Third Edition

MECHANICS OF MATERIALS

CHAPTER

6

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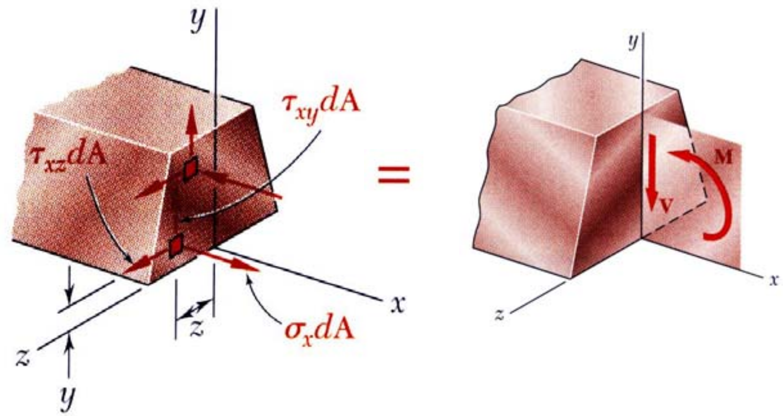
Shearing Stresses in Beams and Thin-Walled Members

Shearing Stresses in Beams and Thin-Walled Members

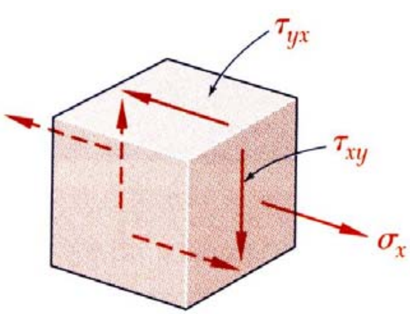
Introduction



- Transverse loading applied to a beam results in normal and shearing stresses in transverse sections.

- Distribution of normal and shearing stresses satisfies

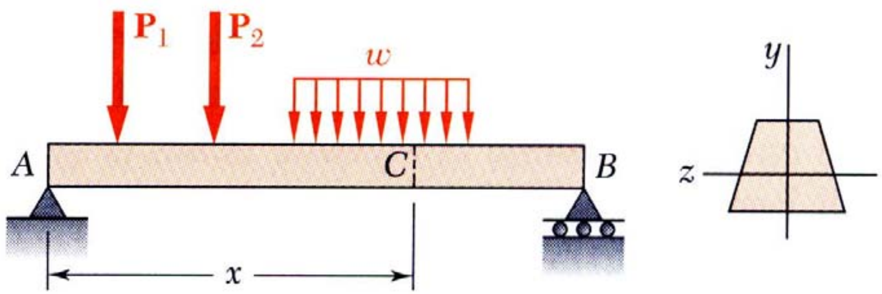
$$F_x = \int \sigma_x dA = 0 \qquad M_x = \int \left(y\tau_{xz} - z\tau_{xy} \right) dA = 0$$
$$F_y = \int \tau_{xy} dA = -V \qquad M_y = \int z\sigma_x dA = 0$$
$$F_z = \int \tau_{xz} dA = 0 \qquad M_z = \int \left(-y\sigma_x \right) = 0$$

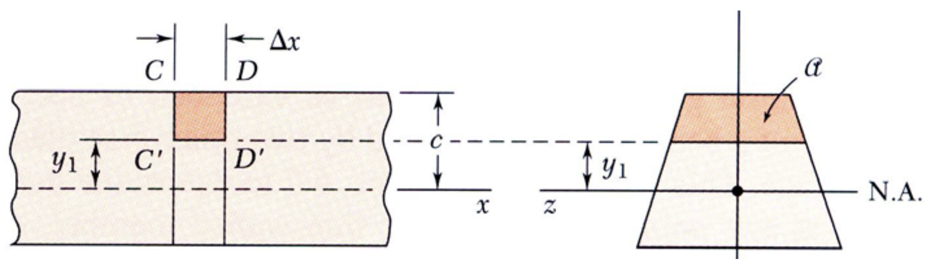


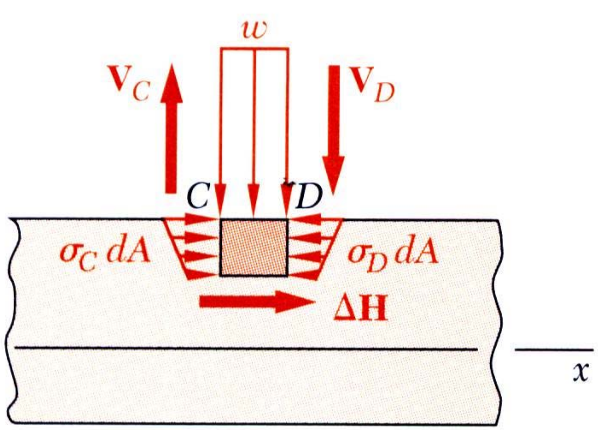
- When shearing stresses are exerted on the vertical faces of an element, equal stresses must be exerted on the horizontal faces

- Longitudinal shearing stresses must exist in any member subjected to transverse loading.

Shear on the Horizontal Face of a Beam Element







- Consider prismatic beam
- For equilibrium of beam element

$$\sum F_x = 0 = \Delta H + \int_A (\sigma_D - \sigma_D)\, dA$$

$$\Delta H = \frac{M_D - M_C}{I} \int_A y\, dA$$

- Note,

$$Q = \int_A y\, dA$$

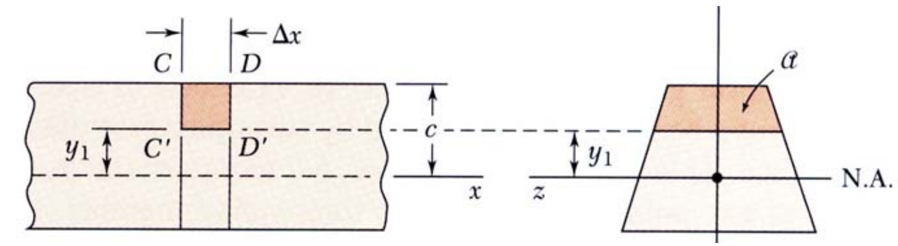
$$M_D - M_C = \frac{dM}{dx}\Delta x = V\,\Delta x$$

- Substituting,

$$\Delta H = \frac{VQ}{I}\Delta x$$

$$q = \frac{\Delta H}{\Delta x} = \frac{VQ}{I} = shear\ \ flow$$

Shear on the Horizontal Face of a Beam Element



- Shear flow,

$$q = \frac{\Delta H}{\Delta x} = \frac{VQ}{I} = shear\ \ flow$$

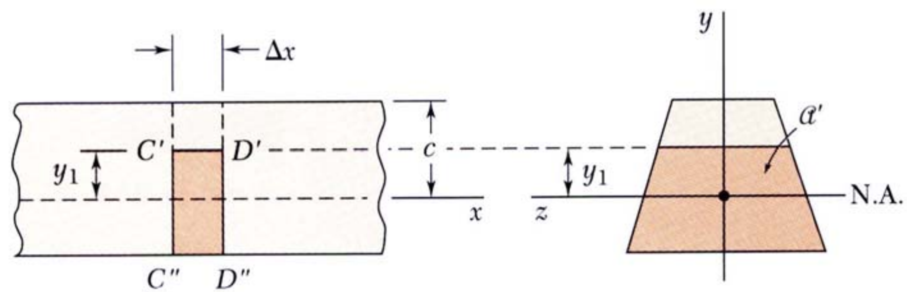
- where

$$Q = \int_A y\,dA$$

$$= \text{first moment of area above } y_1$$

$$I = \int_{A+A'} y^2 dA$$

$$= \text{second moment of full cross section}$$



- Same result found for lower area

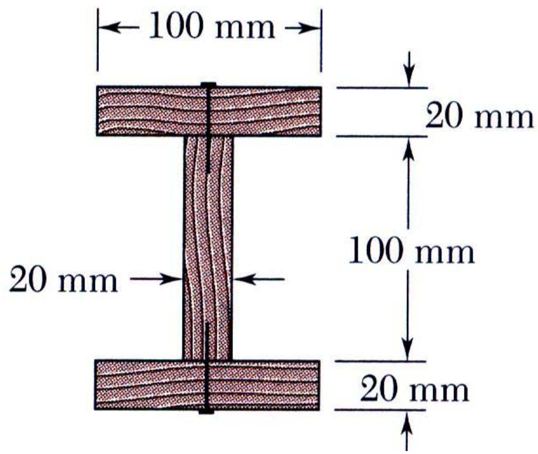
$$q' = \frac{\Delta H'}{\Delta x} = \frac{VQ'}{I} = -q'$$

$$Q + Q' = 0$$

$$= \text{first moment with respect to neutral axis}$$

$$\Delta H' = -\Delta H$$

Example 6.01

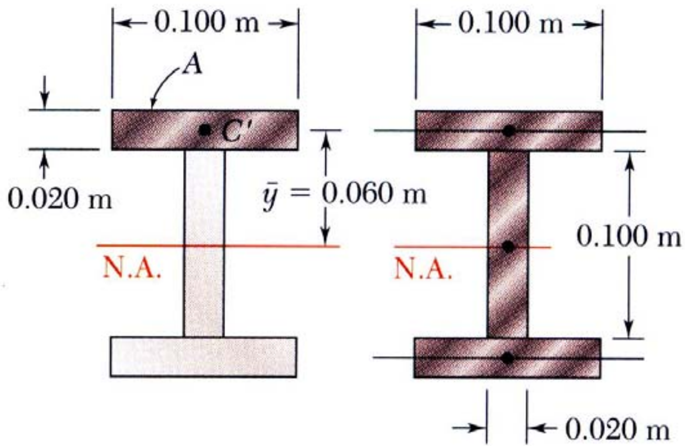


A beam is made of three planks, nailed together. Knowing that the spacing between nails is 25 mm and that the vertical shear in the beam is V = 500 N, determine the shear force in each nail.

SOLUTION:

- Determine the horizontal force per unit length or shear flow q on the lower surface of the upper plank.
- Calculate the corresponding shear force in each nail.

Example 6.01



$$Q = A\bar{y}$$
$$= (0.020\,\text{m} \times 0.100\,\text{m})(0.060\,\text{m})$$
$$= 120 \times 10^{-6}\,\text{m}^3$$

$$I = \tfrac{1}{12}(0.020\,\text{m})(0.100\,\text{m})^3$$
$$+ 2[\tfrac{1}{12}(0.100\,\text{m})(0.020\,\text{m})^3$$
$$+ (0.020\,\text{m} \times 0.100\,\text{m})(0.060\,\text{m})^2]$$
$$= 16.20 \times 10^{-6}\,\text{m}^4$$

SOLUTION:

- Determine the horizontal force per unit length or shear flow q on the lower surface of the upper plank.

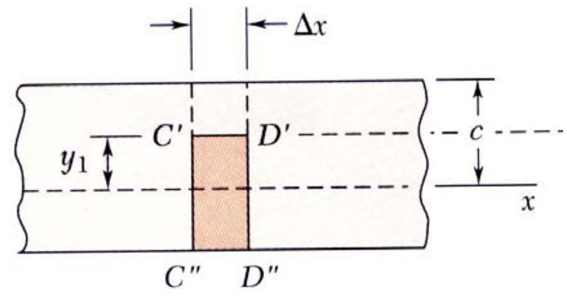
$$q = \frac{VQ}{I} = \frac{(500\,\text{N})(120 \times 10^{-6}\,\text{m}^3)}{16.20 \times 10^{-6}\,\text{m}^4}$$
$$= 3704\,\text{N/m}$$

- Calculate the corresponding shear force in each nail for a nail spacing of 25 mm.

$$F = (0.025\,\text{m})q = (0.025\,\text{m})(3704\,N/m)$$

$$\boxed{F = 92.6\,\text{N}}$$

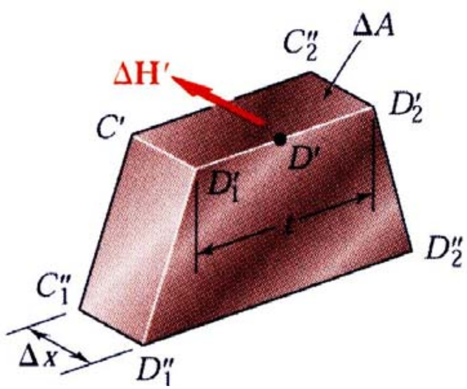
Determination of the Shearing Stress in a Beam



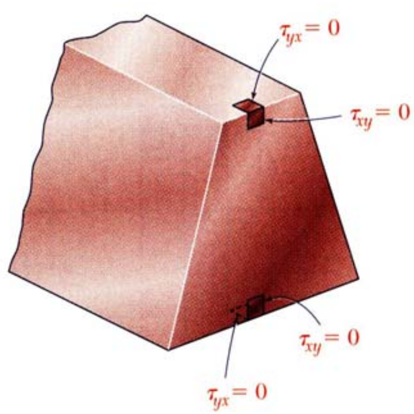
- The *average* shearing stress on the horizontal face of the element is obtained by dividing the shearing force on the element by the area of the face.

$$\tau_{ave} = \frac{\Delta H}{\Delta A} = \frac{q\,\Delta x}{\Delta A} = \frac{VQ}{I}\frac{\Delta x}{t\,\Delta x}$$

$$= \frac{VQ}{It}$$

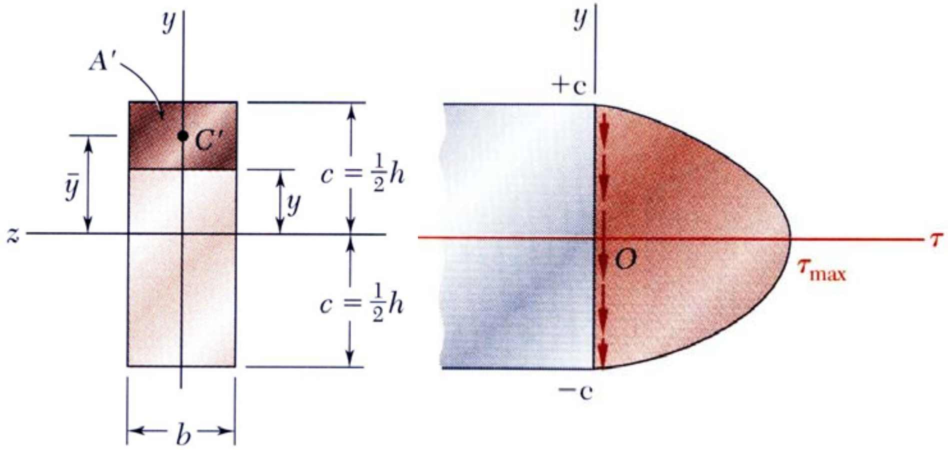


- On the upper and lower surfaces of the beam, τ_{yx}= 0. It follows that τ_{xy}= 0 on the upper and lower edges of the transverse sections.



- If the width of the beam is comparable or large relative to its depth, the shearing stresses at D_1 and D_2 are significantly higher than at D.

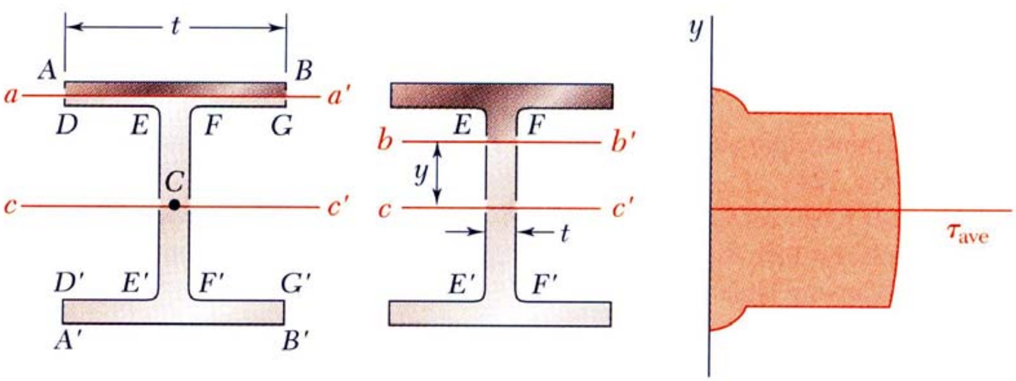
Shearing Stresses τ_{xy} in Common Types of Beams



- For a narrow rectangular beam,

$$\tau_{xy} = \frac{VQ}{Ib} = \frac{3}{2}\frac{V}{A}\left(1 - \frac{y^2}{c^2}\right)$$

$$\tau_{max} = \frac{3}{2}\frac{V}{A}$$

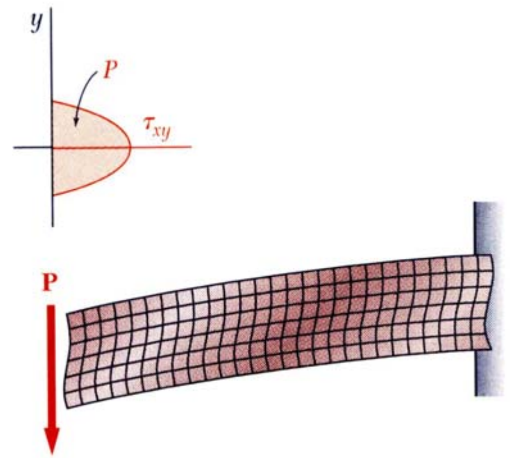


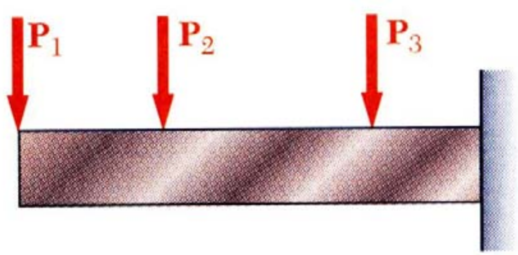
- For American Standard (S-beam) and wide-flange (W-beam) beams

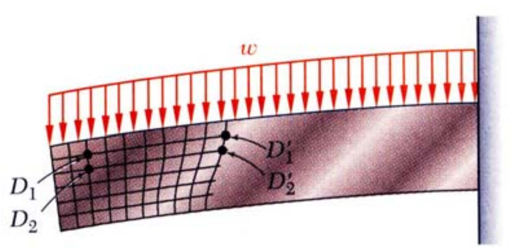
$$\tau_{ave} = \frac{VQ}{It}$$

$$\tau_{max} = \frac{V}{A_{web}}$$

Further Discussion of the Distribution of Stresses in a Narrow Rectangular Beam





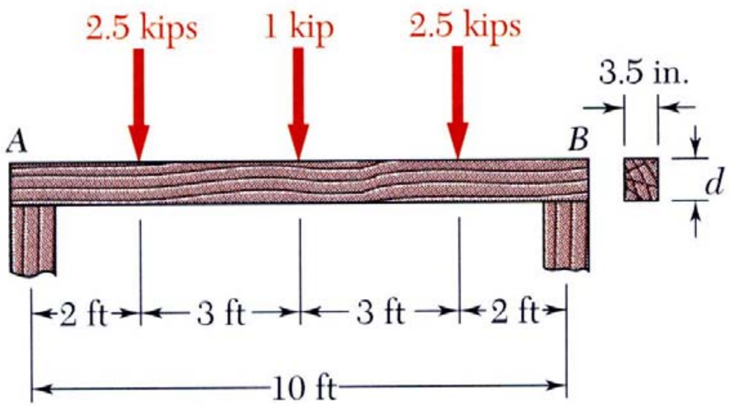


- Consider a narrow rectangular cantilever beam subjected to load P at its free end:

$$\tau_{xy} = \frac{3}{2}\frac{P}{A}\left(1 - \frac{y^2}{c^2}\right) \qquad \sigma_x = +\frac{Pxy}{I}$$

- Shearing stresses are independent of the distance from the point of application of the load.
- Normal strains and normal stresses are unaffected by the shearing stresses.
- From Saint-Venant's principle, effects of the load application mode are negligible except in immediate vicinity of load application points.
- Stress/strain deviations for distributed loads are negligible for typical beam sections of interest.

Sample Problem 6.2



A timber beam is to support the three concentrated loads shown. Knowing that for the grade of timber used,

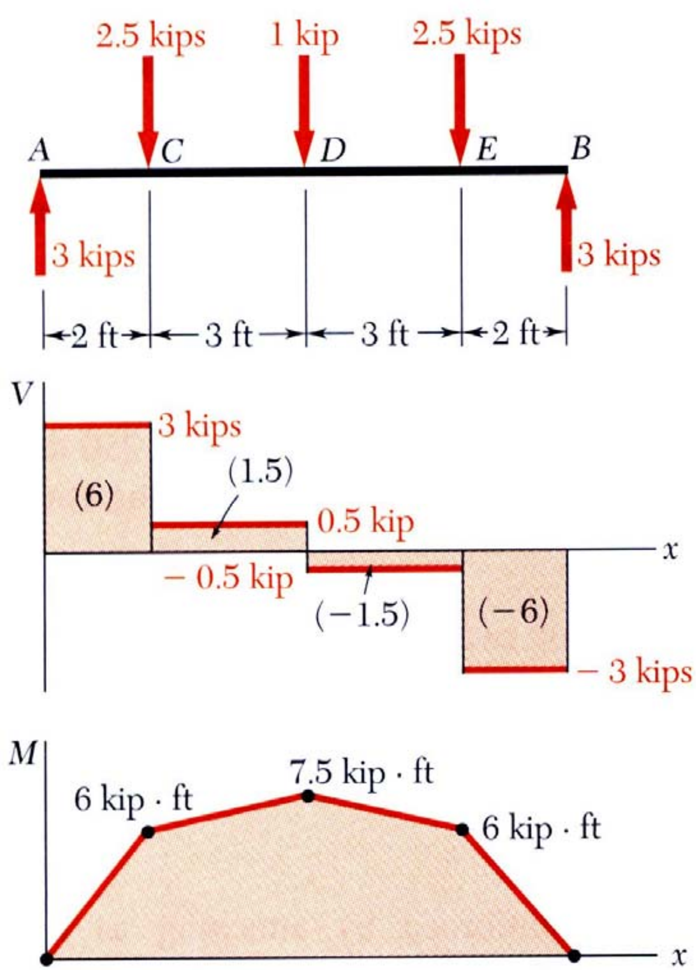
$$\sigma_{all} = 1800\,\text{psi} \qquad \tau_{all} = 120\,\text{psi}$$

determine the minimum required depth *d* of the beam.

SOLUTION:

- Develop shear and bending moment diagrams. Identify the maximums.
- Determine the beam depth based on allowable normal stress.
- Determine the beam depth based on allowable shear stress.
- Required beam depth is equal to the larger of the two depths found.

Sample Problem 6.2



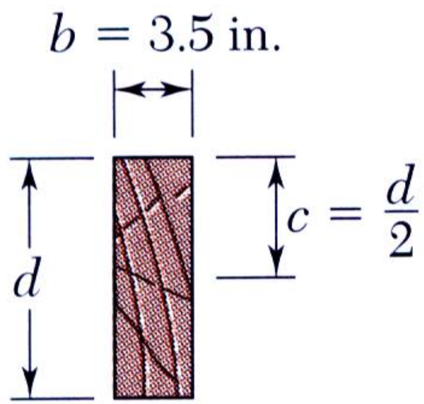
SOLUTION:

Develop shear and bending moment diagrams. Identify the maximums.

$$V_{\text{max}} = 3\,\text{kips}$$

$$M_{\text{max}} = 7.5\,\text{kip} \cdot \text{ft} = 90\,\text{kip} \cdot \text{in}$$

Sample Problem 6.2



$$I = \tfrac{1}{12} b d^3$$

$$S = \frac{I}{c} = \tfrac{1}{6} b d^2$$

$$= \tfrac{1}{6}(3.5\,\text{in.})d^2$$

$$= (0.5833\,\text{in.})d^2$$

- Determine the beam depth based on allowable normal stress.

$$\sigma_{all} = \frac{M_{max}}{S}$$

$$1800\,\text{psi} = \frac{90 \times 10^3\,\text{lb} \cdot \text{in.}}{(0.5833\,\text{in.})d^2}$$

$$d = 9.26\,\text{in.}$$

- Determine the beam depth based on allowable shear stress.

$$\tau_{all} = \frac{3}{2}\frac{V_{max}}{A}$$

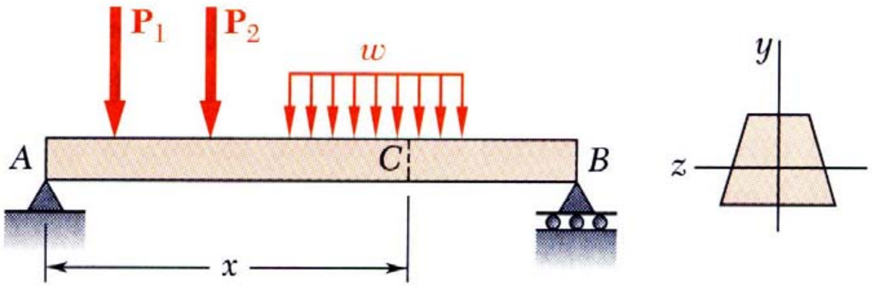
$$120\,\text{psi} = \frac{3}{2}\frac{3000\,\text{lb}}{(3.5\,\text{in.})d}$$

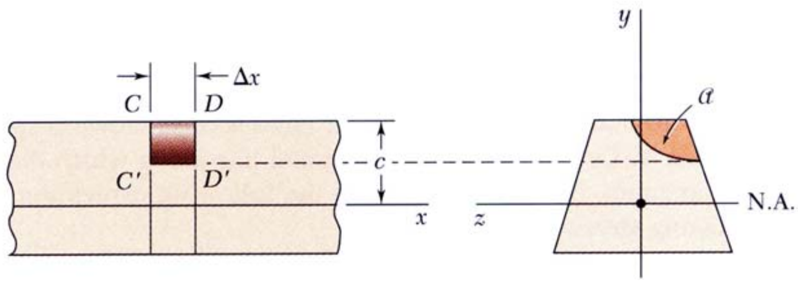
$$d = 10.71\,\text{in.}$$

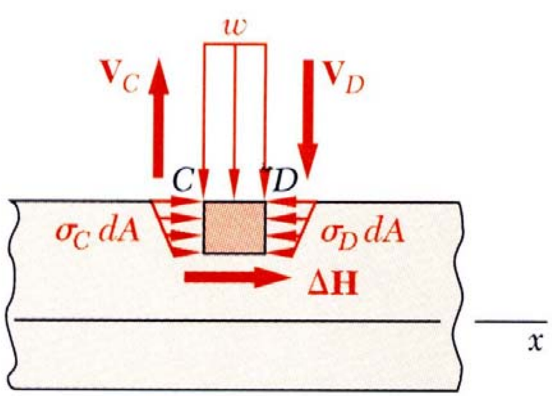
- Required beam depth is equal to the larger of the two.

$$\boxed{d = 10.71\,\text{in.}}$$

Longitudinal Shear on a Beam Element of Arbitrary Shape







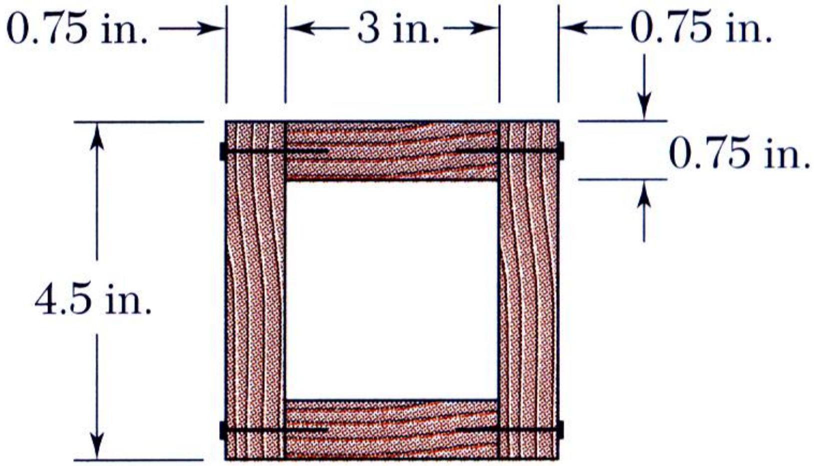
- We have examined the distribution of the vertical components τ_{xy} on a transverse section of a beam. We now wish to consider the horizontal components τ_{xz} of the stresses.
- Consider prismatic beam with an element defined by the curved surface CDD'C'.

$$\sum F_x = 0 = \Delta H + \int_a (\sigma_D - \sigma_C) dA$$

- Except for the differences in integration areas, this is the same result obtained before which led to

$$\Delta H = \frac{VQ}{I}\Delta x \qquad q = \frac{\Delta H}{\Delta x} = \frac{VQ}{I}$$

Example 6.04

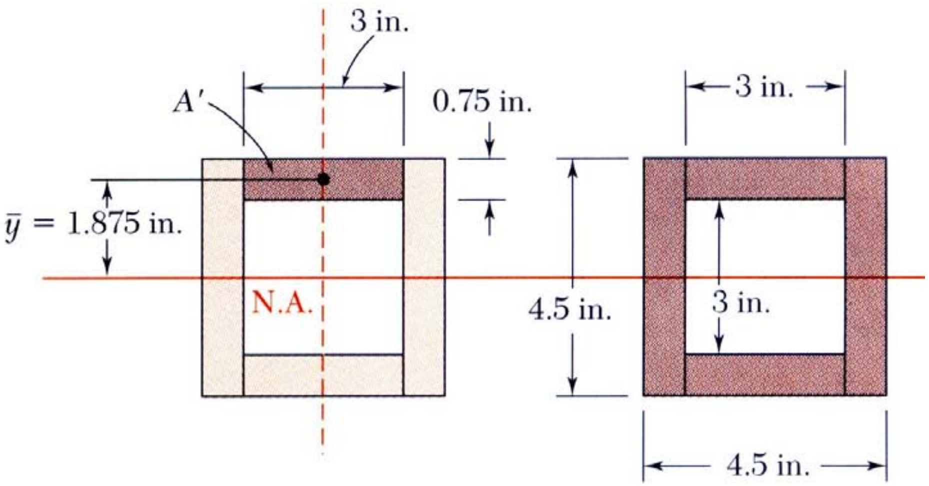


A square box beam is constructed from four planks as shown. Knowing that the spacing between nails is 1.5 in. and the beam is subjected to a vertical shear of magnitude $V = 600$ lb, determine the shearing force in each nail.

SOLUTION:

- Determine the shear force per unit length along each edge of the upper plank.
- Based on the spacing between nails, determine the shear force in each nail.

Example 6.04



For the upper plank,

$$Q = A'y = (0.75\text{in.})(3\text{in.})(1.875\text{in.})$$
$$= 4.22\,\text{in}^3$$

For the overall beam cross-section,

$$I = \tfrac{1}{12}(4.5\text{in})^3 - \tfrac{1}{12}(3\text{in})^3$$
$$= 27.42\,\text{in}^4$$

SOLUTION:

- Determine the shear force per unit length along each edge of the upper plank.

$$q = \frac{VQ}{I} = \frac{(600\,\text{lb})(4.22\,\text{in}^3)}{27.42\,\text{in}^4} = 92.3\frac{\text{lb}}{\text{in}}$$

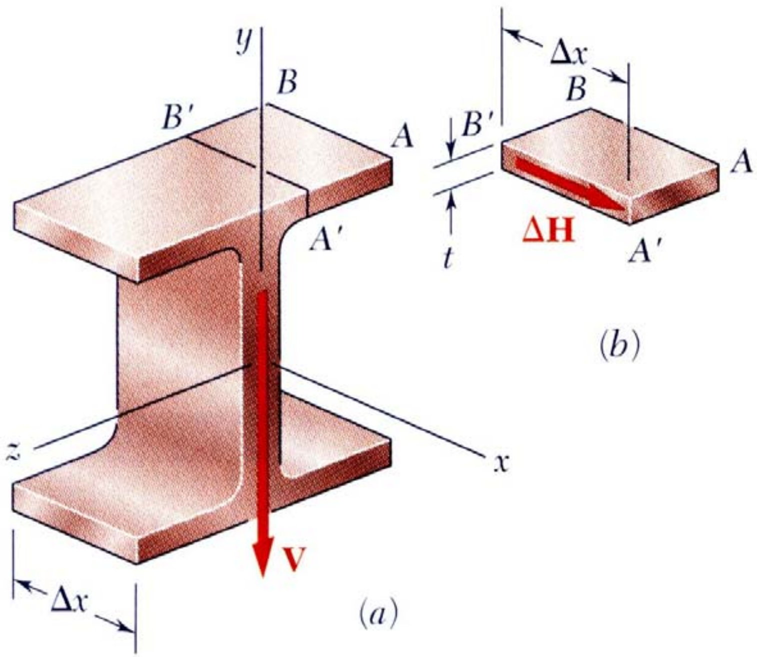
$$f = \frac{q}{2} = 46.15\frac{\text{lb}}{\text{in}}$$
$$= \text{edge force per unit length}$$

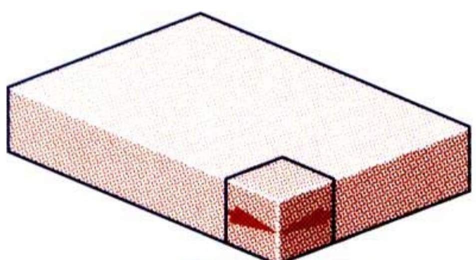
- Based on the spacing between nails, determine the shear force in each nail.

$$F = f\,\ell = \left(46.15\frac{\text{lb}}{\text{in}}\right)(1.75\,\text{in})$$

$$\boxed{F = 80.8\,\text{lb}}$$

Shearing Stresses in Thin-Walled Members





- Consider a segment of a wide-flange beam subjected to the vertical shear *V*.
- The longitudinal shear force on the element is

$$\Delta H = \frac{VQ}{I}\Delta x$$

- The corresponding shear stress is

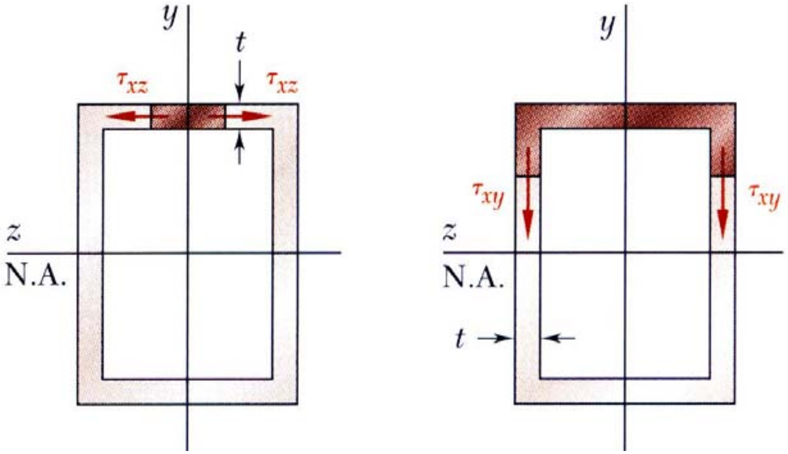
$$\tau_{zx} = \tau_{xz} \approx \frac{\Delta H}{t\,\Delta x} = \frac{VQ}{It}$$

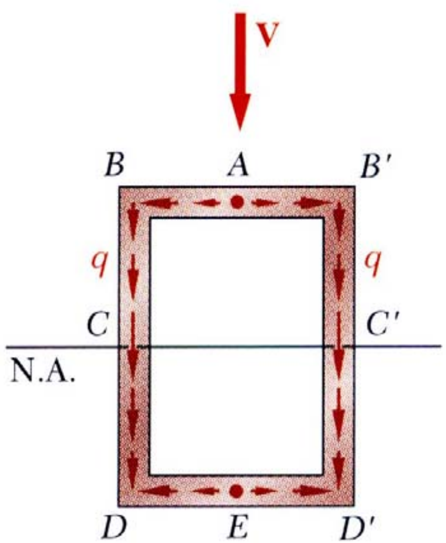
- Previously found a similar expression for the shearing stress in the web

$$\tau_{xy} = \frac{VQ}{It}$$

- NOTE: $\tau_{xy} \approx 0$ in the flanges
  $\tau_{xz} \approx 0$ in the web

Shearing Stresses in Thin-Walled Members





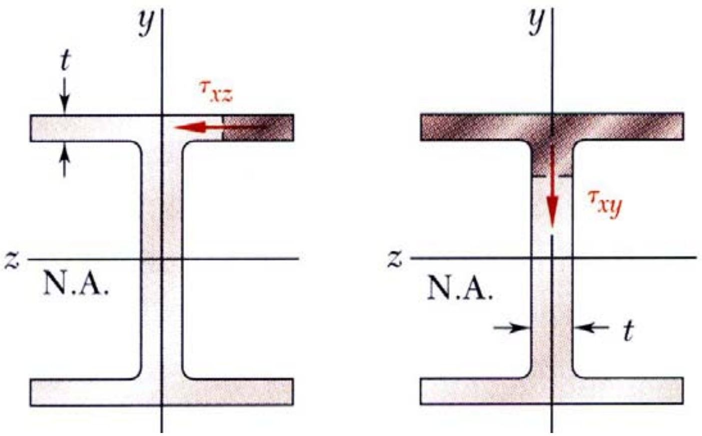
- The variation of shear flow across the section depends only on the variation of the first moment.

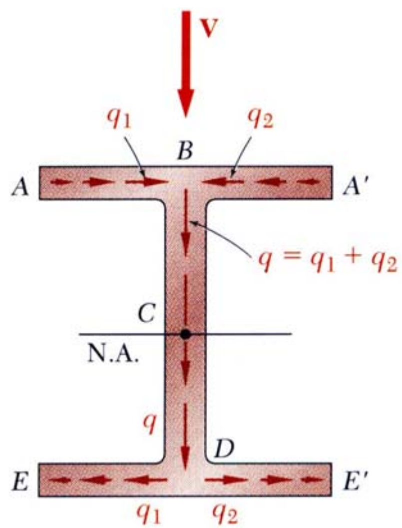
$$q = \tau t = \frac{VQ}{I}$$

- For a box beam, *q* grows smoothly from zero at A to a maximum at *C* and *C'* and then decreases back to zero at *E*.

- The sense of *q* in the horizontal portions of the section may be deduced from the sense in the vertical portions or the sense of the shear *V*.

Shearing Stresses in Thin-Walled Members

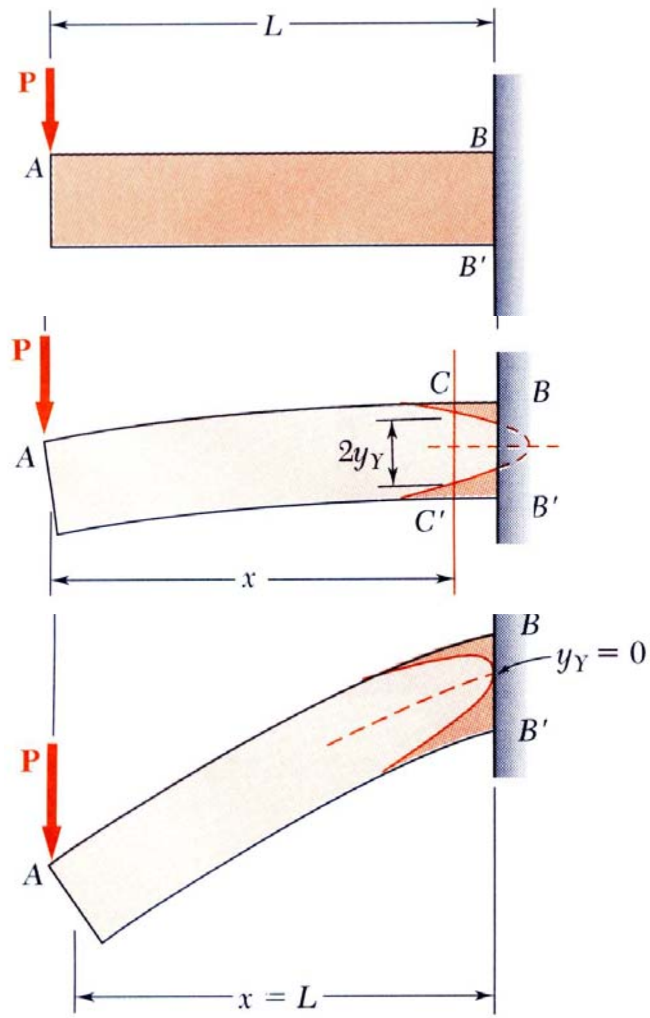




- For a wide-flange beam, the shear flow increases symmetrically from zero at *A* and *A'*, reaches a maximum at *C* and the decreases to zero at *E* and *E'*.

- The continuity of the variation in *q* and the merging of *q* from section branches suggests an analogy to fluid flow.

Plastic Deformations



- Recall: $M_Y = \frac{I}{c}\sigma_Y = \text{maximum elastic moment}$
- For $M = PL < M_Y$, the normal stress does not exceed the yield stress anywhere along the beam.
- For $PL > M_Y$, yield is initiated at *B* and *B'*. For an elastoplastic material, the half-thickness of the elastic core is found from

$$Px = \frac{3}{2}M_Y\left(1 - \frac{1}{3}\frac{y_Y^2}{c^2}\right)$$

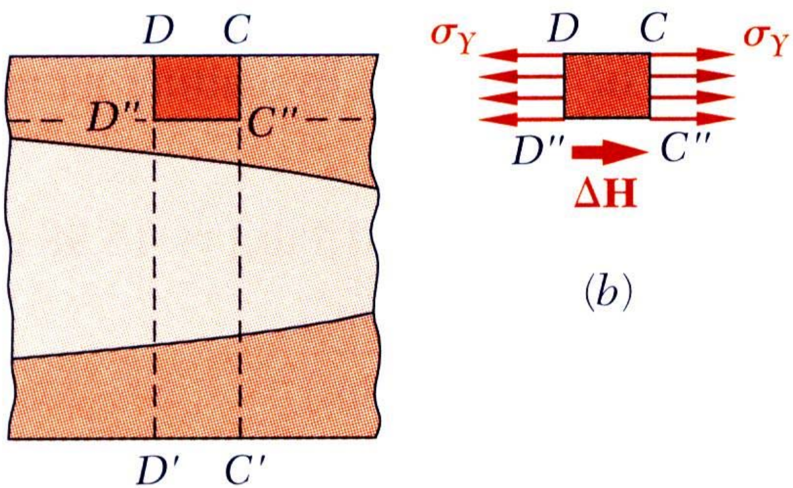
- The section becomes fully plastic ($y_Y = 0$) at the wall when

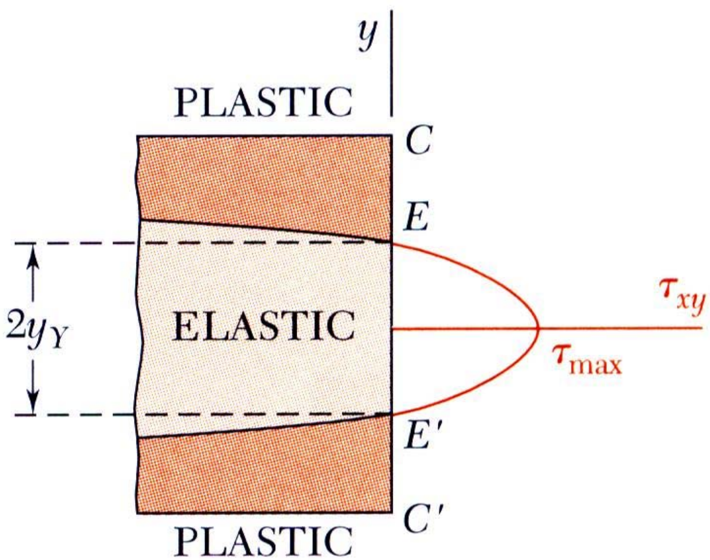
$$PL = \frac{3}{2}M_Y = M_p$$

- Maximum load which the beam can support is

$$P_{\max} = \frac{M_p}{L}$$

Plastic Deformations





- Preceding discussion was based on normal stresses only

- Consider horizontal shear force on an element within the plastic zone,

$$\Delta H = -(\sigma_C - \sigma_D)dA = -(\sigma_Y - \sigma_Y)dA = 0$$

  Therefore, the shear stress is zero in the plastic zone.

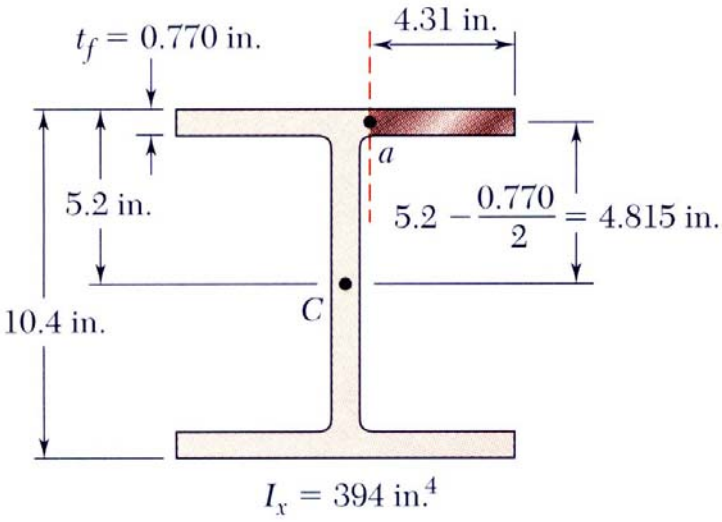
- Shear load is carried by the elastic core,

$$\tau_{xy} = \frac{3}{2}\frac{P}{A'}\left(1 - \frac{y^2}{y_Y^2}\right) \quad \text{where } A' = 2by_Y$$

$$\tau_{max} = \frac{3}{2}\frac{P}{A'}$$

- As *A'* decreases, τ_{max} increases and may exceed τ_Y

Sample Problem 6.3



Knowing that the vertical shear is 50 kips in a W10x68 rolled-steel beam, determine the horizontal shearing stress in the top flange at the point *a*.

SOLUTION:

- For the shaded area,

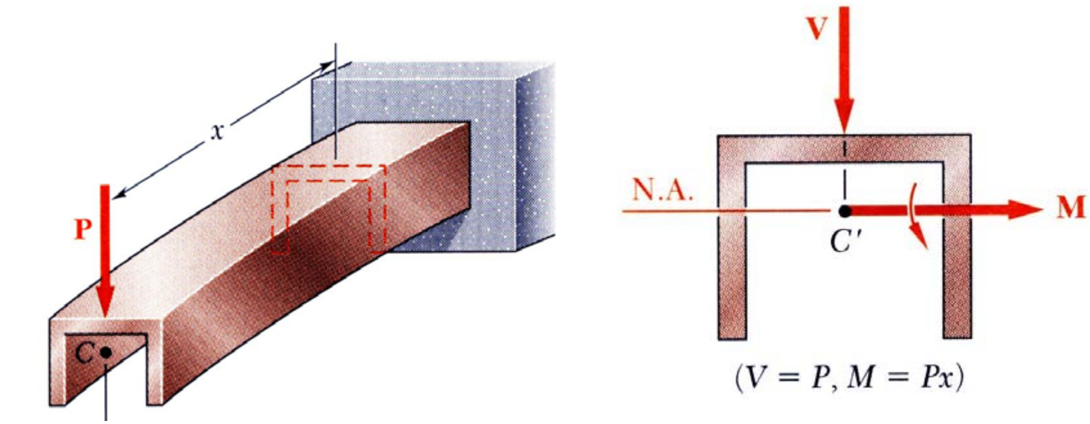
$$Q = (4.31\,\text{in})(0.770\,\text{in})(4.815\,\text{in})$$
$$= 15.98\,\text{in}^3$$

- The shear stress at *a*,

$$\tau = \frac{VQ}{It} = \frac{(50\,\text{kips})(15.98\,\text{in}^3)}{(394\,\text{in}^4)(0.770\,\text{in})}$$

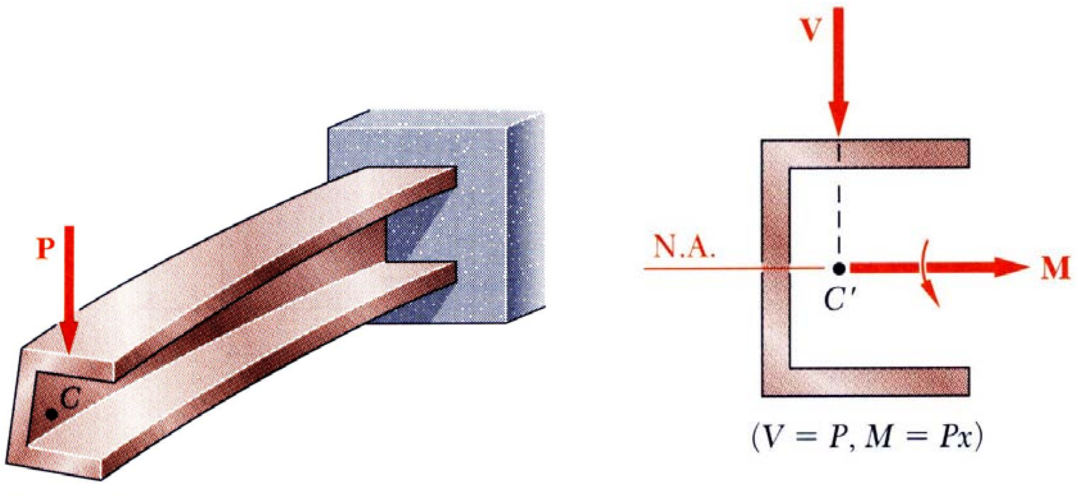
$$\boxed{\tau = 2.63\,\text{ksi}}$$

Unsymmetric Loading of Thin-Walled Members



- Beam loaded in a vertical plane of symmetry deforms in the symmetry plane without twisting.

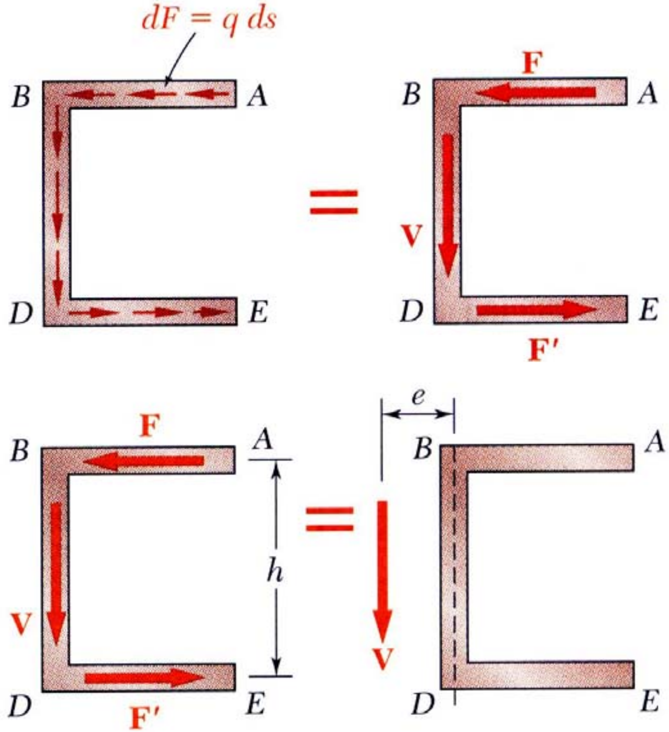
$$\sigma_x = -\frac{My}{I} \qquad \tau_{ave} = \frac{VQ}{It}$$



- Beam without a vertical plane of symmetry bends and twists under loading.

$$\sigma_x = -\frac{My}{I} \qquad \tau_{ave} \neq \frac{VQ}{It}$$

Unsymmetric Loading of Thin-Walled Members

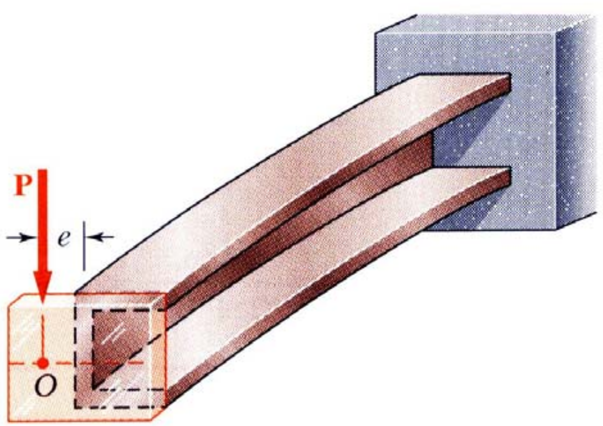


- If the shear load is applied such that the beam does not twist, then the shear stress distribution satisfies

$$\tau_{ave} = \frac{VQ}{It} \quad V = \int_B^D q\,ds \quad F = \int_A^B q\,ds = -\int_D^E q\,ds = -F'$$

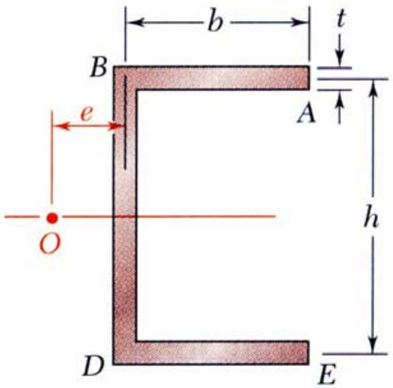
- *F* and *F'* indicate a couple *Fh* and the need for the application of a torque as well as the shear load.

$$Fh = Ve$$



- When the force P is applied at a distance e to the left of the web centerline, the member bends in a vertical plane without twisting.

Example 6.05



- Determine the location for the shear center of the channel section with $b = 4$ in., $h = 6$ in., and $t = 0.15$ in.

$$e = \frac{F\,h}{I}$$

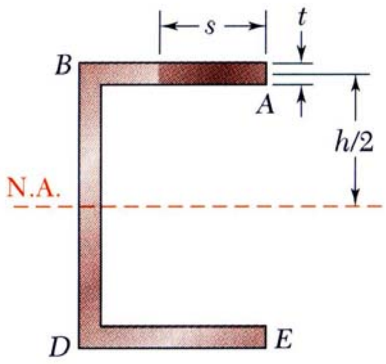
- where

$$F = \int_0^b q\,ds = \int_0^b \frac{VQ}{I}\,ds = \frac{V}{I}\int_0^b st\frac{h}{2}\,ds$$

$$= \frac{Vthb^2}{4I}$$

$$I = I_{web} + 2I_{flange} = \frac{1}{12}th^3 + 2\left[\frac{1}{12}bt^3 + bt\left(\frac{h}{2}\right)^2\right]$$

$$\cong \tfrac{1}{12}th^2(6b+h)$$

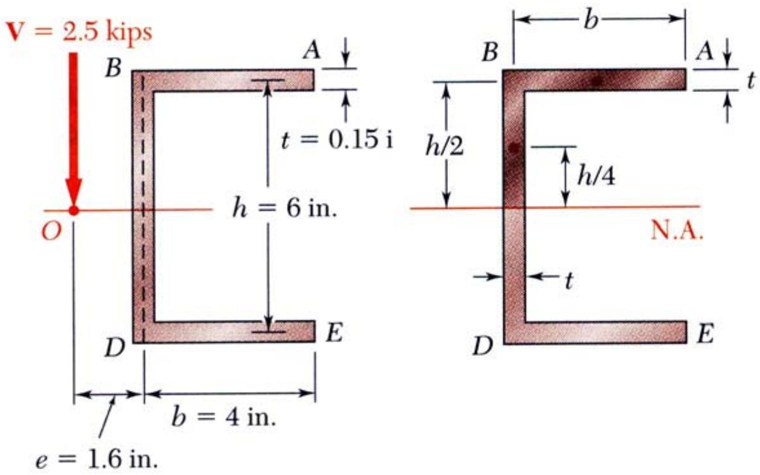


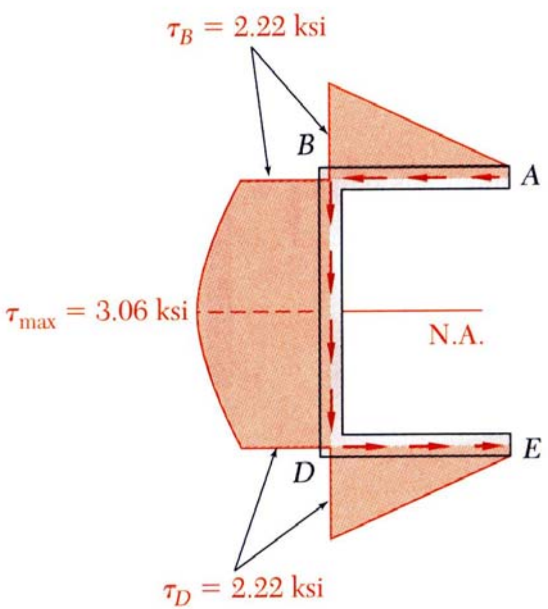
- Combining,

$$e = \frac{b}{2+\dfrac{h}{3b}} = \frac{4\,\text{in.}}{2+\dfrac{6\,\text{in.}}{3(4\,\text{in.})}}$$

$$\boxed{e = 1.6\,\text{in.}}$$

Example 6.06





- Determine the shear stress distribution for $V = 2.5$ kips.

$$\tau = \frac{q}{t} = \frac{VQ}{It}$$

- Shearing stresses in the flanges,

$$\tau = \frac{VQ}{It} = \frac{V}{It}(st)\frac{h}{2} = \frac{Vh}{2I}s$$

$$\tau_B = \frac{Vhb}{2\left(\frac{1}{12}th^2\right)(6b+h)} = \frac{6Vb}{th(6b+h)}$$

$$= \frac{6(2.5\,\text{kips})(4\,\text{in})}{(0.15\,\text{in})(6\,\text{in})(6\times 4\,\text{in} + 6\,\text{in})} = 2.22\,\text{ksi}$$

- Shearing stress in the web,

$$\tau_{\max} = \frac{VQ}{It} = \frac{V\left(\frac{1}{8}ht\right)(4b+h)}{\frac{1}{12}th^2(6b+h)t} = \frac{3V(4b+h)}{2th(6b+h)}$$

$$= \frac{3(2.5\,\text{kips})(4\times 4\,\text{in} + 6\,\text{in})}{2(0.15\,\text{in})(6\,\text{in})(6\times 6\,\text{in} + 6\,\text{in})} = 3.06\,\text{ksi}$$