{s}^2/2\,))\,ds$$
$$- \int_0^{4\sqrt{2}} (.1357\,\mathrm{s} - .0106\,\mathrm{s}^2)\,ds$$
$$= 4.34 + 6.188 - 1.53$$
$$= 9.00$$

plot of q^{C} [MN/m]



Note that the m*wt part is subtracted beacuse we are integrating in the reverse direction. With m* we can calculate q^{C};

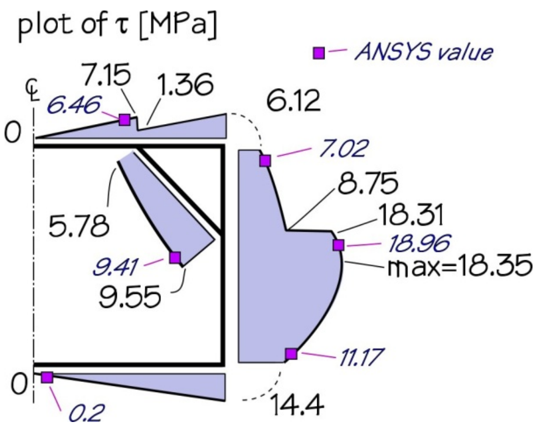
$$q^C = -0.1736 \text{ [MN/m]}$$



We have m* and q^C.

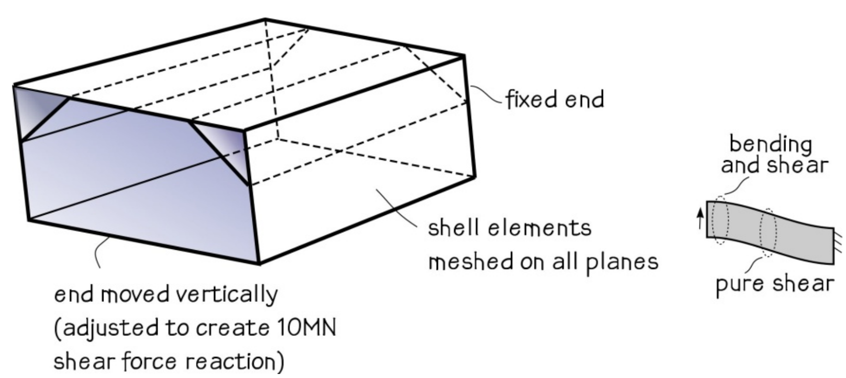
$$q = q^* \pm q^C = 0.2364\ m^* \pm q^C$$

We can plot q^* (solid lines) and q (dashed lines);



and we can plot the shear stress τ;

The values of shear stress have been checked against an ANSYS model, and show good, though not perfect, agreement. A sketch of the ANSYS model is shown below.



See next page for ANSYS results.

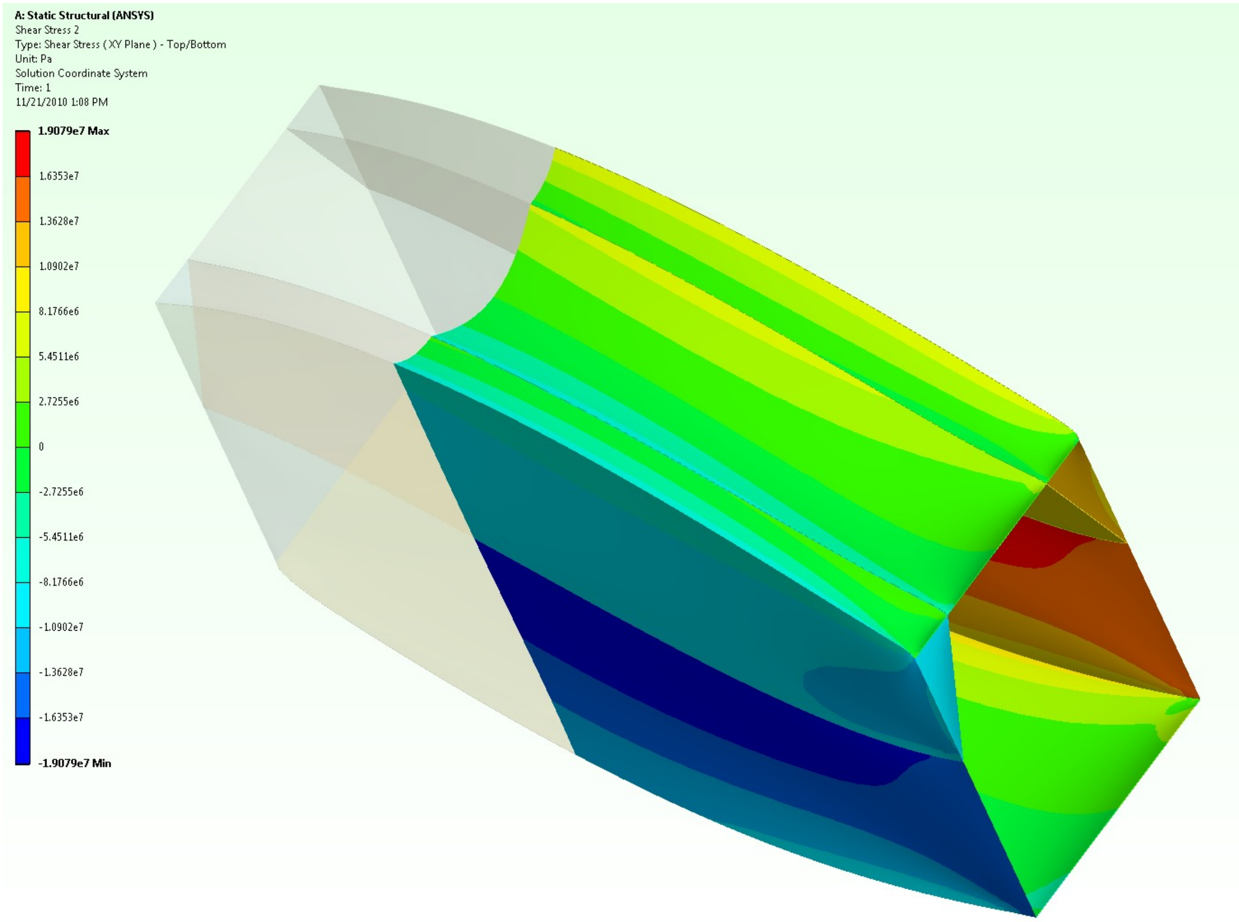


image from ANSYS

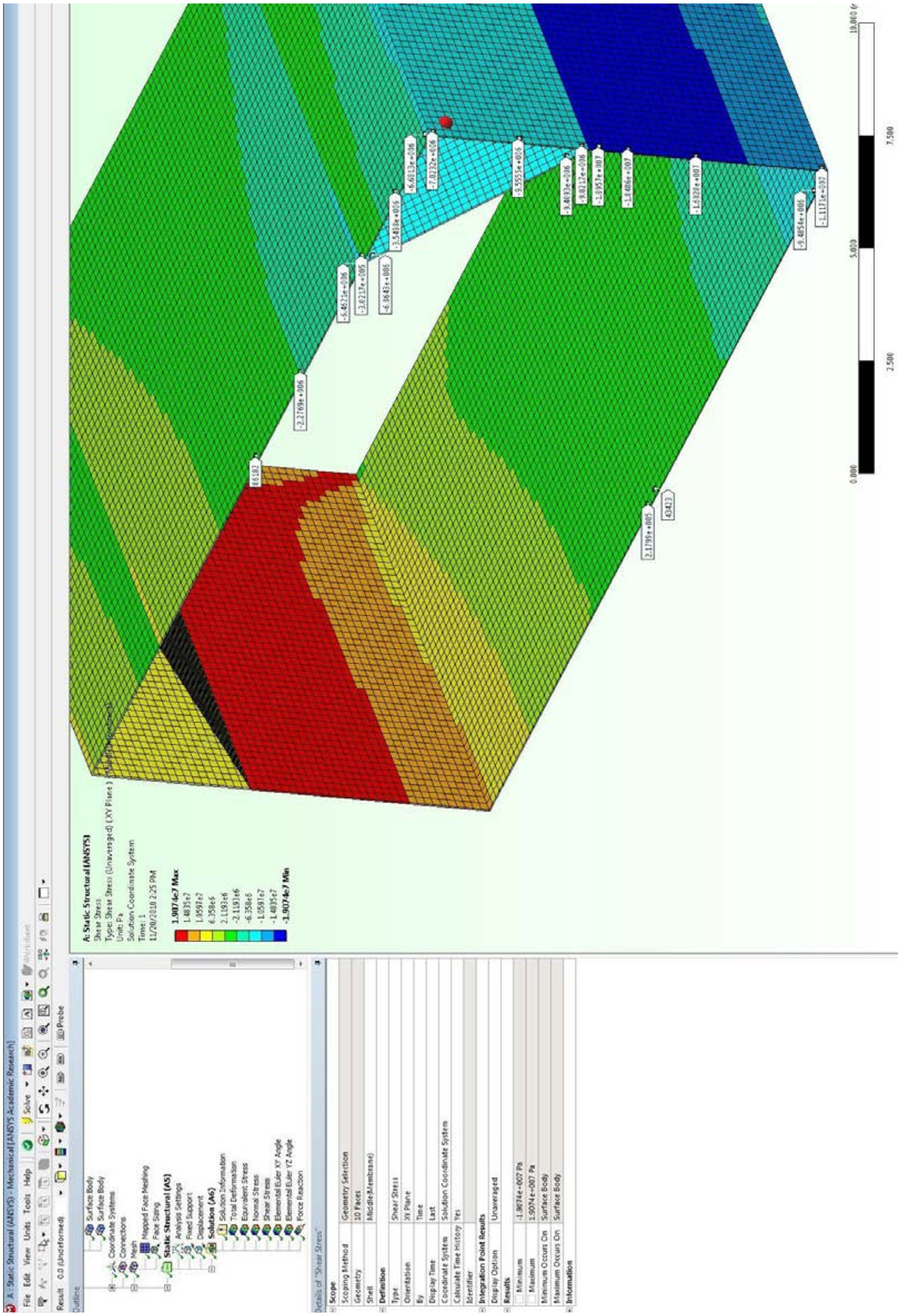
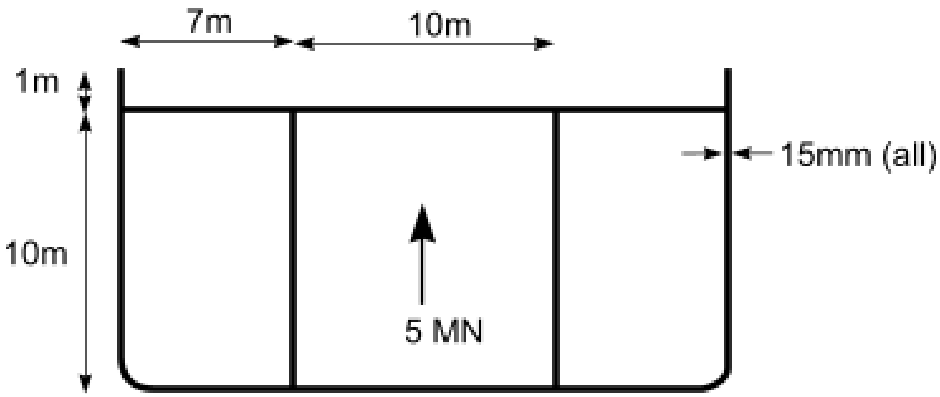


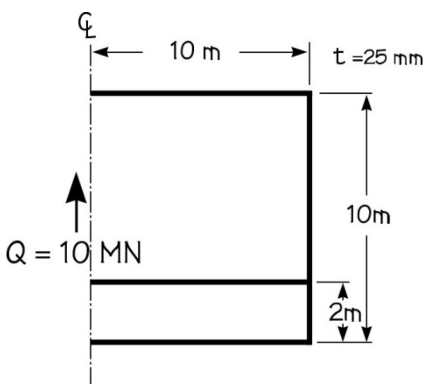
image from ANSYS Workbench

Topic 18: Problems

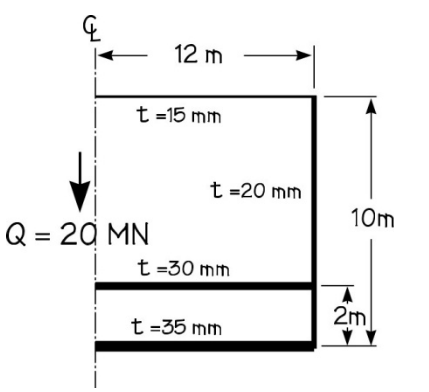
18.1 Solve the shear flow in the following section of a tanker. Ignore the radius of the bilge.



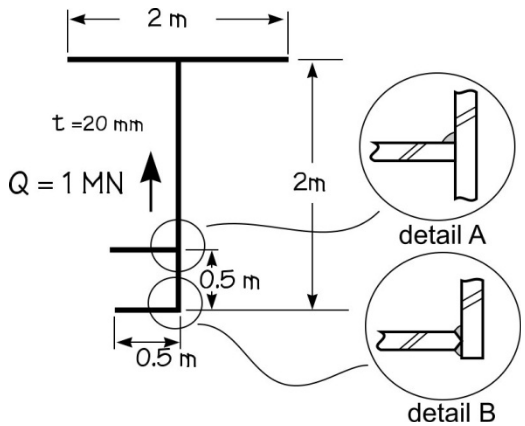
18.2 Solve the shear flow in the following section of a tanker.



18.3 Solve the shear flow in the following section of a tanker.

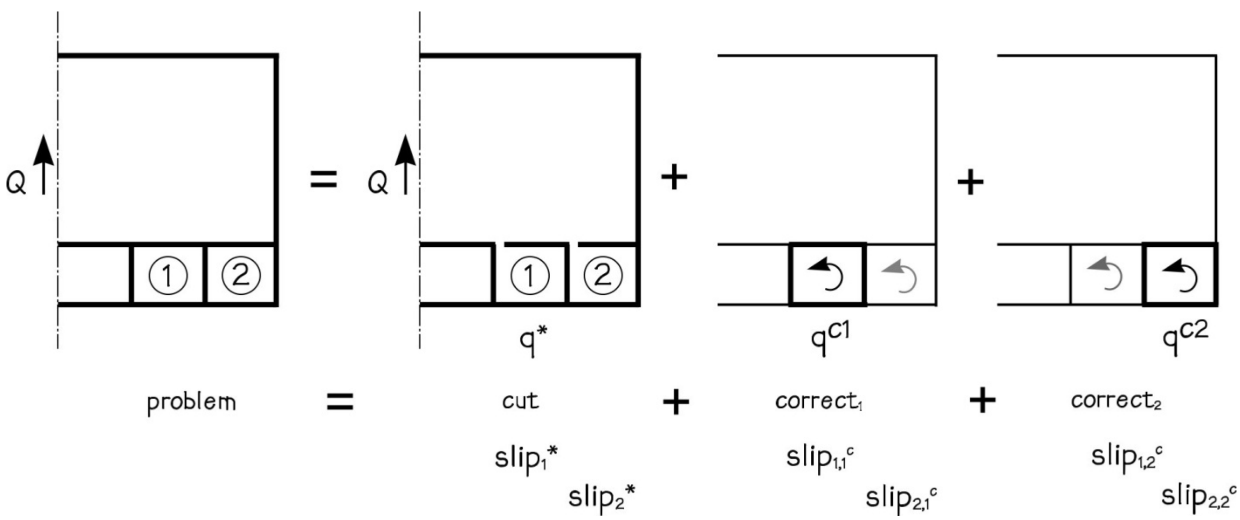


18.1 Solve the shear flow in the following frame section. What are the shear forces transferred through the welds in details A and B (in kN/m)?



Topic 19: Shear Flow in adjacent Closed Cells

In a double sided or double bottom vessel there are often many adjacent closed cells. Adjacent close cells present an added complexity when solving shear flow. The complexity is that the corrective shear flow in once cell causes a corrective slip in the adjacent cell, because of the common side.



When we add a corrective shear flow in one loop we can't help but get some flow and slip in adjacent loops.

Consequently, in order to ensure that we have no net slip at each and all cuts we need to satisfy a set of coupled equations. For example, in the case of two adjacent loops we have;

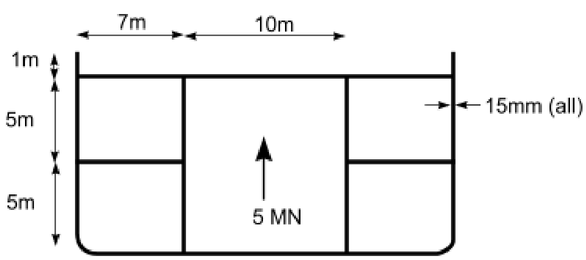
$$\oint_{cell\,1} \frac{q^*}{t} ds + \oint_{cell\,1} \frac{q^{c1}}{t} ds + \oint_{cell\,1} \frac{q^{c2}}{t} ds = 0$$

$$\oint_{cell\ 2} \frac{q^*}{t} ds + \oint_{cell\ 2} \frac{q^{c1}}{t} ds + \oint_{cell\ 2} \frac{q^{c2}}{t} ds = 0$$

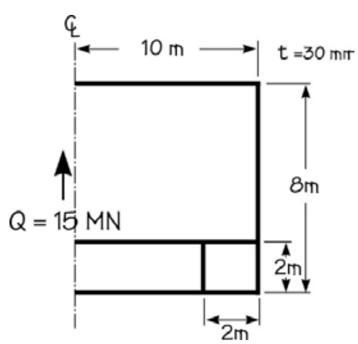
q^{C1} and q^{C2} are unknown constants. q^* is the determinate shear flow in the cut section. For N adjacent closed cells, we have to solve N simultaneous equations.

Topic 19: Problems

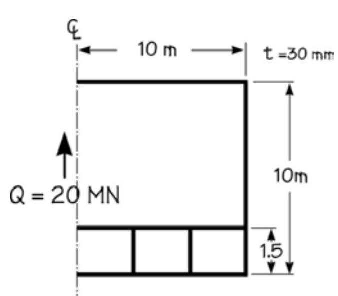
19.1 Solve the shear flow in the following section of a tanker. Ignore the radius of the bilge.



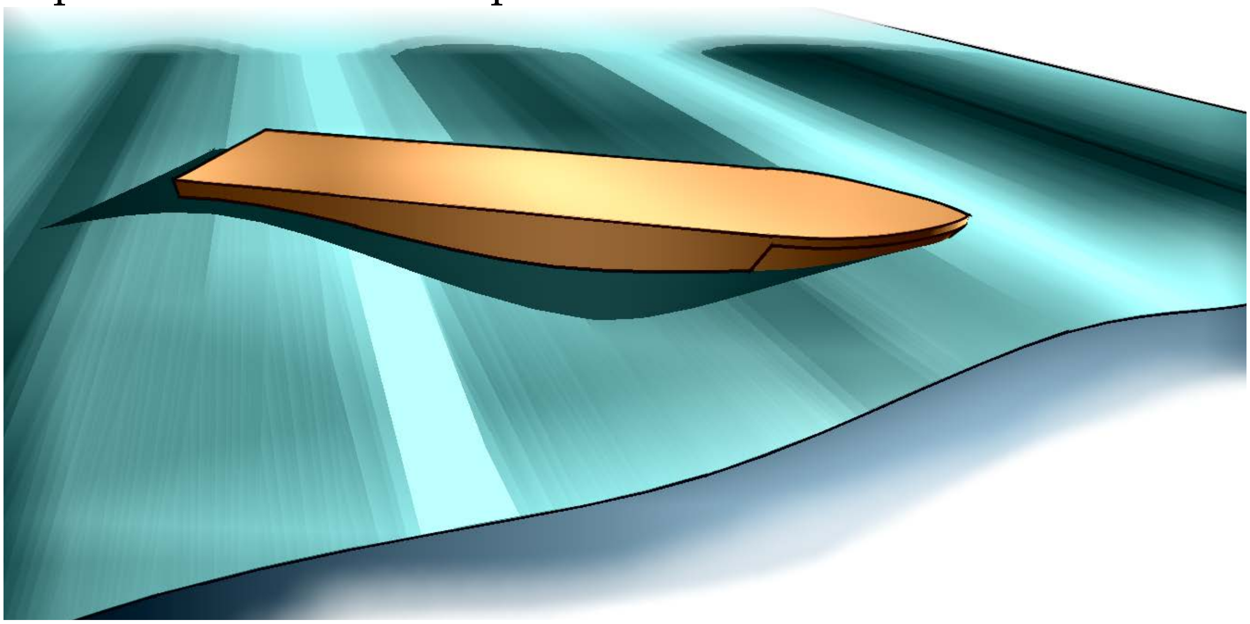
19.2 Solve the shear flow in the following section of a tanker.



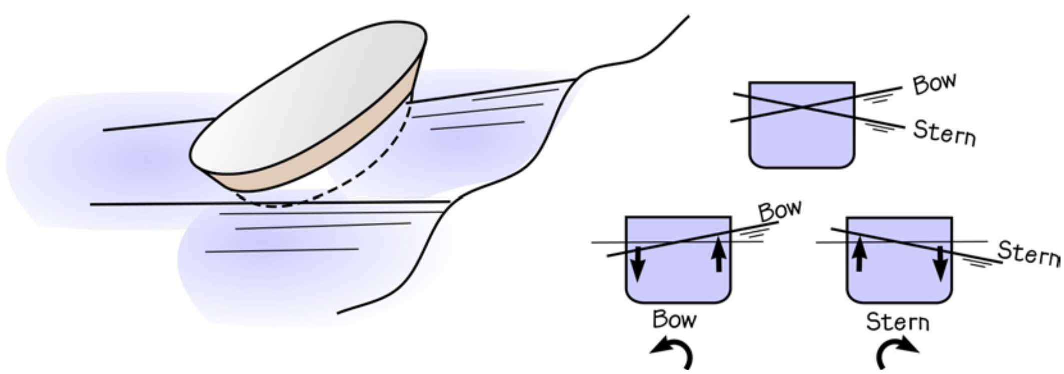
19.3 Solve the shear flow in the following section of a tanker.



Topic 20: Torsion in ships



Ships as a whole and many individual members within ships experience torsion.



The overall design torsional moment is given in various classification society rules;

Torsion – GL Rules

3.5 Torsional moments

The maximum wave induced torsional moment is to be determined as follows:

$$M_{WT\max} = \pm L \cdot B^2 \cdot C_B \cdot c_0 \cdot c_L \cdot \left[0{,}11 + \sqrt{a^2 + 0{,}012}\right] \quad [\text{kNm}]$$

$a = \sqrt{\dfrac{T}{L}} \cdot \dfrac{c_N \cdot z_Q}{B}$

a_{min} = 0,1

c_N = see 3.4

z_Q = distance [m] between shear centre and a level at

$$0{,}2\,\frac{B \cdot H}{T}$$

above the basis

Torsion – NK Rules

2.4 Torsional Moment

2.4.1 Still Water Torsional Moment

The torsional moment in still water, M_{ST}, due to unbalanced loading of containers is estimated using the equation given below as a standard. The distribution of the torsional moment in the longitudinal direction of the ship is to be taken similar to the distribution in waves as given in 2.4.2.

$$M_{ST} = 0.23 L N_R W_C \quad (kN\text{-}m)$$

N_R : Maximum number of rows of cargo containers loaded in the cargo hold

W_C : Mean weight of a 20*ft* container to be loaded. Usually taken as 100*kN*

2.4.2 Wave-induced Torsional Moment

Wave-induced torsional moment can be determined from the equation given below.

$$M_{WT1} = M_{WT} \cdot C_{T1}$$

$$M_{WT2} = M_{WT} \cdot C_{T2}$$

$$M_{WT} = 1.3 C_1 L d_f C_b \cdot (0.65 d_f + e) + 0.2 C_1 L B^2 C_w$$

$$e = e_1 - \frac{d_0}{2}$$

$$e_1 = \frac{(3D_1 - d_1) d_1 t_d + (D_1 - d_1)^2 t_s}{3 d_1 t_d + 2(D_1 - d_1) t_s + B_1 t_b / 3}$$

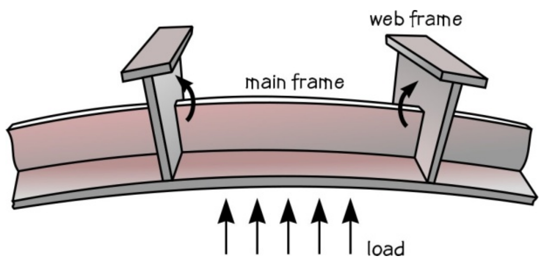
d_0 : Height of double bottom (*m*)

d_1 : Breadth of double side (*m*)

$$D_1 = D_S - \frac{d_0}{2}$$

$$B_1 = B - d_1$$

t_d, t_s, t_b : Mean thickness of deck, side shell and bottom shell plating (*m*) respectively. The range of each is given in Fig. 2.2. The mean plating thickness may also be determined after including the longitudinal strength members within each range.



Local structural torsion can be found throughout ships. Bending of a frame can result in a torsion in a supporting frame.

Torsion Review

Consider a solid circular shaft subject to a torsional moment. The longitudinal axis of the cylinder x axis. A torsion is a moment about the x axis. In such a case we get an ideal torsional response. Every circular cross section remains plane and remains centered on the x axis. Each plane rotates slightly in comparison to its neighboring cross sections. Assume that two planes (1 and 2) are separated by a distance dx. In comparison to their original orientations, the planes are rotated

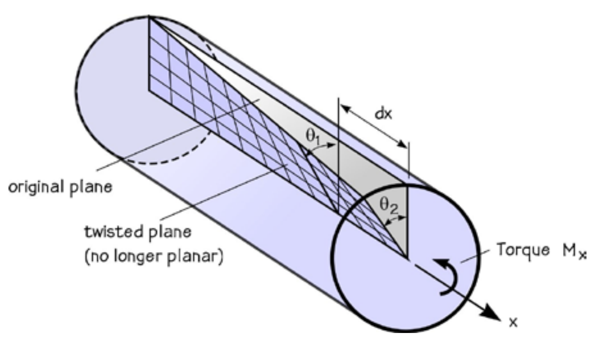
$$d\theta = \theta_1 - \theta_2$$

$$d\theta = \frac{M_x}{G\,J}\,dx$$

or

$$M_x = G\,J\,\frac{d\theta}{dx}$$

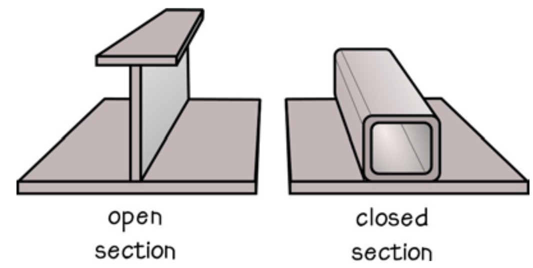
note similarity to the deq. for bending: $M_y = E\,I\,\frac{d\theta}{dx}$

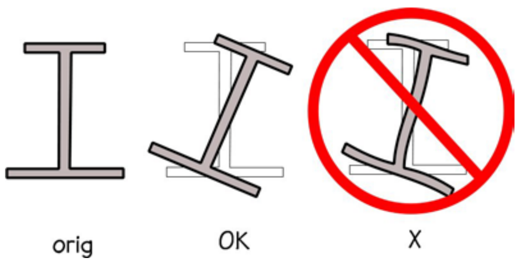


For solid sections like the circular shaft shown at left, the shear stress is;

$$\tau = \frac{M_x r}{J}$$

$$J = \frac{\pi\,r^4}{2}$$





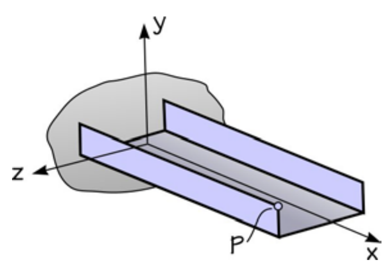
Thin Walled Torsion

Torsion in thin walled sections differs greatly between 'open' and 'closed' sections.

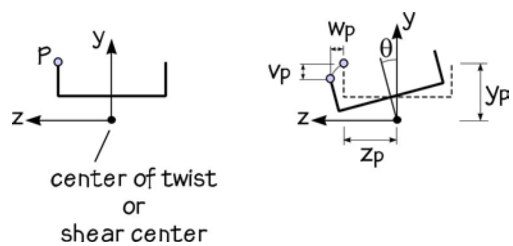
To examine the difference between open and closed sections we first make some simplifying assumptions;

- sections are prismatic
- no in-plane deformation (cross sections only rotate)
- small out of plane deformations (warping)

Thin Walled Torsion – Open Sections



Consider an open section, built-in at its base and subject to a torsion at the free end.

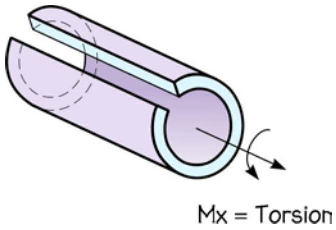


The section rotates about a point called the shear center. Point 'p' moves in the y and z direction due to rotation and in the x direction due to 'warpage'.

The displacements of point 'p'

$$x: \quad u_p = \overbrace{w_n(y,z)}^{warpage\ function} \ \theta' \qquad \theta' = {d\theta}/{dx}$$

$$\left.\begin{aligned} y: &\quad v_p = -\theta\, z_p \\ z: &\quad w_p = \theta\, y_p \end{aligned}\right\} small\ rigid\ body\ rotations$$



For ideal open sections with no warping restraint;

$$M_x = G\,J\,\frac{d\theta}{dx}$$

J = St. Venant torsional constant

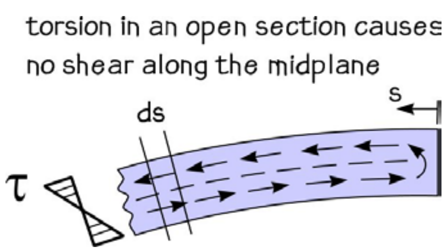
For an open section;

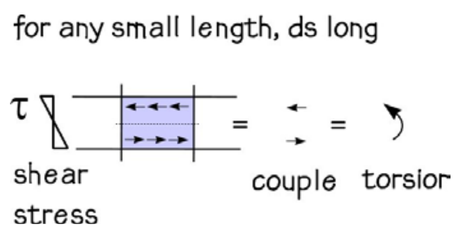
$$J = \frac{1}{3}\int_0^b t^3\,ds$$

For example, for a pipe of thickness t, radius r, cut longitudinally;

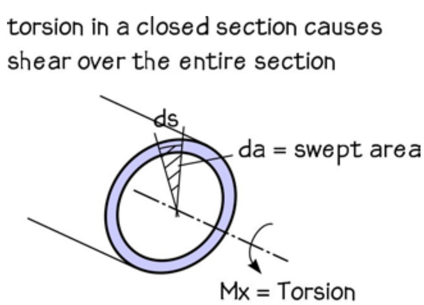
$$b = 2\pi r$$

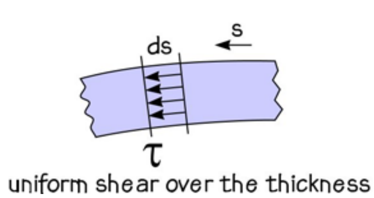
$$J = \frac{1}{3}\int_0^{2\pi r} t^3\,ds = \frac{2\pi\,r\,t^3}{3} = 2.09\,r\,t^3$$





Thin Walled Torsion – Closed Sections



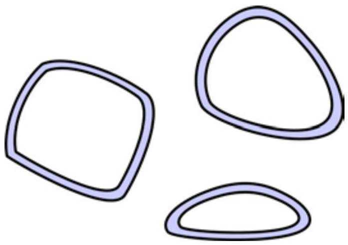


Closed sections carry torsion in an entirely different way from open sections. Because the loop is closed, shear can flow around the loop. The shear stress is uniform over the full thickness of the wall. The shear flow is also constant over the full loop. Once again;

$$M_x = G\,J\,\frac{d\theta}{dx}$$

We can also write;

$$M_x = \int_0^{2\pi\,r} \tau\,t\;\cdot r\,ds$$
$$= \int_0^{2\pi\,r} 2q\,da$$
$$note\!:\ \tau\,t = q,\ \ r\,ds = 2\,da$$



As q is constant we can write;

$$M_x = 2\,q \int_0^{2\pi r} da$$
$$= 2\,qA$$

where;
A = enclosed area of the loop

For a pipe (a circle);

$$M_x = 2\,q\,\pi\,r^2$$

Using the general formula for torsion;

$$\tau = \frac{M_x\,r}{J_{closed}} \qquad (similar\ to\ \sigma = \frac{M_y\,c}{I_{na}})$$

We can use this to find J_{closed}

$$J_{closed} = \frac{M_x\,r}{\tau} = \frac{2\,q\,\pi\,r^3}{\tau} = 2\,\pi\,t\,r^3$$

Compare this to J_{open};

$$J_{open} = \frac{2}{3}\,\pi\,r\,t^3$$

For example, consider a pipe of 1m dia., with a 10mm wall thickness;

$$J_{closed} = 2\,\pi\,t\,r^3 = 2\,\pi\,\cdot 0.01\,\cdot 1^3 = 0.062\,m^4$$

$$J_{open} = \frac{2}{3}\,\pi\,r\,t^3 = \frac{2}{3}\,\pi\,\cdot 1\,\cdot 0.01^3 = 2x10^{-6}\ \ m^4$$

$$J_{open} = \frac{J_{closed}}{J_{open}} = 29{,}600$$

The difference is so dramatic that it is easily illustrated by seeing what happens when a cardboard tube (eg paper coffee cup) is cut open longitudinally.

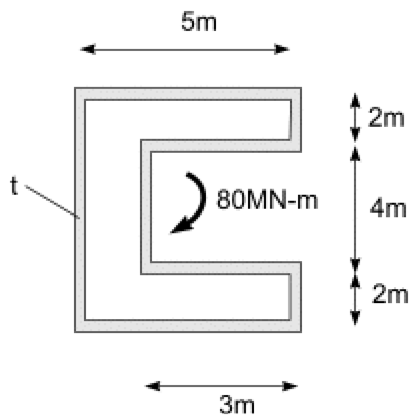
Thin Walled Torsion – warpage restraint

Warpage is the term to describe axial displacements due to torsion. In a closed circular section the axial symmetry prevents all warpage. In an open circular section, the warpage is unrestrained (ie. The section is free to warp), so no warpage stresses arise.

In sections with corners such as a box section, the twist of one face is, to a degree, incompatible with the twist of the connecting face. Each face wants to warp differently, but is constrained at the corner. This results in stresses on both faces. The treatment of these effects requires the use of warpage functions. This topic will not be considered any further here. We will limit our attention to simple torsion theory.

Topic 20: Problems

20.1 A hollow closed section is made of plate of uniform thickness 't' . A torsional moment of 80 MN-m is applied. To have the maximum shear stress equal to 135 MPa, what value should *t* be?



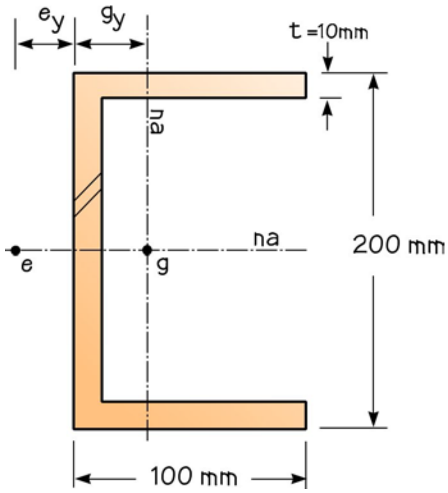
Topic 21: Shear Center and Shear Lag in Ship Structures

Topsides supports on an FPSO

Introduction

In this Chapter we will

- Discuss the idea of the shear center of a frame
- Describe the idea of shear lag and the notion of effective width.

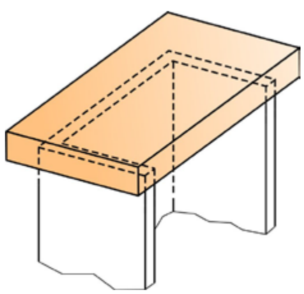


Consider a channel section. A channel is a common structural profile, but one that is asymmetric on one axis. The center of area (centroid) and the 'shear center' are not at the same location.

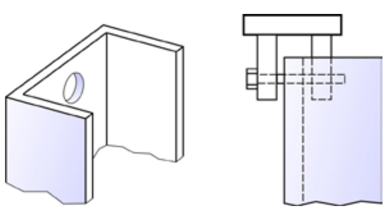
What is the centroid? For one thing, it is a property of the cross sectional area. But what does it mean for the channel section? If we were to want to use the section as a column and apply an axial force that would only compress (and not bend) the column, we would apply the force at the centroid 'g'. This is because a uniform stress in the cross section would have a 'center of force' at 'g'.

To find 'g' we use the standard formulations;

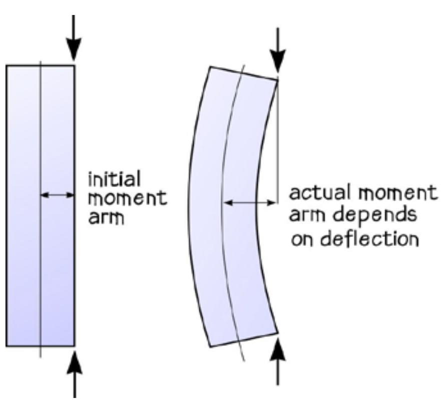
$$g_y = \frac{\Sigma\, a\, y}{A} = \frac{1800 \cdot 5 + 2000\ \cdot 50}{3800} = 28.7\ mm$$



If the end of the column had an end cap, the load would naturally find its way to the centroid.



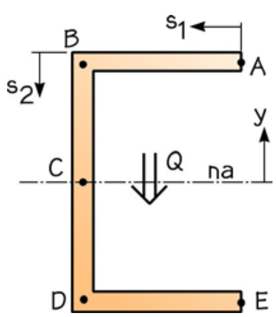
However if the end were connected with a bold through the web, the load would be applied off the centroid and the axial load would cause bending. In this case the end load would not only cause bending, but the bending deflection would increase the moment arm to further increase the bending. This is a kind of self-excited response called the p-delta effect, and is the subject of a special analysis.



The above discussion is about axial loads. What is the connection to shear? The connection is the idea of the shear center. When a load is applied at the shear center of a beam, the load will only cause shear and bending, and no torsion. If the load is applied anywhere else, a torsion will result.

Consider a shear force Q =10000 N applied to the channel section on the previous page.

$$I_y = \frac{1}{12}\,(100\cdot 200^3 - 90\cdot 180^3)$$
$$= 22{,}927{,}000\ \ mm^4$$



We will need Q/I;

$$\frac{Q}{I_y} = 0.0004362$$

Now we find the values of m. On the top flange;

$$m_1 = \int_0^{s_1} y\,t\,ds = 950\ s_1$$

$$q_1 = \frac{Q}{I_y} m_1 = 0.4144\ s_1$$

So at B;

$$q_B = 0.4144\ \cdot 95 = 39.36\ N/m$$

The force on the top flange is;

$$F_{tf} = \int_0^{95} q_1\,ds$$

$$\text{or} \qquad = \frac{1}{2} q_B\ 95$$
$$= 1870\ N$$

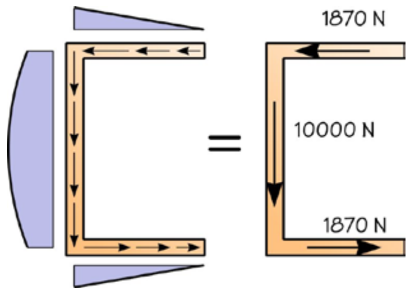
In the web;

$$q_2 = 39.36 + \frac{Q}{I_y}\int_0^{s_2} y\,t\,ds$$
$$= 39.36 + .0004362\ \cdot 10\ \int_0^{s_2} (95 - s_2) ds$$
$$= 39.36 + .0004362\ \cdot 10\ (95\ s_2 - \frac{s_2^2}{2})$$

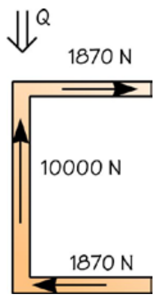
The force on the web is;

$$F_w = \int_0^{190} q_2\ ds$$
$$= 39.36\cdot 190 + .004362\left(95\frac{190^2}{2} - \frac{190^3}{6}\right)$$

$$= 9978\ \ (\cong 10{,}000\,)\ \ OK$$

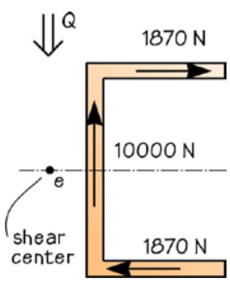


The lower flange is symmetrical with the upper and will have a shear flow of the same magnitude but opposite in direction.



The shear flow as drawn shows the directions of shear in the direction of the applied force. If we think instead of the reaction to the applied force, we have the sketch at left.

In this case the applied force is shown pushing directly down on the web. In this case the vertical forces oppose each other and produce no moment. However, the horizontal forces, while equal in magnitude, are separated by 190mm and produce a couple of 1879 x 190 = 355300 N-mm. This couple is a torsion acting on the section.

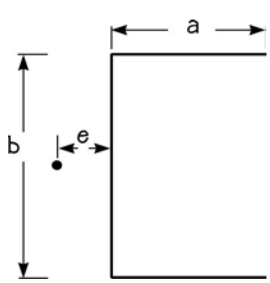


In order to eliminate the torsion, we would need to apply the load Q at the shear center 'e' to the left of the web. We can find the location of 'e' as follows;

$$e_y = \frac{F_{tf} \cdot 190}{Q} = \frac{35530}{10000}$$
$$= 35.53\ mm\ (to\ cent.of\ web)$$

General formula for shear centers of channels

The following derivation is only valid for symmetrical channels with constant wall thickness.



$$t = constant, \ll a$$

$$I = \frac{1}{12} t\, b^3 + 2\, a\, t\, \left(\frac{b}{2}\right)^2$$

$$q_1 = \frac{Q}{I} \int_0^{s_1} y\ t\ ds$$
$$= \frac{Q}{I} y\ t\ s_1$$

The force in the top flange;

$$F_f = \overline{q_1}\, a = \frac{Q}{I} y\, t \frac{a}{2}\, a$$

$$= \frac{Q\, b\, a^2\, t}{4\, I}$$

Now we can find ‘e’ by setting the sum of the torsional moments to zero. The flange forces create one couple and the applied load, opposed by the reaction in the web, creates another couple. The two couples will sum to zero when the load is applied at the shear center.

$$Q\, e = F_f\, b \quad (balance\ moments)$$

$$e = \frac{Q\, b\, a^2\, t}{4\, I} \frac{b}{Q} = \frac{b^2 a^2 t}{4\, I}$$

$$e = \frac{b^2 a^2 t}{4} \frac{1}{\frac{1}{12} t\, b^3 + 2\, a\, t\, \left(\frac{b}{2}\right)^2}$$

$$e = \frac{a}{2\left(\frac{b}{6a} + 1\right)}$$

For the previous example

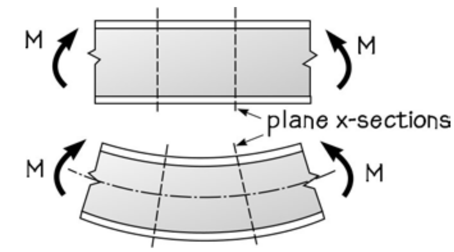
$$a = 95, \qquad b = 190$$

$$e = \frac{95}{2\left(\frac{190}{6 \cdot 95} + 1\right)} = 35.6\, mm$$

(Q? – why would there be a slight difference between the above result and the previous example?)

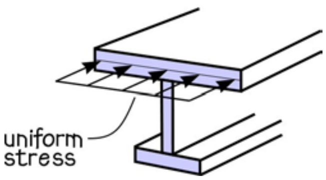
Shear Lab / Effective Width

We normally assume that bending in a frame of a ship or the hull girder can be modeled with what we call ‘simple beam theory’. This means that we assume that as the beam bends, plane sections remain plane. When we make this assumption, we



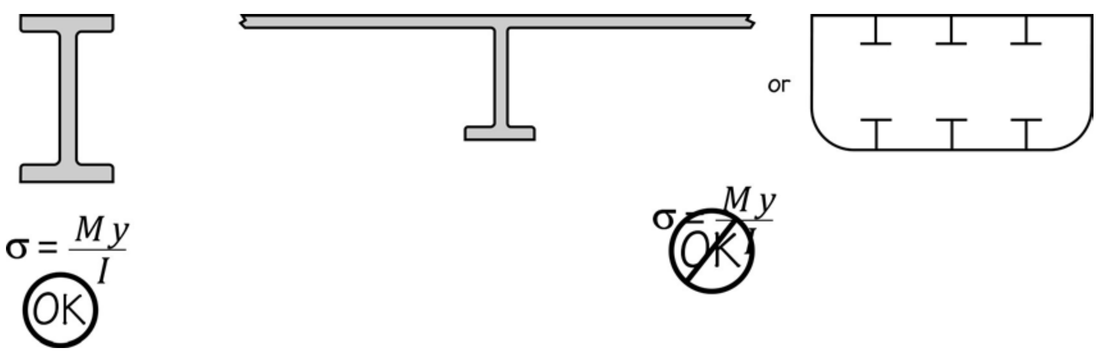
are implicitly assuming that the flange is uniformly compressed (or stretched), and that the compressive or tensile stresses are uniform in the flanges. Recall that 'standard' formula;

$$\sigma = \frac{M\,y}{I}$$

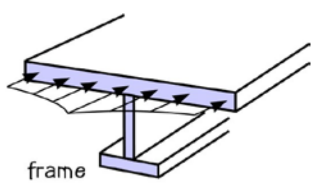


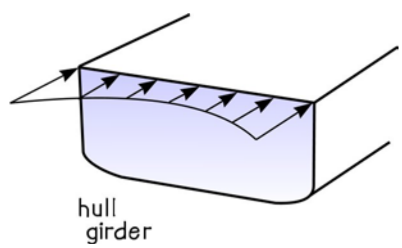
This formula says that all stresses at the same value of y will be the same (i.e. all stresses in the flange are the same!).

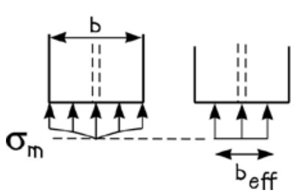
While the simple beam assumption is ok for beams with relatively narrow flanges, the assumption is not valid for sections with wide flanges such as are sometimes found in ships.



In the case of very wide flanges, the compressive stresses drop off away from the web.

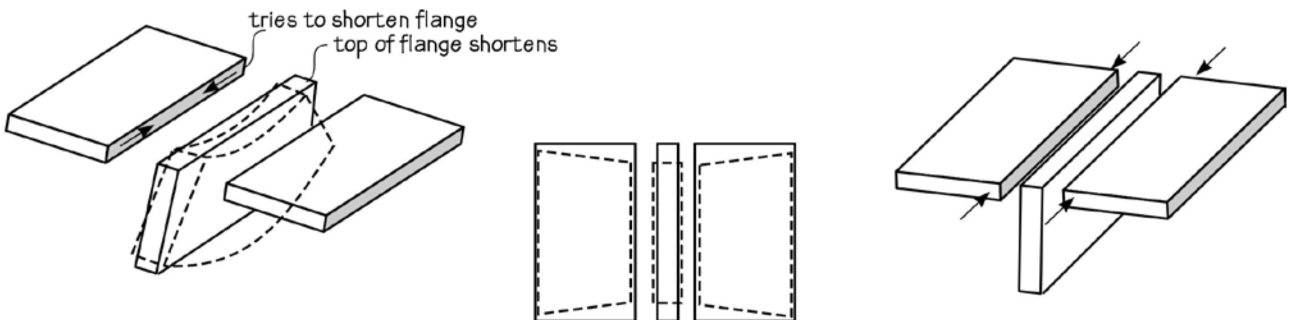






To find the true pattern of flexural stress in a wide flange beam, and the consequent effective width, is a complex analysis, easily done in a finite element model, but difficult to obtain analytically. The idea of the behavior is presented below.

When we a lateral load (a bending load) to a beam or ship frame, the web carries the load and tends to bend. The flange is attached at the edges of the web and as the web bends, its edge shortens (or lengthens) and tends to pull the flange with it. To pull on the flange, a shear stress of applied to the edge of the flange. As shown in the sketch, the flange is acted upon along its edge. Its as if the flange is pinched along its edge, causing the flange to compress more near the web and less away from the web.



Unfortunately there are no general analytical solutions for shear lag and effective width. Certain approximate solutions have been postulated (see PNA, VI, pp 247-250)

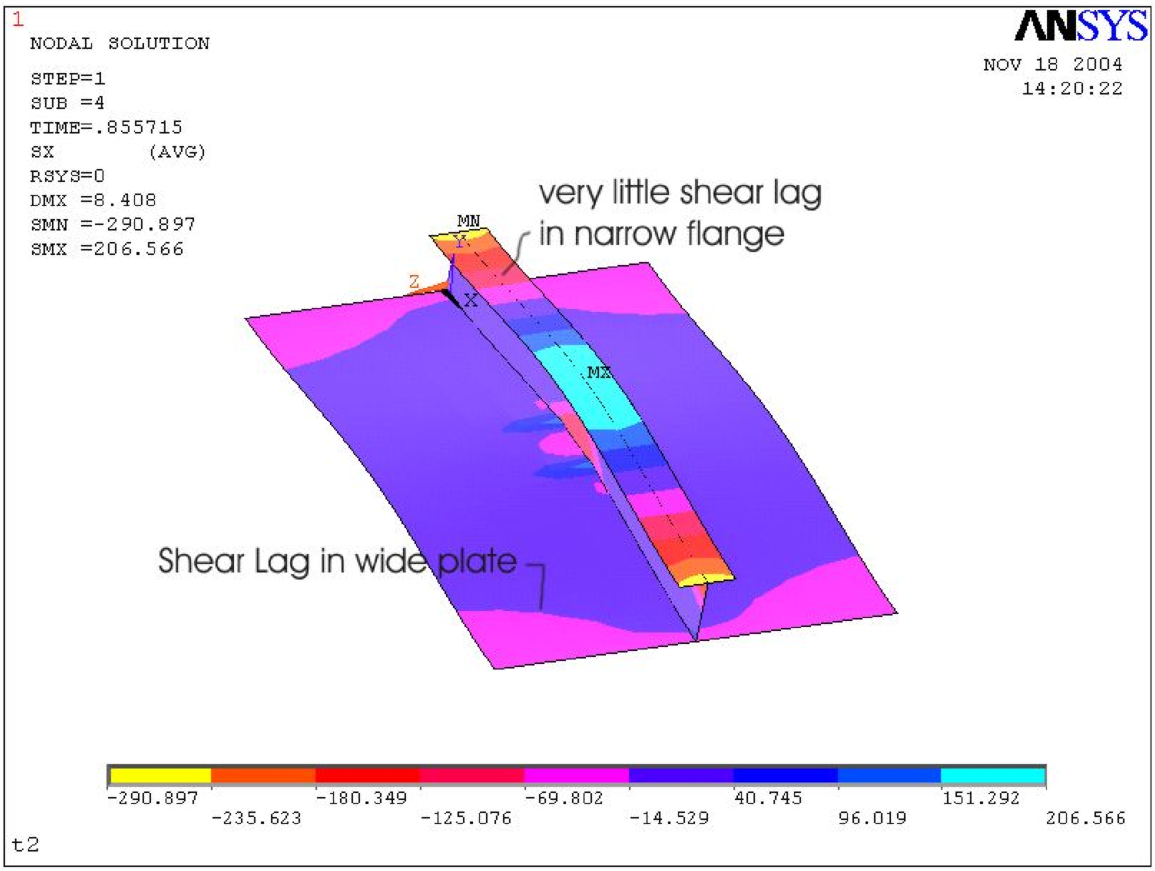
Shear lag and diminished effective width are most important in cases of ;

- wide flanges (large b)
- short frames (small L/b)
- proximity to free ends
- proximity to concentrated loads

Finite element programs, when shell or brick elements are used to model the frames, will naturally show the shear lag effects.
There have been experiments on hull girder models that have shown not only a variation in deck stresses, but actual stress reversals. This means that even when the average deck stress is compressive, there may be a part of the deck (at center) where the stresses are tensile, with the

deck edges in exaggerated compression. (PNA p 250)

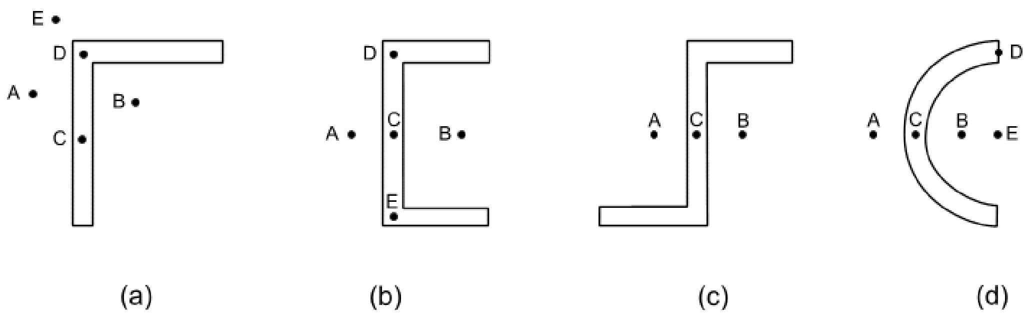
Classification society rules have various approaches to account for effective width. We will consider deck plate buckling in the next ship structures course (6003). In that case we will consider another type of effective width of plating, but one that describes a buckled plate's reserve capacity.



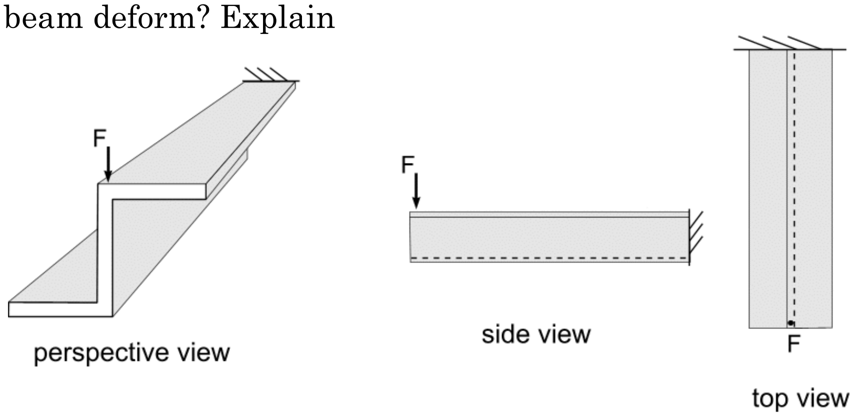
ANSYS analysis results

Topic 21: Problems

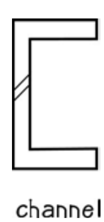
21.1 The following figure shows 4 x-sections. Identify the location of the shear center in each case (i.e. which letter?). You should sketch the shear flow to help identify the location.



21.2 When the vertical force F is applied to this section, how will the cantilever beam deform? Explain



21.3 Where is the shear center of a 300 x 150 x 15fl x 10w mm ?



Topic 22: Plate Bending

Wexford Ireland

Introduction

In this chapter we will

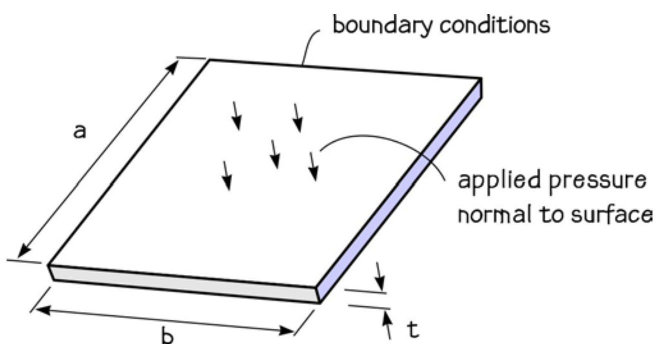
- Discuss the mechanics of plate bending

~~~~~~~~~~~~~~~~~~~~~
~~~~~~~~~~~~~~~~~~~~~

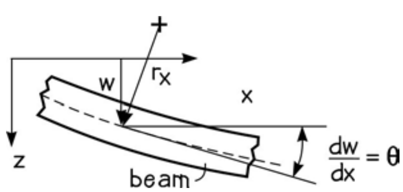
Plates are the essential structural components in ships. Almost all the structural weight in ships is from the shell plating, the bulkheads, decks and webs of large frames.

This section will examine the lateral deformation of a single plate panel subject to a uniform pressure. We will limit our problem as follows;

- rectangular plate
- constant thickness (t<<a, b)
- simple edge conditions (fixed, pinned, free)
- linear elastic material behavior
- steel material (isotropic, homogeneous)
- pressure normal to surface
- no membrane stresses (no in-plane stress)
- $\sigma_z \ll \sigma_x, \sigma_y$



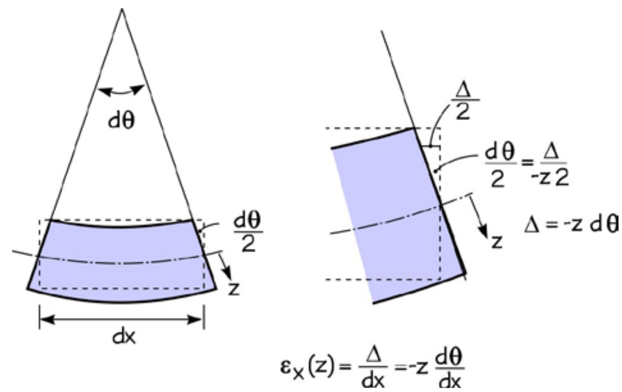
Recall that with beams we describe the deformation and strains as follows;

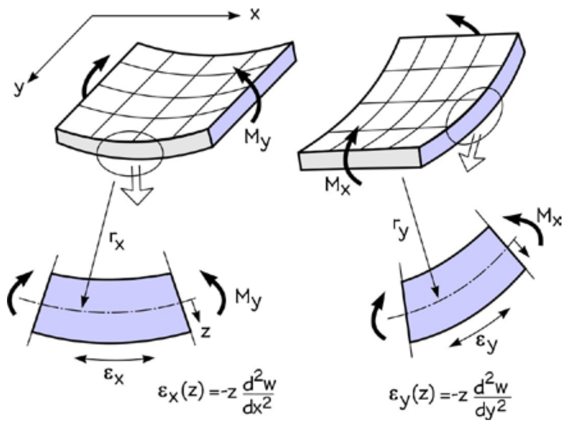


$$deflection = w$$

$$slope = \theta = \frac{dw}{dx}$$

$$curvature = \rho_x = \frac{1}{r_x} = \frac{d\theta}{dx} = \frac{d^2w}{dx^2}$$

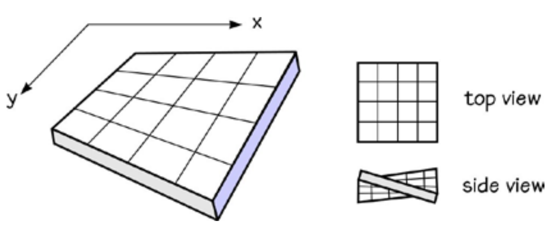




Plates can bend in 3 ways;

- x-bending
- y-bending
- twist

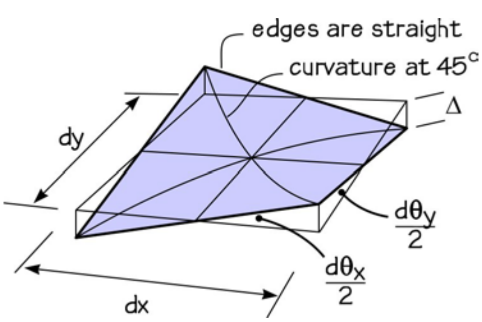
X and y bending are similar to beam bending.



Recall that there are no membrane stresses, therefor no x and y stresses at the mid-plane. Stresses only arise from bending, and are equal, opposite and maximum on the bottom and top of the plate.

Twist is a behavior that does not occur in beams, although it is something like torsion.

Twist causes a shear strain in the top (and bottom) of the plate, and results in curvature on 45° diagonals. When we twist a *dx* x *dy* portion of a plate we get;



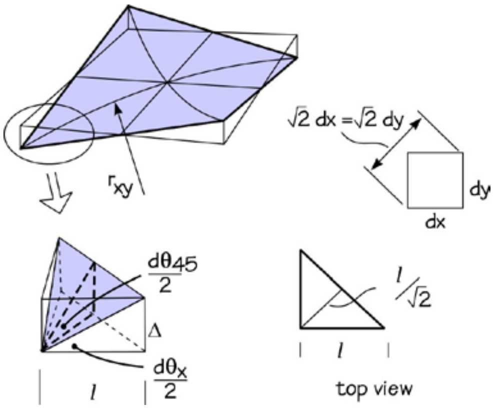
$$\Delta = \frac{d\theta_y}{2} \cdot \frac{dy}{2} = \frac{d\theta_x}{2} \cdot \frac{dx}{2}$$

therefore

$$\frac{d\theta_y}{dx} = \frac{d\theta_x}{dy} = \frac{d}{dx}\left(\frac{dw}{dy}\right) = \frac{d}{dy}\left(\frac{dw}{dx}\right) = \frac{d^2w}{dx\,dy}$$

The above equation can be stated as;
the change in x-slope with change in y
= the change in y-slope with change in x

$$= \frac{d^2w}{dx\,dy}$$

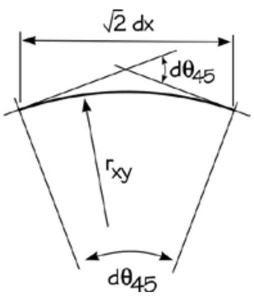


What about the curvature on 45° diagonals?

$$\frac{d\theta_x}{2} = \frac{\Delta}{l} \quad , \quad \frac{d\theta_{45}}{2} = \frac{\Delta}{l/\sqrt{2}} = \sqrt{2}\frac{\Delta}{l} = \sqrt{2}\frac{d\theta_x}{2}$$

$$d\theta_{45} = \sqrt{2}\, d\theta_x \;\left(= \sqrt{2}\, d\theta_y\right)$$

Consider a view of the plate normal to the 45° diagonal.

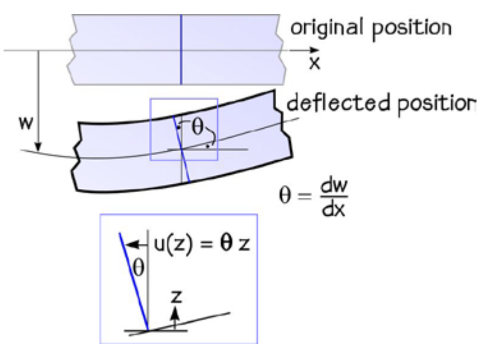


$$\sqrt{2}\, dy = r_{xy}\, d\theta_{45} = r_{xy}\sqrt{2}\, d\theta_x$$

$$\frac{1}{r_{xy}} = \frac{d\theta_x}{dy} \left(= \frac{d\theta_y}{dx}\right)$$

$$= \frac{d}{dy}\left(\frac{dw}{dx}\right) = \frac{d^2w}{dx\, dy}$$

We now have a variety of relationships for deflection, curvature and strain.

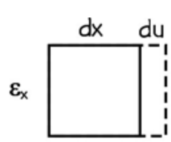


The x direction movement 'u' is the result of bending deflection w in the y direction.

$$u = -z\frac{dw}{dx}$$

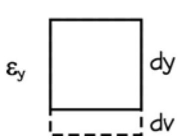
We can find the strain from derivatives of the movement;

$$\varepsilon_x = \frac{du}{dx} = -z\frac{d^2w}{dx^2}$$

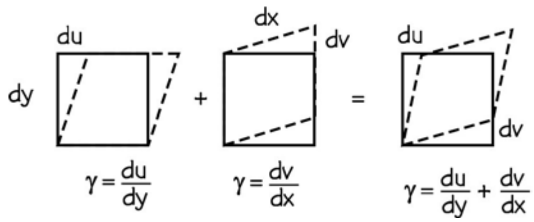


In the y direction the movement is called 'v';

$$v = -z\frac{dw}{dy}$$



$$\varepsilon_y = \frac{dv}{dy} = -z\frac{d^2w}{dy^2}$$

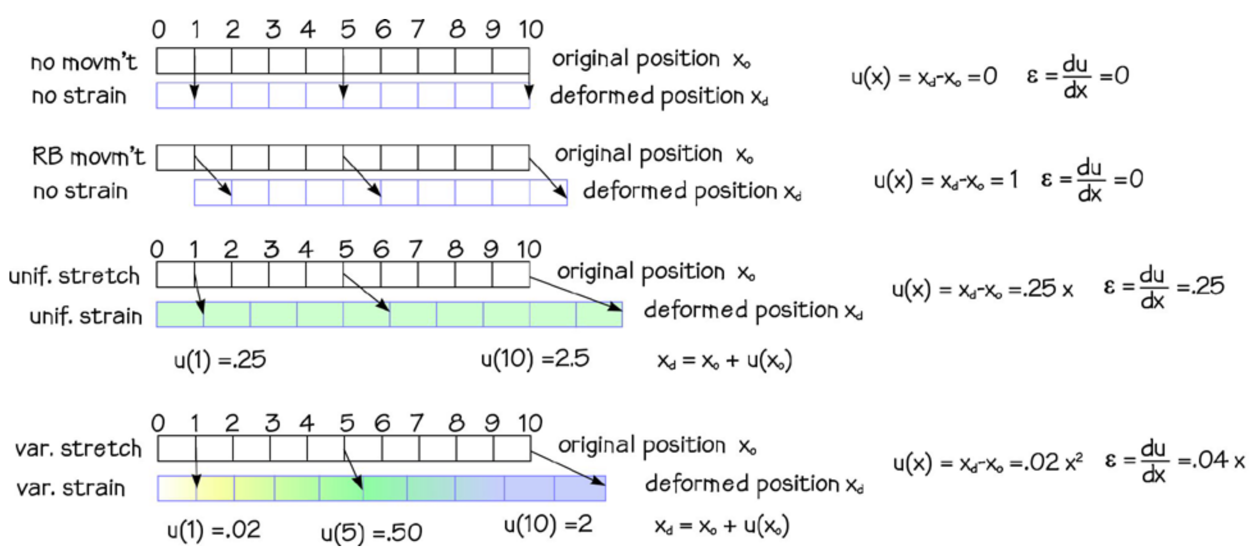


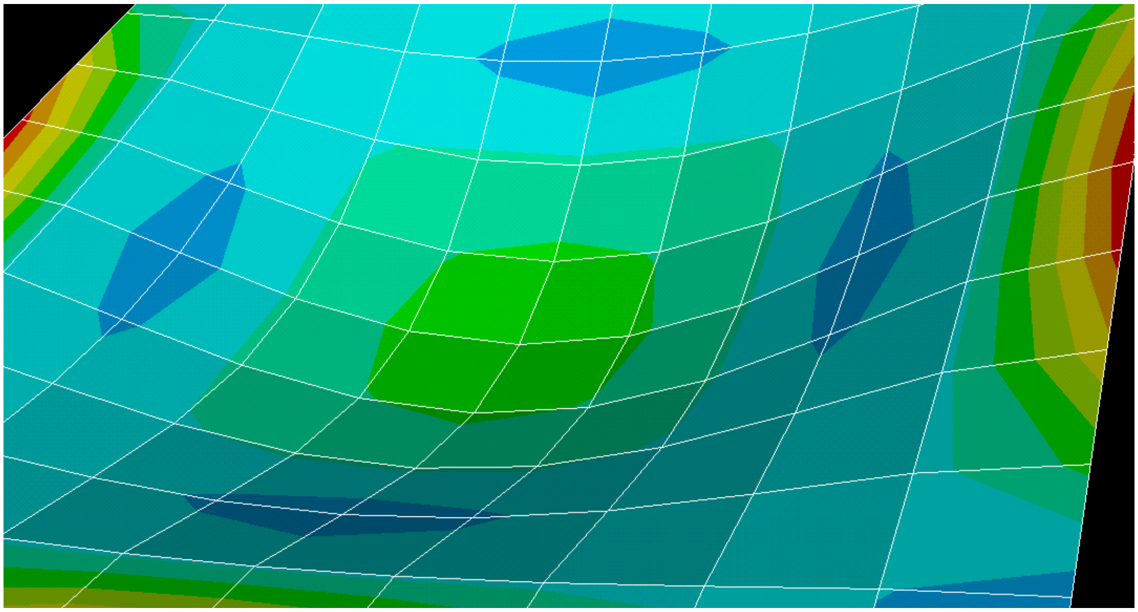
When u and v vary in x and y we can get shear strains.

$$\gamma = \frac{du}{dy} + \frac{dv}{dx}$$

CONCEPT: displacement field

In solid mechanics it is useful to describe how all points move relative to their original positions as a 'displacement field'. In the example below we just consider how points along an x axis move. We call the movement in the x direction u. A point at some original position x_o moves to a displaced position x_d. The displacement $u = x_d - x_o$. we describe u as a function of x, or $u(x)$. We could also write this as $u(x_o)$ because we think of the displacement as dependent on the original position. If all points move the same amount, then $u(x) = constant$. In such case the derivative of the displacement field is zero and there is no strain anywhere. We call this 'rigid body movement'. If the movement is a linear function of the x coordinate, (such as $(x) = c + k\ x$) then the derivative of the displacement field is k and the strain is k everywhere. The sketch below illustrates the concept. The concept can be extended to 2D and 3D problems.





ANSYS analysis results

For

$$\gamma = \frac{du}{dy} + \frac{dv}{dx}$$

we can use our definitions of u and v to get;

$$\gamma = \frac{d}{dy}\left(-z\frac{dw}{dx}\right) + \frac{d}{dx}\left(-z\frac{dw}{dy}\right)$$

$$= -2\,z\frac{d^2w}{dx\,dy}$$

or

$$\gamma = -2\,z\frac{1}{r_{xy}}$$

$$\varepsilon_x = -\,z\frac{d^2w}{dx^2} = -z\frac{1}{r_x}$$

$$\varepsilon_y = -\,z\frac{d^2w}{dy^2} = -z\frac{1}{r_y}$$

We can use the 2D version of Hooke's Law to get the stresses.

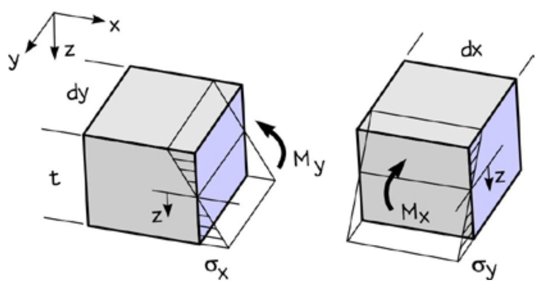
$$\sigma_x = \frac{E}{(1-\nu^2)}\left(\varepsilon_x + \nu\,\varepsilon_y\right)$$

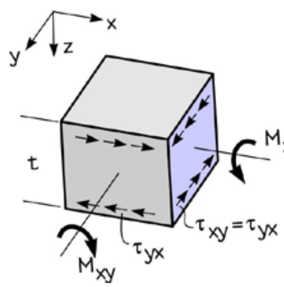
$$= -z\frac{E}{(1-\nu^2)}\left(\frac{d^2w}{dx^2} + \nu\frac{d^2w}{dy^2}\right)$$

$$\sigma_y = \frac{E}{(1-\nu^2)}\left(\varepsilon_y + \nu\,\varepsilon_x\right)$$
$$= -z\frac{E}{(1-\nu^2)}\left(\frac{d^2w}{dy^2} + \nu\frac{d^2w}{dx^2}\right)$$

$$\tau_{xy} = \frac{E}{(1+\nu)}\gamma_{xy} = -z\frac{E}{(1+\nu)}\frac{d^2w}{dx\,dy}$$

Clearly when $z = 0$ (middle of plate), all stresses vanish. Also, there are no average in-plane stresses, only bending moments and torsion.





$$M_x = \int_{-t/2}^{t/2} \sigma_y\, z\, dz$$

$$M_y = \int_{-t/2}^{t/2} \sigma_x\, z\, dz$$

$$M_{xy} = \int_{-t/2}^{t/2} \tau_{xy}\, z\, dz$$

By using the expressions for σ_x , σ_y and τ_{xy} we can write;

$$M_x = -D\left(\frac{d^2w}{dy^2} + \nu\,\frac{d^2w}{dx^2}\right)$$
$$M_y = -D\left(\frac{d^2w}{dx^2} + \nu\,\frac{d^2w}{dy^2}\right)$$
$$M_{xy} = M_{yx} = -D(1-\nu)\,\frac{d^2w}{dx\,dy}$$

where

$$D = \frac{E\,t^3}{12\,(1-\nu^2)}$$

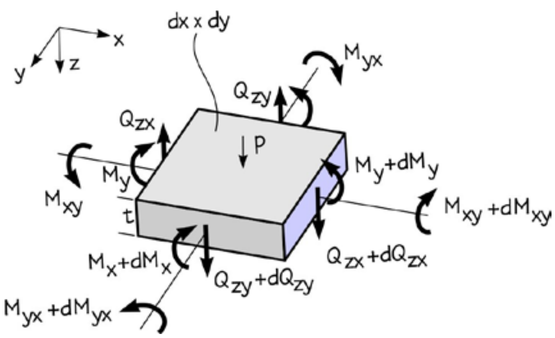
The derivation of these equations is as follows;

$$M_x = \int_{-t/2}^{t/2} \sigma_y\, z\, dz$$
$$= \frac{E}{(1-\nu^2)}\left(\frac{d^2w}{dy^2} + \nu\frac{d^2w}{dx^2}\right)\int_{-t/2}^{t/2} -z^2\, dz$$

$$= \frac{E}{(1-\nu^2)}\left(\frac{d^2w}{dy^2} + \nu\frac{d^2w}{dx^2}\right) \quad -\frac{z^2}{3}\Bigg|_{-t/2}^{t/2}$$
$$= -\frac{E\,t^3}{12\,(1-\nu^2)}\left(\frac{d^2w}{dy^2} + \nu\frac{d^2w}{dx^2}\right)$$

So far we have expressions for stress and strain (2 axial and shear) and for moments (2 bending and torsion) expressed as the derivatives of the deflection w.

We now want to derive the differential equation relating the deflection to load. The load is a pressure acting normal to the plate. Consider a small section of the plate subject to a uniform pressure p.



Summing the vertical forces ;

$$\Sigma\, F_{vert} = 0$$

$$p\;dx\;dy + (Q_{zx} + \;dQ_{zx})dy - Q_{zx}dy$$
$$+ \left(Q_{zy} + \;dQ_{zy}\right)dx - Q_{zy}dx = 0$$

$$p\;dx\;dy + dQ_{zx}dy \;+ dQ_{zy}dx = 0$$

$$p + \frac{dQ_{zx}}{dx} \;+ \frac{dQ_{zy}}{dy} = 0$$

Summing moments about x axis (about center of plate) ;

$$\Sigma\, M_x = 0$$

$$dM_{xy}dy + dM_x dx - \;Q_{zy}dx\;dy = 0$$

now divide by dy;

$$dM_{xy} + \frac{dM_x}{dy}dx - \;Q_{zy}dx = 0$$

and by dx;

$$\frac{dM_{xy}}{dx} + \frac{dM_x}{dy} - \;Q_{zy} = 0$$

which gives;

$$Q_{zy} = \frac{dM_{xy}}{dx} + \frac{dM_x}{dy}$$

Using the previous expressions for M_{xy} and M_x we can write;

$$Q_{zy} = \frac{d}{dx}\left(-D(1-\nu)\ \frac{d^2w}{dx\,dy}\right) + \frac{d}{dy}\left(-D\ \left(\frac{d^2w}{dy^2} + \nu\ \frac{d^2w}{dx^2}\right)\right)$$

$$Q_{zy} = \frac{d}{dx}\left(-D\ \frac{d^2w}{dx\,dy} + D\ \nu\ \frac{d^2w}{dx\,dy}\right) + \frac{d}{dy}\left(-D\frac{d^2w}{dy^2} - D\ \nu\ \frac{d^2w}{dx^2}\right)$$

$$Q_{zy} = -D\ \frac{d^3w}{dx^2\,dy} + D\ \nu\frac{d^3w}{dx^2dy} - D\ \frac{d^3w}{dy^3} - D\ \nu\ \frac{d^3w}{dx^2dy}$$

$$Q_{zy} = -D\ \frac{d^3w}{dx^2\,dy} - D\ \frac{d^3w}{dy^3}$$

Similarly;

$$Q_{zx} = -D\ \frac{d^3w}{dx\,dy^2} - D\ \frac{d^3w}{dx^3}$$

Now, using

$$p + \frac{dQ_{zx}}{dx} + \frac{dQ_{zy}}{dy} = 0$$

we can write;

$$p + \frac{d}{dx}\left(-D\ \frac{d^3w}{dx\,dy^2} - D\ \frac{d^3w}{dx^3}\right) + \frac{d}{dy}\left(-D\ \frac{d^3w}{dx^2\,dy} - D\ \frac{d^3w}{dy^3}\right) = 0$$

which simplifies to;

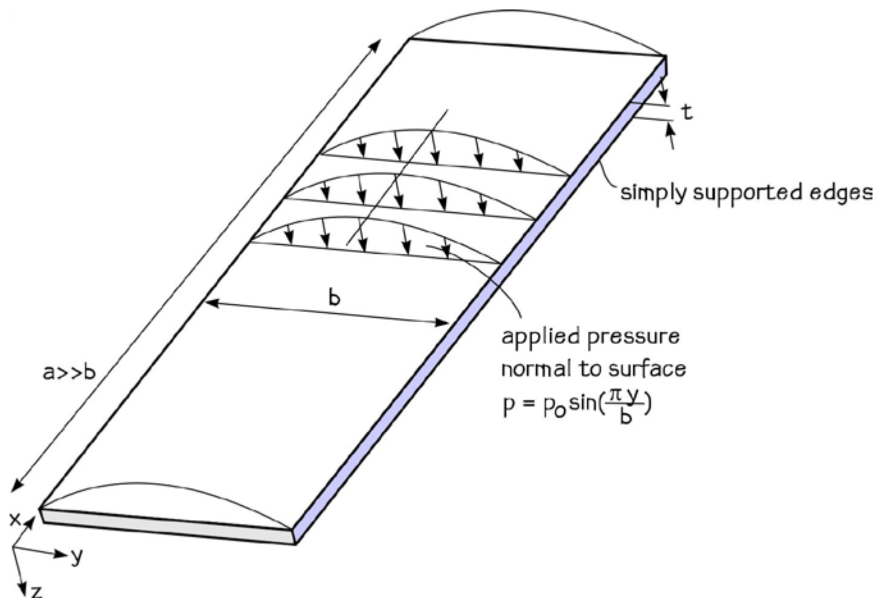
$$\frac{p}{D} = \frac{d^4w}{dx^2\,dy^2} + \frac{d^4w}{dx^4} + \frac{d^4w}{dx^2\,dy^2} + \frac{d^4w}{dy^4}$$

and can be written in the short hand got a general 4th derivative in 2 dimensions;

$$\frac{p}{D} = \Delta^4 w$$

Note the similarity to the differential equation for a beam of $p = EI\, w''''$. Now we need to solve $p = D\, \Delta^4 w$ for the appropriate boundary conditions to get $w(x,y)$ and the other results (stress, moments etc.)

Example #1:
A long plate, simply supported with a pressure in the shape of a half sine wave.



Check the pressure equation;

$$p(y=0) = 0 \quad (at\ edge) \qquad OK$$
$$p(y=b/2) = p_o \sin\left(\frac{\pi}{2}\right) = p_o \quad (at\ center) \quad OK$$

Note that nothing varies along the x axis, so all derivatives of x are zero. Therefore, the differential equation becomes;

$$\frac{p}{D} = \frac{d^4 w}{dy^4}$$

assume the solution has the form;

$$w(y) = C\ \sin\left(\frac{\pi\, y}{b}\right)$$

so

$$p_o \sin\left(\frac{\pi y}{b}\right) = D\ \frac{d^4}{dy^4}\left(C\ \sin\left(\frac{\pi\ y}{b}\right)\right)$$

which becomes;

$$p_o \sin\left(\frac{\pi y}{b}\right) = D\ C\ \left(\frac{\pi}{b}\right)^4 \left(\sin\left(\frac{\pi y}{b}\right)\right)$$

and lets us solve for C;

$$C = \frac{p_o}{D}\left(\frac{b}{\pi}\right)^4$$

which gives the deflection as;

$$w(y) = \frac{p_o}{D}\left(\frac{b}{\pi}\right)^4 \sin\left(\frac{\pi\ y}{b}\right)$$

with

$$w_{max} = \frac{p_o}{D}\left(\frac{b}{\pi}\right)^4$$

The stress can be found using;

$$\sigma_y = -z\frac{E}{(1-\nu^2)}\left(\frac{d^2w}{dy^2} + \nu\frac{d^2w}{dx^2}\right)$$

which simplifies to;

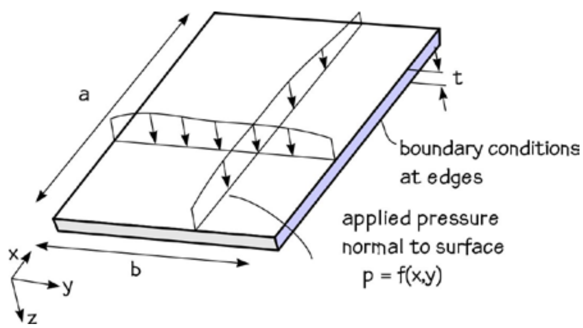
$$\sigma_y(z,y) = -z\frac{E}{(1-\nu^2)}\left(\frac{d^2w}{dy^2}\right)$$

The stress at the top of the plate , @ z = t/2 ;

$$\begin{aligned}\sigma_{y,top}(y) &= -\frac{t}{2}\frac{E}{(1-\nu^2)}\left(\frac{d^2w}{dy^2}\right)\\ &= \frac{t}{2}\frac{E}{(1-\nu^2)}\ \frac{p_o}{D}\left(\frac{b}{\pi}\right)^4\left(\frac{\pi}{b}\right)^2 \sin\left(\frac{\pi\ y}{b}\right)\\ &= \frac{6}{\pi^2}\left(\frac{b}{t}\right)^2 p_o \sin\left(\frac{\pi\ y}{b}\right)\end{aligned}$$

The stress as the edge is;

$$\sigma_{y,top}(0) = 0$$



The stress in the center is;

$$\sigma_{y,top}(b/2) = \frac{6}{\pi^2}\left(\frac{b}{t}\right)^2 p_o$$

Similarly, we can find;

$$\sigma_{x,top}(b/2) = \frac{\nu\, 6}{\pi^2}\left(\frac{b}{t}\right)^2 p_o$$

General Plate Problems

The solution for a general plate problem requires the solution of the 4th order partial differential equation;

$$\frac{p(x,y)}{D} = \Delta^4 w(x,y)$$

Such solutions can be complex, even for simple load patterns. Even in the case;

$$p(x,y) = p_o \quad (i.e\ \ uniform\ pressure)$$

The solution is found by expressing the load as a Fourier equation;

$$p(x,y) = p_o = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{mn} \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{m\pi y}{b}\right)$$

where

$$a_{mn} = \frac{16\, p_o}{\pi^2\, m\, n} \qquad \begin{matrix} m = 1,3,5\ \ldots \\ n = 1,3,5\ \ldots \end{matrix}$$

$$a_{mn} = 0 \qquad m\ or\ n = even$$

For this load pattern and simply supported edges, the deflected shape can be derived as;

$$w(x,y) = \frac{1}{\pi^4 D}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_{mn}}{\left(\frac{n^2}{b^2}+\frac{m^2}{a^2}\right)}\sin\left(\frac{m\pi x}{a}\right)\,\sin\left(\frac{m\pi y}{b}\right)$$

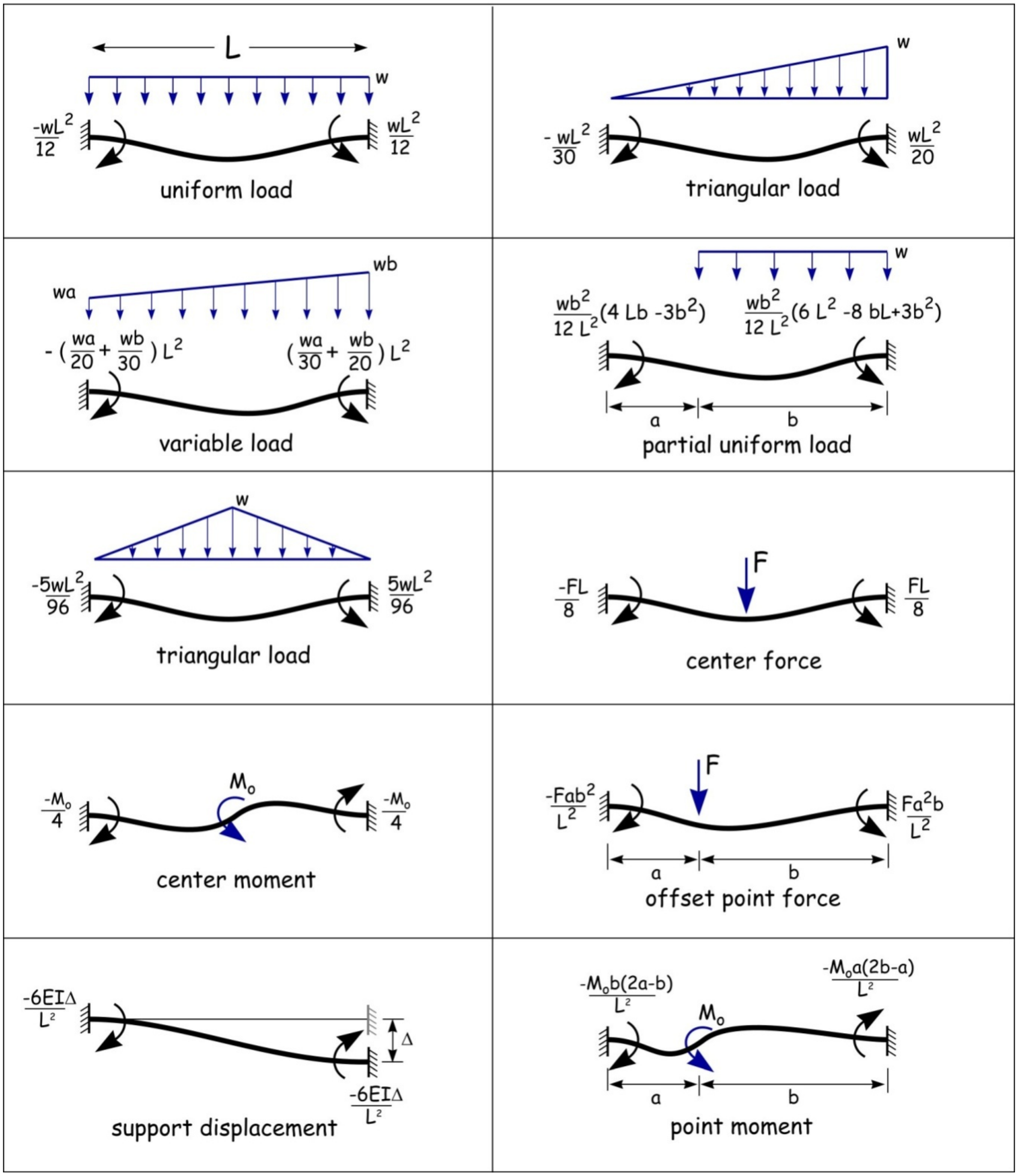
Note that a sine pattern of load has been shown to produce a sine pattern of response. So a group of sine shaped loads will produce a group of sine shaped responses. Hence the Fourier approach should work. It all depends on the elegance of super-position (hurray for Hooke!)

We will leave the general solution of more complex problems to a specialized course in palates and shells. See Hughes for solutions to some typical problems.

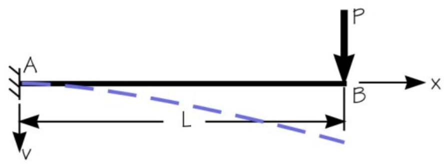
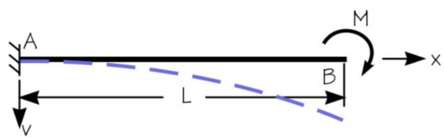
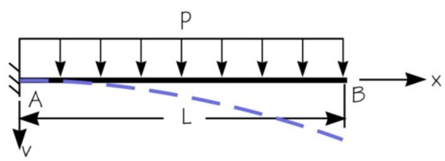
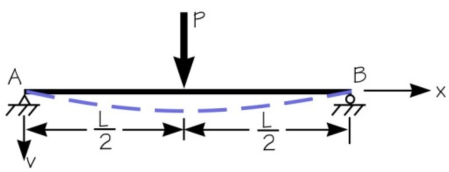
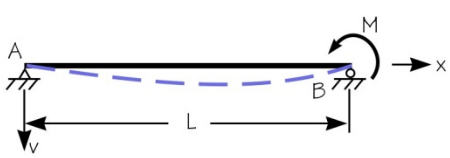
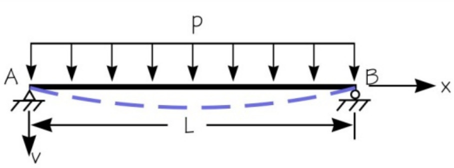
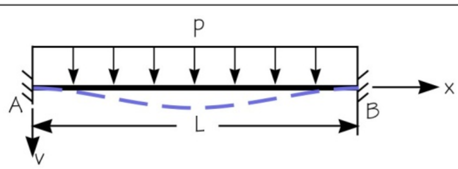
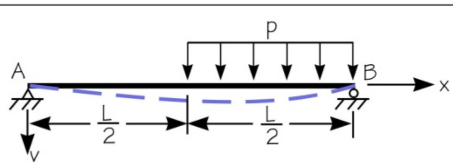
Topic 22: Problems

Appendix

Fixed End Moments



Deflection and Slopes of Beams

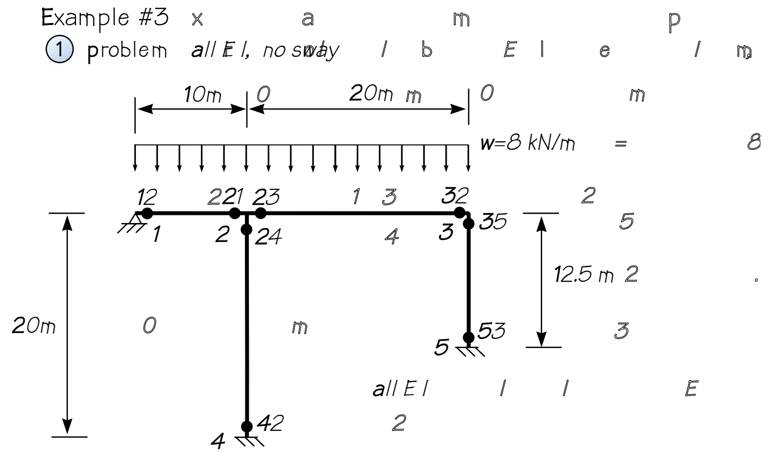
Loading	Deflection	Slope
	$v = \frac{Px^2}{6EI}(3L - x)$ $v_{max} = v_B = \frac{PL^3}{3EI}$	$\theta_B = \frac{PL^2}{2EI}$
	$v = \frac{Mx^2}{2EI}$ $v_{max} = v_B = \frac{ML^2}{2EI}$	$\theta_B = \frac{ML}{EI}$
	$v = \frac{px^2}{24EI}(6L^2 - 4Lx + x^2)$ $v_{max} = v_B = \frac{pL^4}{8EI}$	$\theta_B = \frac{pL^3}{6EI}$
	$v = \frac{Px^2}{48EI}(3L^2 - 4x^2)$ $v_{max} = \frac{PL^3}{48EI}$ @ $x=L/2$	$\theta_A = -\theta_B = \frac{PL^2}{16EI}$
	$v = \frac{Mx}{6EI\,L}(L^2 - x^2)$ $v_{max} = \frac{ML^2}{9\sqrt{3}\,EI}$ @ $x=L/\sqrt{3}$	$\theta_A = \frac{ML}{8EI}$ $\theta_B = -\frac{ML}{3EI}$
	$v = \frac{px}{24EI}(L^3 - 2Lx^2 + x^3)$ $v_{max} = \frac{5\,pL^4}{384\,EI}$ @ $x=L/2$	$\theta_A = -\theta_B = \frac{pL^3}{24EI}$
	$v = \frac{px^2}{24EI}(L - x)^2$ $v_{max} = \frac{pL^4}{384\,EI}$ @ $x=L/2$	$\theta_A = \theta_B = 0$
	$v_{cent} = \frac{3\,pL^4}{256\,EI}$ @ $x=L/2$	$\theta_A = \frac{3\,pL^3}{128\,EI}$ $\theta_B = \frac{-pL^3}{12\,EI}$

Typical spreadsheet to solve Moment Distribution problems.

Moment Distribution Method

table of values

param.	value	units
L1	10	m
L2	20	m
L3	20	m
L4	12.5	m
EI1	1	rel
EI2	1	rel
EI3	1	rel
EI4	1	rel
w	8	kN/m



	e12	e21	e23	e24	e42	e32	e35	e53
α	1.0	0.5	0.250	0.250	0.0	0.3846	0.6154	0.0
FEM	-66.7	66.7	-266.7	0.0	0.0	266.7	0.0	0.0
net	66.7	200.0			0.0	-266.7		0.0
dist	66.7	100.0	50.0	50.0	0.0	-102.6	-164.1	0.0
CO	50.0	33.3	-51.3	0.0	25.0	25.0	0.0	-82.1
eEM	50.0	200.0	-267.9	50.0	25.0	189.1	-164.1	-82.1
net	-50.0	17.9			-25.0	-25.0		82.1
dist	-50.0	9.0	4.5	4.5	0.0	-9.6	-15.4	0.0
CO	4.5	-25.0	-4.8	0.0	2.2	2.2	0.0	-7.7
eEM	4.5	184.0	-268.3	54.5	27.2	181.7	-179.5	-89.7
net	-4.5	29.8			-27.2	-2.2		89.7
dist	-4.5	14.9	7.5	7.5	0.0	-0.9	-1.4	0.0
CO	7.5	-2.2	-0.4	0.0	3.7	3.7	0.0	-0.7
eEM	0.0	196.0	-260.1	64.0	32.0	183.8	-183.8	-91.9

F21y 59.6
F12y 20.4
F23y 83.8
F42x 5
F42y 143.4
F32y 76.2
F53x -22
F53y 83.8