PROBABILISTIC ASPECTS OF OCEAN WAVES

by

J.A. Battjes

The material in this report has been prepared as notes for lectures on
"Engineering aspects of ocean waves", given at the seminar on "Safety of
Structures under Dynamic Loading", Trondheim, June 23 tot July 1, 1977;
it will be published in the proceedings of that seminar.

CONTENTS

ENGINEERING ASPECTS OF OCEAN WAVES

J.A. Battjes   Delft University of Technology   Netherlands

1. Purpose and scope

In the lectures on "Engineering Aspects of Ocean Waves", the aspects
to be considered will be restricted to those which are relevant for the
dynamics of structures under wave loading. Furthermore, since the hydro-
dynamics of given wave (and current) fields near structures are dealt with
elsewhere in this seminar, only the probabilistic specification of the inci-
dent wave fields will be considered here, both for the short term and the
long term.

For an indication of the treatment to be adopted, it should be borne
in mind that in the lectures at this seminar knowledge of basic probability
theory and of the theory of random vibrations should be assumed, and
that recent developments should be included. In view of this and of
the available lecture time, a rather general overview of short-term and
long-term statistics will be given, while one area of recent developments
has been selected for a somewhat more detailed treatment - namely, the
short-term joint probability distribution of zero-crossing wave height
and period.

2. Short-term and long-term descriptions

In the description of sea states, it is meaningful to distinguish
various time (and length) scales.

In the so-called short-term description one adopts a time scale
such that on that scale the sea state is approximately stationary in
the statistical sense, while for purposes of sampling it should be

sufficient for statistically significant results.

The short-term variability of the wave field is characterized by means of various probabilistic and spectral distribution functions, with their associated parameters.

In the long-term description the aim is to characterize the long-term variability of the parameters of the short-term distributions.

3. Short-term description of sea states

3.1. Generalities

In the following, the variable water surface elevation above some reference plane is taken to be representative for a sea state. Particle velocities, pressures etc. are considered as dependent variables which need not be specified in addition to the surface elevation itself.

In short-term descriptions, the water surface elevation above the chosen reference plane is treated as a random process, approximately stationary in time and homogeneous in the horizontal coordinates. The specification of this process requires knowledge of the joint probability distributions of the elevations at an arbitrary number of arbitrary times and places. These probability distributions in turn can be characterized through their moments of various orders, the lowest-order ones being the most important.

For a process with zero mean, the second-order moments (or equivalently, the covariances) are the lowest order moments of interest, and therefore the most important of the whole hierarchy.

For a process which is stationary in time and homogeneous in the horizontal coordinates, all statistical properties depend on time- and space-intervals only, altogether three independent variables. The Fourier transform of the autocovariancefunction with respect to these variables yields the three-dimensional spectral variance density function (in the frequency-wave number domain). Reduced versions of this general three-dimensional spectral density function (which it is virtually impossible to measure) are obtained by projecting it on a sub-space of the frequency-wave number domain, such as the frequency axis, the wave number plane etc. Another reduction of the multi-dimensionality of the spectrum is obtained under the assumption of linearity of the wave field, in which case frequency and wave number are coupled through the linear dispersion equation.

A consequence of considering the wave field as a linear superposition of an uncountably infinite number of spectral components is that it can be treated as a Gaussian process. Such processes are fully specified by their mean value (usually assumed zero) and their autocovariancefunction, or by their mean and the spectral variance density function. This assumption is usually made in theoretical considerations of the probability structure of the wave field.

If the nonlinear coupling of the spectral components cannot be neglected then higher order moments than those of the second order are also required for a full specification of the wave field, or for an investigation of the degree of nonlinearity. The Fourier transform of the third-order moment function is called the bispectrum; it has been measured in a number of cases (refs. 23 and 28), but its use has so far been very limited. Fourth- and higher order moments and their Fourier transforms have not yet been measured to the author's knowledge. There is much room for research here, particularly in view of the importance of extreme sea states, for which the assumption of linearity is least valid.

It should be noted that even if the variance spectrum is not quite sufficient for a specification of the wave field, due to nonlinearities, it still remains eminently useful, because of its high information content in a condensed form.

Although the autocovariancefunction and the corresponding spectrum are formally equivalent with respect to their information content, the use of the spectral format for the presentation of that information is usually much to be preferred for a number of reasons, apart from those of computational efficiency:
- the spectrum localizes the contributions to the variance of the process in terms of frequency and wave number, and it thereby gives more insight into the underlying structure of the process than is possible through the autocovariancefunction;
- as a corollary, the structure of a given process, as revealed in its spectrum, usually can be more simply explained in terms of causative factors than in the case of the autocovariancefunction;
- the calculation of the effects of linear operators on the process is far simpler in the spectral domain (algebraic multiplication) than through the use of covariances (convolutions);
- the statistical theory of the sampling distribution of estimates from a finite sample, and the results obtained, are less complicated for spectra than they are for covariances.

In the following we shall first deal with some probabilistic properties of sea waves of a given variance spectrum, and after that with some aspects of the spectra themselves, particularly for wind-driven waves.

3.2. Probabilistic description of sea states

In this section we shall deal with some statistical properties of the water surface elevation at a fixed point, for a given sea state. Emphasis will be placed on level crossings, crest heights, wave heights and wave periods. Most of the theoretical results to be mentioned apply to a stationary Gaussian process with zero mean.

The value of the water surface elevation in a given vertical above its mean is written as $\underline{h}(t)$, in which t is time and the underscore denotes a random variable. The spectral variance density function of $\underline{h}(t)$ is written as $E(f)$, in which f is frequency in cycles per unit time. $E(f)$ is defined such that its integral over all positive values of f equals the variance of $\underline{h}(t)$. The moments of $E(f)$ about $f = 0$ are denoted by m_n:

$$m_n = \int_0^\infty f^n E(f) df. \tag{3.1}$$

3.2.1. Level crossings

Consider the upcrossings of an arbitrary level h by the process $\underline{h}(t)$. In other words, consider the events $\{\underline{h}(t) = h$ and $\underline{\dot{h}}(t) > 0\}$, in which a dot denotes a differentiation with respect to t. Let $\underline{n}(h;t_1,t_2)$ be the number of such events in a time interval (t_1,t_2). Considering an infinitesimal interval $(t,t+\delta)$, Rice (ref. 43) showed that

$$\Pr\{\underline{n};t,t+\delta t) = 1\} = (m_2/m_0)^{\frac{1}{2}}\exp(-h^2/2m_0)\delta t + o(\delta t) \tag{3.2}$$

and that

$$\Pr\{\underline{n}(h;t,t+\delta t) > 1\} = o(\delta t), \tag{3.3}$$

in which $o(\delta t)$ denotes any function of δt which with decreasing δt goes faster to zero than δt itself.

It follows from (3.2) that the expected number of upcrossings of $\underline{h}(t)$ through the level h per unit time, written as $\lambda(h)$, is given by

$$\lambda(h) = (m_2/m_0)^{\frac{1}{2}} \exp(- h^2/2m_0) . \tag{3.4}$$

In particular, the expected number of upcrossings through the level h = 0 ("zero upcrossings") per unit time is

$$\lambda(0) = (m_2/m_0)^{\frac{1}{2}}. \tag{3.5}$$

The time interval between consecutive zero-upcrossings, often called the "zero-crossing wave period", is a random variable, written as $\underline{T}$. Is expected value, the mean zero-crossing period, is written as T_z; it equals the reciprocal of $\lambda(0)$:

$$T_z = E(\underline{T}) = \lambda(0)^{-1} = (m_0/m_2)^{\frac{1}{2}}. \tag{3.6}$$

The shape of the probability distribution of $\underline{T}$ is rather sensitive to variations in the shape of the spectrum. Some approximate results for narrow spectra will be mentioned in conjunction with the joint probability distribution of zero-crossing wave height and period.

Extreme value

A quantity of interest is the probability that $\underline{h}(t)$ shall not exceed some value h in a given time interval (t_1, t_2). If $\underline{h}_{max}(t_1, t_2)$ is the maximum value of $\underline{h}(t)$ in such interval then obviously

$$\Pr\{\underline{h}(t) \le h \text{ for } t_1 \le t \le t_2\} = \Pr\{\underline{h}_{max}(t_1, t_2) \le h\}. \tag{3.7}$$

Various methods can be employed to estimate this probability. In the present context it is natural to consider the probability of non-occurrence of crossings of the level h by $\underline{h}(t)$, and to use $\Pr\{\underline{n}(h; t_1, t_2) = 0\}$ for an estimate of $\Pr\{\underline{h}_{max}(t_1, t_2) \le h\}$.

The expected value of $\underline{n}$ for a time interval of duration D is

$$E\{\underline{n}(h; t_1, t_1+D)\} = \lambda(h)D. \tag{3.8}$$

The probability distribution of $\underline{n}(h; t_1, t_1+D)$ is assumed to be of the Poisson-type, which is asymptotically correct for large $h/\sqrt{m_0}$ (ref. 13).

It then follows that

$$Pr\{\underline{n} = 0\} \simeq e^{-E\{\underline{n}\}},$$ (3.9)

or

$$Pr\{\underline{h}_{max}(t_1, t_1+D) \leq h\} \simeq e^{-\lambda(h)D} = e^{-\lambda(0)De^{-h^2/2m_0}},$$ (3.10)

provided $h/\sqrt{m_0}$ is sufficiently large. This is not a severe restriction from a practical point of view, since one is generally interested in large values of $h/\sqrt{m_0}$ anyway.

It should be noted that the preceding results did not require the assumption of a narrow spectrum. Furthermore, the validity of

$$Pr\{\underline{h}_{max}(t_1, t_1+D) \leq h\} \simeq e^{-E\{\underline{n}(h;t_1,t_1+D)\}}$$ (3.11)

is not restricted to stationary processes. For non-stationary processes it is only necessary to replace (3.8) by

$$E\{\underline{n}(h;t_1,t_1+D)\} = \int_{t_1}^{t_1+D} \lambda(h,t)dt.$$ (3.12)

3.2.2. Maxima, crest heights, and wave heights

Before dealing with the distributions of the heights of maxima etc., some more general remarks will be made concerning the relations between the appearance of the realizations of $\underline{h}(t)$ and its variance density spectrum.

Consider the maxima of $\underline{h}(t)$. The average time interval between consecutive maxima is written as T_m. The ratio of this to the mean zero-upcrossing period T_z, written as

$$r = T_m/T_z \qquad (0 \leq r \leq 1),$$ (3.13)

is a parameter which in some sense is a measure of the degree of irregularity of the process $\underline{h}(t)$. This has already been pointed out in 1921 by Taylor in his pioneering work on the statistical theory of turbulence (ref. 47).

Narrow-banded processes have realizations which have the appearance

6

of a slowly modulated sine curve, without positive minima or negative maxima, in which case $r \simeq 1$.

Broad-banded processes have realizations with a less regular appearance, with numerous positive minima and negative maxima, in which case r can be much less than one. It can even approach zero.

The proportion of positive maxima (α, say) is closely related to r. If the process $\underline{h}(t)$ is statistically symmetrical about its mean value, then a geometrical argument (ref. 10) shows that

$$\alpha = \tfrac{1}{2}(1 + r). \tag{3.14}$$

Another way of looking at these aspects is to consider the correlation between the function value $\underline{h}(t)$ and its second derivative, say its curvature, $\underline{\ddot{h}}(t)$. The coefficient of linear correlation between these (ρ) can be expressed in terms of the moments of the variance spectrum as follows:

$$\rho = -\, m_2/(m_0 m_4)^{\frac{1}{2}}. \tag{3.15}$$

If the spectrum is narrow then the right-hand side of (3.15) is near to -1. Considering the corresponding realizations, with the appearance of a slowly modulated sine curve, we see that $\underline{h}(t)$ and $\underline{\ddot{h}}(t)$ are strongly negatively correlated, in agreement with (3.15). With increasing spectral width, the right-hand side of (3.15) diminishes, while the realizations display an increasing proportion of negative maxima and a corresponding decrease of the absolute value of the correlation between $\underline{h}(t)$ and $\underline{\ddot{h}}(t)$, in agreement with (3.15).

It should be noted that the preceding remarks are in no way restricted to Gaussian processes. However, to proceed further by theory it is of course most convenient to introduce such a restriction. If this is done then (3.6) can be used, for instance. Applying it to $\underline{\dot{h}}(t)$ gives the result

$$T_m = (m_2/m_4)^{\frac{1}{2}}, \tag{3.16}$$

in which case

$$r = m_2/(m_0 m_4)^{\frac{1}{2}} = -\,\rho, \tag{3.17}$$

which (for Gaussian processes) links the two viewpoints presented above.

7

Maxima

We now consider the heights of the maxima of $\underline{h}(t)$, denoted by $\underline{h}_m$ (see figure 1).
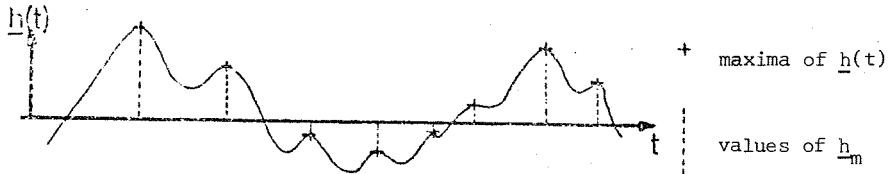

Figure 1 – Definition sketch for heights of maxima

The distribution of $\underline{h}_m$ was first derived by Rice (ref. 43). It has been further elaborated and compared with ocean wave data by Cartwright and Longuet-Higgins (ref. 10). It will here be reproduced for easy reference.

The co-cumulative probability of the normalized height of the maxima, defined by

$$\underline{\eta}_m = \underline{h}_m / \sqrt{m_0},\qquad\qquad(3.18)$$

can be written as

$$q_{\underline{\eta}_m}(\eta;\varepsilon) \equiv \Pr\{\underline{\eta}_m > \eta\} = \tfrac{1}{2}\operatorname{erfc}(\frac{\eta}{\sqrt{2}\varepsilon}) + \tfrac{1}{2}re^{-\frac{1}{2}\eta^2}\operatorname{erfc}(-\frac{r\eta}{\sqrt{2}\varepsilon}),\qquad(3.19)$$

in which

$$\varepsilon^2 = 1 - m_2^2/(m_0 m_4) = 1 - r^2\qquad (0 \le \varepsilon \le 1),\qquad\qquad(3.20)$$

and

$$\operatorname{erfc}(x) \equiv \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2}\,dt\,.\qquad\qquad(3.21)$$

The proportion of positive maxima is

$$\alpha = q_{\underline{\eta}_m}(0;\varepsilon) = \tfrac{1}{2}(1 + r),\qquad\qquad(3.22)$$

8

in agreement with (3.14).

If for some constant $\varepsilon < 1$ we increase η, then the complementary error functions in (3.19) converge rapidly to 0 and 2 respectively, so that

$$q_{\underline{\eta}_m}(\eta;\varepsilon) \simeq r\, e^{-\frac{1}{2}\eta^2} = (T_m/T_z)e^{-\frac{1}{2}\eta^2} \quad \text{for } \eta r/\varepsilon \gg 1. \tag{3.23}$$

In the narrow-spectrum case, $\varepsilon \to 0$ and $r \to 1$, in which case the inequality in (3.23) is fulfilled for all η, so that

$$\begin{aligned} q_{\underline{\eta}_m}(\eta;0) &= 1 && \text{for } \eta \leq 0 \\ &= e^{-\frac{1}{2}\eta^2} && \text{for } \eta > 0, \end{aligned} \tag{3.24}$$

which is the well-known Rayleigh distribution (refs. 2, 33). This result can also be established more directly by noting that narrow-band processes have virtually no positive minima, so that there is just one maximum exceeding h for each upcrossing of h by $\underline{h}(t)$, for all $h > 0$. By the same token, all the maxima are positive, so that

$$\Pr\{\underline{h}_m > h\} = \frac{\lambda(h)}{\lambda(0)} = e^{-h^2/2m_0} \quad \text{for } h > 0, \tag{3.25}$$

provided the spectrum is sufficiently narrow.

An empirical correction to the distribution (3.23) has been proposed by Jahns and Wheeler (ref. 32), who point to significant non-Gaussian behavior of $\underline{h}(t)$ in water of restricted depth, leading to probabilities of occurrence of high maxima far in excess of those predicted by (3.23). The expression proposed by them for the range $h > 3(m_0)^{\frac{1}{2}}$ can be written as

$$\Pr\{\underline{h}_m > h\} = r\, e^{-\frac{h^2}{2m_0}}\left\{1 - B_1\frac{h}{d}\left(B_2 - \frac{h}{d}\right)\right\}, \tag{3.26}$$

in which d is the mean water depth, and B_1 and B_2 are empirical coefficients estimated to have the values (4.0 ± 0.2) and (0.60 ± 0.02), respectively. The ratio $\sqrt{m_0}/d$ ranged up to about 0.1 in the data on which (3.26) is based.

Crest heights

The approximate distribution function (3.23) of the maxima $\underline{h}_m$ suggests a Rayleigh-like behaviour, provided we consider not all

the maxima, but only a fraction (T_m/T_z) thereof. In particular, we now choose to consider the so-called crest-height $\underline{h}_c$ (normalized: $\underline{n}_c = \underline{h}_c/\sqrt{m_0}$), which is defined as the largest value (the largest maximum) of $\underline{h}(t)$ on an interval between a zero-upcrossing and the next zero-downcrossing of $\underline{h}(t)$ (see figure 2).
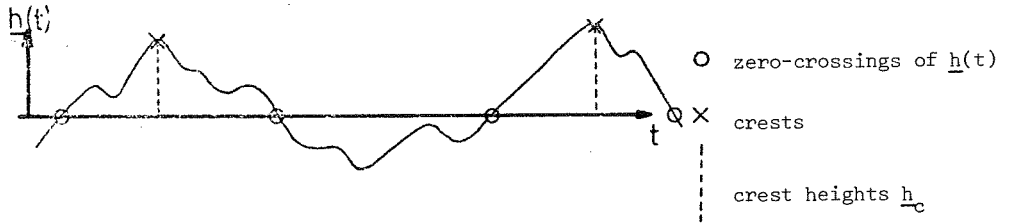

Figure 2 - Definition sketch for crest heights

For each crest height exceeding a level h, there may be more than one upcrossing of $\underline{h}(t)$ through the level h. But with increasing $h/\sqrt{m_0}$, the probability of such an event approaches zero. And since the expected number of crest heights per unit time equals the expected number of zero-upcrossings per unit time, we have

$$\Pr\{\underline{h}_c > h\} \simeq \frac{\lambda(h)}{\lambda(0)} = e^{-\frac{1}{2}h^2/m_0} \quad \text{for } h/\sqrt{m_0} \gg 1, \qquad (3.27)$$

so that the crest heights are approximately Rayleigh-distributed in the higher range, for all $\varepsilon > 1$.

A more refined estimate of the distribution of $\underline{h}_c$ than that given above has been presented by Bonneau (ref. 5). He accounts for the fact that there may be more than one upcrossing of $\underline{h}(t)$ through h for each crest height exceeding h, the excess number being equal to the number of minima between the levels 0 and h minus the number of maxima in the same range, both in the time interval (t_1, t_2) defining the crest being considered (see figure 3).
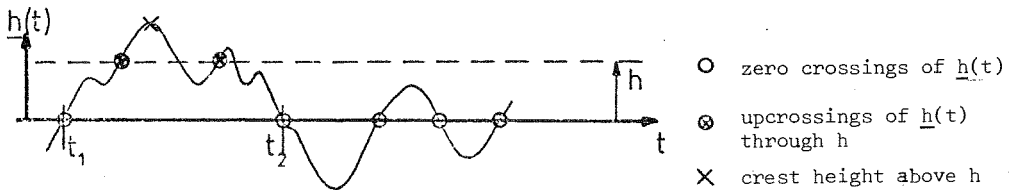

Figure 3 - Sketch showing relation between numbers of extremes and level crossings

10

If δ denotes the expected value of this excess number, then it follows from the definitions adopted that

$$\lambda(h) = \lambda(0)\Pr\{\underline{h}_c > h\}(1 + \delta), \tag{3.28}$$

or, switching to the normalized crest height $\underline{n}_c = \underline{h}_c/\sqrt{m_0}$:

$$q_{\underline{n}_c}(\eta;\varepsilon,\ldots) \equiv \Pr\{\underline{n}_c > \eta\} = \frac{e^{-\frac{1}{2}\eta^2}}{1 + \delta}. \tag{3.29}$$

Bonneau derives some rigorous constraints on the variations of δ with η and ε. By making some ad hoc assumption in addition he arrives at an expression for δ which is here written in the form

$$\delta(\eta;\varepsilon) = \frac{1}{2r}\left[e^{-\frac{1}{2}\eta^2}\{1 - r\ \mathrm{erf}(\frac{r\eta}{\sqrt{2}\varepsilon})\} - \mathrm{erfc}(\frac{\eta}{\sqrt{2}\varepsilon})\right]. \tag{3.30}$$

An approximation to (3.30) is

$$\delta(\eta;\varepsilon) \simeq \frac{1-r}{2r}\ e^{-\frac{1}{2}\eta^2} \quad \text{for } (r/\varepsilon)\eta \gg 1. \tag{3.31}$$

Bonneau's correction to (3.27) is mainly relevant to the intermediate range of crest heights, since it was made to vanish for both very small and very large crest heights ($\delta < 0.02$ for all $\eta > 2.5$ and $\varepsilon < 0.8$). It is therefore not of much practical importance for estimating extreme conditions. Note also that Bonneau's correction to (3.27) is within the framework of the Gaussian model for $\underline{h}(t)$, whereas this is not the case for the empirical correction to (3.23) proposed by Jahns and Wheeler (eq. 3.26).

Extreme value

We shall in this paragraph reconsider the extreme value of $\underline{h}(t)$ in a finite duration, this time using the crest heights for the parent population.

Let $\underline{n}$ denote the number of crest heights occurring in the time interval $(t,t+D)$, and $\underline{m}$ the number of these which exceed h. We then have

$$N \equiv E(\underline{n}) = \lambda(0)D \qquad (3.32)$$

and

$$E(\underline{m}) = N \, Pr\{\underline{h}_c > h\}. \qquad (3.33)$$

We now consider the largest crest height in the time interval $(t_1, t_1 + D)$, written as $\underline{h}_{c,max}(t_1, t_1 + D)$. In view of the definitions adopted, we have

$$Pr\{\underline{h}_{c,max}(t_1, t_1+D) \leq h\} = Pr\{\underline{m} = 0\}. \qquad (3.34)$$

Usually, only such cases are of interest in which $N \gg 1$ and $Pr\{\underline{h}_c > h\} \ll 1$. If furthermore the individual crest heights are assumed to be stochastically independent, then $\underline{m}$ is Poisson-distributed, and

$$Pr\{\underline{m} = 0\} = e^{-E(\underline{m})}, \qquad (3.35)$$

so that

$$Pr\{\underline{h}_{c,max}(t_1, t_1+D) \leq h\} = e^{-E(\underline{m})}. \qquad (3.36)$$

The condition of stochastic independence of the individual crest heights is actually unnecessarily restrictive. Watson (ref. 51) has shown that (3.36) also holds in the limit $N \to \infty$ if the crest heights are "m-dependent", which means that $\underline{h}_c^{(j)}$ and $\underline{h}_c^{(k)}$ are independent if $|j - k| > m$, where $\underline{h}_c^{(j)}$ is the j-th crest height in the sequence. It seems very reasonable to make such an assumption.

It should also be noted that (3.36) holds for non-stationary processes as well, provided $E(\underline{m})$ is estimated as the time integral of $\lambda(0)Pr\{\underline{h}_c > h\}$, in which both factors can vary with t. This has been considered in detail by Borgman (ref. 7).

The restriction to relatively large values of h implies that theoretically (in the Gaussian model) (3.27) holds, in which case (3.33) becomes

$$E(\underline{m}) = Ne^{-h^2/2m_0} = \lambda(h)D \qquad (3.37)$$

(for stationary processes), so that

$$Pr\{h_{c,max}(t_1,t_1+D) \le h\} = e^{-Ne^{-h^2/2m_0}},$$ (3.38)

which corresponds to (3.10). A similar result is also obtained for the largest of all the maxima (h_m). This case has been investigated in detail by Cartwright (ref. 9), who also gave an empirical verification of the validity of the results. About 15 years later, Ochi (ref. 39), apparently unaware of Cartwright's work, considered the problem of the distribution of the largest of the positive maxima, which needless to say again gave the same result.

An interesting and practically important point in these results is that they do not depend on ε; m_0 and m_2 are the only spectral parameters required. The distribution (3.38) is therefore the same as for the case of a narrow spectrum, for which it has first been presented by Longuet-Higgins (ref. 33). Thus, the expressions derived by Longuet-Higgins for the expected value and the mode of h_c carry over without modification to cases of arbitrary spectrum shapes.

The theory of extreme value statistics (ref. 20) has been applied by Cartwright (ref. 9) to the distribution (3.23). For large N the dispersion of $h_{m,max}/(m_0)^{\frac{1}{2}}$ about its mean value is relatively small, and that of the second largest value of h_m is even smaller. A similar statement applies to the extreme values of h_c. For this reason an estimate of m_0 can be made with a reasonably small coefficient of variation, just on the basis of an observation of N and of the highest one or two values of h_m (or h_c) (refs. 9, 50). The fact that this procedure gives an unbiased estimate of m_0 has recently been re-confirmed by Haring et al (ref. 21), on the basis of surface records made under a variety of storm conditions. This is of particular interest in view of the fact that Haring et al do concur with Jahns and Wheeler's conclusion (ref. 32) that the theoretical parent distribution of h_m (3.23) or of h_c (3.27) would grossly underestimate the actual probability of occurrence of the relatively high maxima. It seems worthwhile to make further study of these seemingly contradictory findings.

Wave height

A wave height H is here defined as the total range of $h(t)$ in a time interval between two consecutive zero-upcrossings of $h(t)$ (see figure 4).
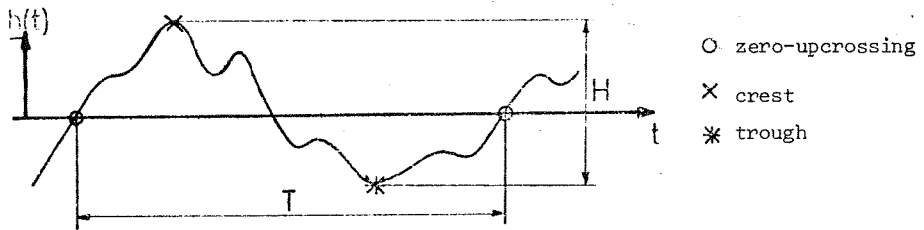
13

Figure 4 - Definition sketch for zero-upcrossing height and period

In the case of a narrow spectrum, the largest depth of a trough between zero-crossings is almost equal in absolute value to the height of the crest immediately preceding it, so that $\underline{H} \simeq 2\underline{h}_c$. Since $\underline{h}_c$ is Rayleigh-distributed, so is $\underline{H}$:

$$\Pr\{\underline{H} > H\} = \exp(-H^2/8m_0). \tag{3.39}$$

With increasing spectral width, the correlation between consecutive crest heights and trough depths diminishes. Even though these quantities separately may in theory still be approximately Rayleigh-distributed, at least the larger ones, there is no theoretical ground for the same statement applied to $\underline{H}$. Yet, there is much empirical evidence in support of a Rayleigh-distribution for $\underline{H}$ (see e.g. refs. 8 , 27, 49 and 53), and its use is generally deemed to be justified, though with a scale parameter for the heights which is often found to be between 5% and 10% less than that of eq. 3.39 (see e.g. ref. 27). Similarly, the distribution of the largest wave height in a sample is generally found to be adequately described by the double-exponential type given in eq. 3.38.

An exception to the rather general acceptance of the Rayleigh-distribution for the wave heights is made by Jahns and Wheeler (ref. 32), who point to the less-than-100% correlation between crest height and following trough depth in broad-band Gaussian processes, and who find it necessary to correct for this, with the effect of decreasing the exceedance probabilities of the larger wave heights. Their results are corroborated by those of Haring et al (ref. 21), who give the following empirical distribution:

$$\Pr\{\underline{H} > H\} = \exp \left\{ -\frac{H^2}{8m_0} \left(C_1 + C_2 \frac{H}{(m_0)^{\frac{1}{2}}} \right) \right\}. \tag{3.40}$$

Compared to the Rayleigh distribution based on the narrow-band, Gaussian model (eq. 3.39), the scale parameter is changed (if $C_1 \neq 1$), and there is an increasing deviation from a Rayleigh-behavior with increasing $H/(m_0)^{\frac{1}{2}}$ (if $C_2 \neq 0$). The authors give no explicit information about

14

the magnitude of C_1 and C_2. The combined effect is such that the wave height with an exceedence probability of 10^{-3} is about 10% less than predicted on the basis of (3.39).

Regarding the more general question of the applicability of the theoretical results as given above (based on the Gaussian model) to empirical data, no unique and definitive answer can be given, at least from a purely statistical point of view. Considering the question in a broader context, the reliability of input data becomes important, as well as the sensitivity of results of subsequent calculations to variations in the results discussed here. It is the author's opinion that such discrepancies as may exist in the short-term descriptions of a sea state of given parameters are generally insignificant compared to the uncertainties in those parameters themselves.

Wave groups

The occurrence of wave groups is of importance in various applications. In the context of dynamics of structures one can think of the build-up of the response to a sequence of high waves (response with the frequency of the waves), or of the response to low-frequency drift forces (response with the frequency of the wave groups).

For a review of theoretical and empirical results on the length of groups of wave heights exceeding a certain threshold, the reader is referred to Goda (ref. 19), who presented such a review at the BOSS Symposium in 1976. Suffice it here to say that the empirical frequencies of occurrence of long groups of large wave heights were considerably in excess of the available theoretical estimates.

3.2.3. Joint distribution of heights and periods

The effects of periodic waves on structures vary both with the height and the period of the wave train. An estimate of such effects for wind-generated waves is conveniently made in the spectral domain, if linearity can be assumed. An estimate of nonlinear effects is often made in the time domain on a wave-by-wave basis. In this approach the random wave motion is supposedly characterized by a joint distribution of heights and periods. From this, a distribution of the effect being considered is calculated, assuming that the same transformation applies as in the case of periodic waves. This

hypothesis of equivalency has been checked and found valid for
run-up of waves breaking on dike slopes (ref. 3). It has also widely
been used for the calculation of nonlinear wave forces on piles. It
is obvious that at least in procedures such as sketched above, knowledge
of the joint distribution of heights and periods is necessary. As mentioned in
the introduction, the recent developments in this area will be dealt with
in some detail.

    We shall in the following consider the joint distribution of $(\underline{H},\underline{T})$,
in which $\underline{T}$ is defined as the duration of a time interval between two
consecutive zero-upcrossings of $\underline{h}(t)$, and $\underline{H}$ is the total range of $\underline{h}(t)$
on that interval (see figure 4). In the following the normalized variables
$\underline{\xi} = \frac{1}{2}\underline{H}/\sqrt{m_0}$ and $\underline{\tau} = \underline{T}/E\{\underline{T}\} = \underline{T}/\bar{T}$ are used.

Bretschneider distribution

    Until recently the most widely used joint distribution of $\underline{H}$ and
$\underline{T}$ was that due to Bretschneider (ref. 8), who found that the Rayleigh
distribution would apply not only to $\underline{H}$ but also to $\underline{T}^2$, at least for
wind-driven waves. He also found that in developed seas $\underline{H}$ and $\underline{T}$ were
virtually uncorrelated. Assuming that this implies stochastic independence,
the joint probability density of $(\underline{\xi},\underline{\tau})$ becomes

$$p(\xi,\tau) = p(\xi)p(\tau), \qquad\qquad (3.41)$$

in which

$$p(\xi) \quad = \xi e^{-\frac{1}{2}\xi^2} \qquad\qquad (3.42)$$

and

$$p(\tau) \quad = 4\{\Gamma(\tfrac{5}{4})\}^4 \tau^3 e^{-\{\Gamma(\frac{5}{4})\}^4 \tau^4} \simeq 2.7\tau^3 e^{-0.675\tau^4}. \qquad\qquad (3.43)$$

    While there is some empirical support of (3.42),as discussed in
a previous paragraph, and to some extent also of (3.43) (see e.g. refs.
8, 48, 53), the joint density (3.41) has hardly been tested empirically.
A fairly recent attempt has been made by Earle et al (ref. 16), using
data from hurricane "Camille". They conclude that "extreme wave proba-
bilities are not accurately estimated using independent Rayleigh dis-
tributions". They ascribe this in part to the assumption of independence,
which they find to be not always correct.

    A two-dimensional Rayleigh-distribution with arbitrary degree of
correlation between $\underline{H}$ and $\underline{T}$ has been used tentatively by Battjes (refs.
3 , 4). This might possibly give a better fit to the data used by

16

Earle et al, but this has not been checked.

It was noted above that there is some empirical evidence in support of (3.43), in wind-driven waves. While this is true, there are also numerous cases in which the period distribution is quite different. This is not surprising inasmuch as the shape of the period distribution is sensitive to variations in the shape of the wave spectrum, much more so than the shape of the wave height distribution. This implies that it is rather meaningless to try to search for a common shape of the period distributions of waves with quite different spectral shapes. One must select certain categories of spectra, such as very narrow ones, or spectra for pure wind-driven seas.

Leaving aside the question of the goodness-of-fit of (3.43) and (3.41), these distributions still have the limitation of being purely hypothetical at worst, or purely empirical at best. There are however at least two theoretically derived joint amplitude-period distributions, both based on the model of a stationary Gaussian process with a narrow spectrum. One is due to Wooding (ref. 52) and Longuet-Higgins (refs. 34, 35), another to Cavanie et al (ref. 11).

Solution by Wooding and Longuet-Higgins

In 1955, Wooding derived a joint probability density function of wave frequency and amplitude for the case of a narrow spectrum. Shortly afterwards, a similar result was derived independently by Longuet-Higgins (ref. 34) in his work on random surfaces. More recently, Longuet-Higgins (ref. 35) reformulated the solution and its derivation, and compared it with some empirical data.

The derivation is based on a consideration of the envelope $\underline{R}(t)$ and the associated phase function $\underline{\chi}(t)$ of the process $\underline{h}(t)$, and their first derivatives. The joint probability density of these can be found by using Rice's results on the statistics of the envelope of narrow-band processes (ref. 43). It reduces to a relatively simple form if the mean frequency of the spectrum, defined by

$$\bar{f} = m_1/m_0,$$ (3.44)

is used as the central frequency in the definition of the envelope.

The joint probability density of $\underline{R}(t)$ and $\dot{\underline{\chi}}(t)$ is used to estimate the joint probability density of the wave amplitude $\underline{a}$ and period $\underline{T}$, using the approximations $\underline{a} \approx \underline{R}$ and $\underline{T} \approx 2\pi/\dot{\underline{\chi}}$. These approximations,

17

and the subsequent transformation from $\underline{\dot{\chi}}$ to $\underline{T}$, are based on the assumption that the spectral width parameter ν, defined as

$$\nu = (m_0 m_2/m_1^2 - 1)^{\frac{1}{2}}, \qquad (3.45)$$

is small. Terms of $0(\nu^2)$ are in fact neglected.

The normalized variable $\underline{\xi}$ is now defined as

$$\underline{\xi} = \underline{a}/(m_0)^{\frac{1}{2}}, \qquad (3.46)$$

and a normalized and reduced period $\underline{\zeta}$ as

$$\underline{\zeta} = \frac{\bar{f}\,\underline{T} - 1}{\nu} . \qquad (3.47)$$

The joint probability density of $(\underline{\xi},\underline{\zeta})$ is then given by

$$p(\xi,\zeta) = \frac{\xi^2}{(2\pi)^{\frac{1}{2}}} \exp\{- \tfrac{1}{2}\xi^2(1 + \zeta^2)\}. \qquad (3.48)$$

The marginal densities are

$$p(\xi) = \xi \exp(- \tfrac{1}{2}\xi^2), \qquad (3.49)$$

which is the Rayleigh density, as expected, and

$$p(\zeta) = \tfrac{1}{2}(1 + \zeta^2)^{-3/2}. \qquad (3.50)$$

It follows that $E\{\underline{\zeta}\} = 0$, so that $\bar{T} \equiv E\{\underline{T}\} = (\bar{f})^{-1}$. The interquartile ranges of $\underline{\zeta}$ and $\underline{T}$ are given by

$$IQR(\underline{\zeta}) = 2/\sqrt{3} \text{ and } IQR(\underline{T}) = 2\nu\bar{T}/\sqrt{3}. \qquad (3.51)$$

The conditional density of $\underline{\zeta}$ is

$$p(\zeta|\xi) = \frac{p(\xi,\zeta)}{p(\xi)} = \frac{\xi}{(2\pi)^{\frac{1}{2}}} \exp(- \tfrac{1}{2}\xi^2\zeta^2), \qquad (3.52)$$

which is Gaussian, with zero mean, standard deviation ξ^{-1}, and inter-quartile range $1.35\xi^{-1}$. It follows that the conditional mean of $\underline{T}$ equals $(\bar{f})^{-1}$, independent of ξ, and that the higher the waves, the more likely it is that their period is close to the mean value.

Longuet-Higgins compares his theoretical results with measured data presented by Bretschneider (ref. 8) for a sequence of 399 consecutive. pairs of values of $H/\bar{H}$ and $\underline{T}^2/\overline{T^2}$. In this analysis, ν is estimated from (3.51), using the interquartile range of the observed values of $(T^2/\overline{T^2})^{\frac{1}{2}}$ as an estimate of $IQR(\underline{T}/\bar{T})$.

The theoretical marginal densities (3.49) and (3.50) are found to provide a "reasonable fit" to the data.

Two tests of the validity of the joint density are given. The conditional mean of $T/\bar{T}$ (actually, Longuet-Higgins uses $(T^2/\overline{T^2})^{\frac{1}{2}}$) is found to be scattered somewhat about 1, but without trend with $H/\bar{H}$, in accordance with the theory. The predicted conditional interquartile range is also in good agreement with the data.

The following comments are made.

1. Strictly speaking, (3.50) implies a non-zero probability of negative wave periods, but this probability becomes vanishingly small ($\approx \frac{1}{4}\nu^2$) as $\nu \to 0$.

2. The mean value of $\underline{T}$ is found to be m_0/m_1. This differs from the exact value of the mean zero-upcrossing interval $T_z = (m_0/m_2)^{\frac{1}{2}}$ derived by Rice, the relative difference being equal to $(1 + \nu^2)^{\frac{1}{2}} - 1$, or to $\frac{1}{2}\nu^2$ if $\nu^2 \ll 1$. Since the derivation is based on approximations to $0(\nu)$, there is no inconsistency. The point is only that one is here again reminded of the approximate character of the solution (within the framework of Gaussian processes), as in item (1).

3. Due to the fact that for large $|\zeta|$, $p(\zeta)$ as given by (3.50) falls off only as $|\zeta|^{-3}$, the second moment of $p(\zeta)$ diverges, so that the (root) mean square, the variance, and the standard deviation of $\underline{T}$, are not defined. This is the case for all ν, no matter how close to zero. This is a somewhat surprising conclusion, which serves to indicate that the theoretical density is not valid for large deviations of $\underline{T}$ from its mean.

4. In the comparison of theory with empirical data, the parameters of the distribution had to be estimated from the observed values of heights and periods themselves, since no variance spectrum of the record was available. As noted by Longuet-Higgins, this means that the theoretical relations between these parameters and the spectral moments m_0, m_1 and m_2 have not been checked.

5. Bretschneider's work, from which Longuet-Higgins took the data, contains only one measured joint density of heights and periods (squared), but several marginal cumulative distributions (ref. 8 , fig. 4.25 and fig. 4.26). The author has compared the cumulative distribution of wave periods based on (3.50), which is given by

19

$$P(\zeta) = \int_{-\infty}^{\zeta} p(\zeta')d\zeta' = \tfrac{1}{2} + \frac{\zeta}{2(1 + \zeta^2)^{\frac{1}{2}}} \ , \qquad\qquad (3.53)$$

to these data, for various assumed values of ν. The results are presented
in figure 5, from which it appears that the overall shapes of the
empirical distributions are better described by Bretschneider's equation

$$P(\tau) = \Pr\{\underline{T}/\bar{T} < \tau\} = 1 - e^{-\,0.675\tau^4} \ , \qquad\qquad (3.54)$$

which was in fact based on these data, than by Longuet-Higgins'
eq. (3.53). By choosing a proper value of $\underline{T}$ and $IQR(\underline{T})$, and therefore
of ν, the curves based on (3.53) can be made to agree with any of
the given empirical distributions in the middle range, but they will
still diverge strongly from the data in the tails, displaying far
greater probabilities of occurrence of very small and of very large
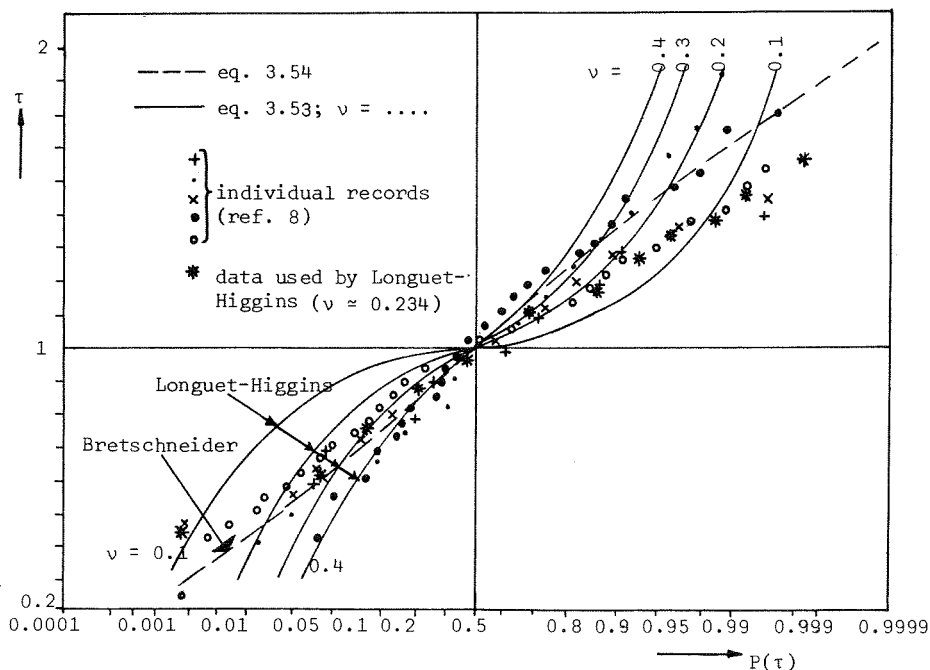periods than the data (or than 3.54). This is qualitatively in agreement
with item 3 above.

*Figure 5 - Comparison of theoretical cumulative distributions of relative
wave period to empirical data*

20

Solution by Cavanie, Arhan and Ezraty

Another approach to the theoretical derivation of the joint probability density of wave amplitude and period in narrow-band Gaussian processes has been taken by Cavanie et al (ref. 11,1). They consider the positive maxima ($\underline{h}_m+$) of $\underline{h}(t)$, and define a wave amplitude $\underline{a}$ and period $\underline{T}$ from the value of $\underline{h}(t)$ and $\underline{\ddot{h}}(t)$ at these maxima, using the same relations as for a pure sine function:

$$\underline{a} = \underline{h}_m+ \text{ and } \underline{T} = 2\pi(-\underline{h}/\underline{\ddot{h}})^{\frac{1}{2}}\Big|_{\underline{h} = \underline{h}_m+} \qquad (3.55)$$

Since $p(h,\ddot{h})$ is known (from Rice's work), $p(a,T)$ can be calculated. The resulting joint density of $\underline{\xi} = \underline{a}/(m_0)^{\frac{1}{2}}$ and $\underline{\tau} = \underline{T}/\bar{T}$ can be written as

$$p(\xi,\tau;\varepsilon) = \frac{2\alpha^3 \xi^2 \tau^{-5}}{(2\pi)^{\frac{1}{2}}\varepsilon(1-\varepsilon^2)\mu^4} \exp\left[-\frac{\xi^2 \tau^{-4}}{2\varepsilon^2 \mu^4}\{(\mu^2\tau^2 - \alpha^2)^2 + a^2\alpha^4\}\right] \qquad (3.56)$$

in which α, a and μ are given functions of ε. The coefficient α is the proportion of positive maxima, given by

$$\alpha = \tfrac{1}{2}\{1 + (1 - \varepsilon^2)^{\frac{1}{2}}\}, \qquad (3.57)$$

while

$$a^2 = \varepsilon^2/(1 - \varepsilon^2), \qquad (3.58)$$

and μ is the ratio of the mean value $\underline{T}$, as defined in (3.55), to the mean time interval between positive maxima ($\underline{T}_m+$):

$$\mu = \frac{E\{\underline{T}\}}{E\{\underline{T}_m+\}} = \mu(\varepsilon), \qquad (3.59)$$

in which

$$E\{\underline{T}_m+\} = \frac{1}{\alpha}(m_2/m_4)^{\frac{1}{2}}. \qquad (3.60)$$

The function $\mu(\varepsilon)$ has been calculated numerically by Cavanie et al, and is presented in tabular form in reference 1. For $\varepsilon = 0$, $\mu = 1$, while in the range ($0 \le \varepsilon \le 0.95$), μ deviates less than 7% from 1.

The marginal density of $\underline{\xi}$ as defined above is simply the density of the positive maxima, which can be obtained directly from the Rice distribution (3.19) for all the maxima. It will not be given here. Suffice it to say that it approaches the Rayleigh density for sufficiently large values of $r\xi/\varepsilon$.

The marginal density of $\underline{\tau}$ is

$$p(\tau;\varepsilon) = \frac{\alpha^3 a^2 \mu^2 \tau}{\{(\mu^2\tau^2-\alpha^2)^2 + a^2\alpha^4\}^{3/2}} \qquad (3.61)$$

The authors have made a comparison of their theory to measured values of $\underline{H}$ and $\underline{T}$, defined according to the zero-upcrossing convention (figure 4). A total of 182 records were used, selected on the criterion that they should have been made during storm conditions. The spectral width parameter ε for these records had a mean value of 0.865 and a standard deviation of 0.031 (a surprisingly small value).

The frequencies of occurrence of $\underline{H}/\sqrt{m_0}$ and $\underline{T}/\bar{T}$ in certain intervals for all the records were lumped in one figure, and compared to the theoretical joint density function 3.56 (modified according to $\underline{H} = 2\underline{a}$). The theoretical shape appears to agree well with the data.

The marginal distributions were also checked. The shape of the period distribution is well predicted in a range of approximately $0 < T/\bar{T} < 1.6$ (about 90% of the waves), but the empirical distribution displays a sort of cut-off in the range $1.6 < T/\bar{T} < 2.0$, in contrast to the theoretical distribution (example: for $\varepsilon = 0.865$, $\Pr\{\underline{T}/\bar{T} > 2\} \simeq 0.03$ theoretically; observed frequency $\simeq 0.002$).

It should be noted that for values of ε as large as 0.8 or more one can hardly expect the theory by Cavanie et al to be realistic with regard to the wave periods. This is certainly true for the prediction of the mean zero-upcrossing period, with which the theoretical period, calculated on the basis of eq. 3.55, bears but little resemblance, unless ε is small. The ratio

$$\frac{E\{\underline{T}\}}{T_z} = \frac{\mu}{\alpha}\frac{T_m}{T_z} = \frac{\mu(\varepsilon)r(\varepsilon)}{\frac{1}{2}\{1+r(\varepsilon)\}}, \qquad (3.62)$$

in which $\underline{T}$ is defined by (3.55), is approximately 0.7 for $\varepsilon = 0.8$ ($\mu(0.8) \simeq 0.93$, ref. 1). Nevertheless, the theoretical distribution of the normalized variable $\underline{\tau} = \underline{T}/E\{\underline{T}\}$ may be useful for the description of the distribution of the normalized zero-upcrossing intervals even for large values of ε, but the theoretical foundation of such eventuality is of course not very strong any more.

Comparison

It would be of interest to perform statistical tests of goodness-of-fit to investigate further the theoretical distributions proposed by Longuet-Higgins and by Cavanie et al. Lacking such a basis for a quantitative comparison, at the moment of writing, only a limited, qualitative comparison will be made.

The marginal distributions of the wave amplitudes are almost the same. Longuet-Higgins starts a priori from the envelope, which is always Rayleigh-distributed, while Cavanie et al consider the positive maxima, allowing for more maxima than zero-upcrossings. As noted above, the distribution of these positive maxima approximates to the Rayleigh distribution with decreasing spectral width and/or increasing value of the heights of the maxima being considered.

For a gross comparison of the marginal densities of $\underline{\tau} = \underline{T}/E\{\underline{T}\}$, reference is made to figure 6, which shows plots of (3.61) for $\varepsilon = 0.50$ and 0.70, of (3.50) for $\nu = 0.25$ and 0.35, and also of (3.43).
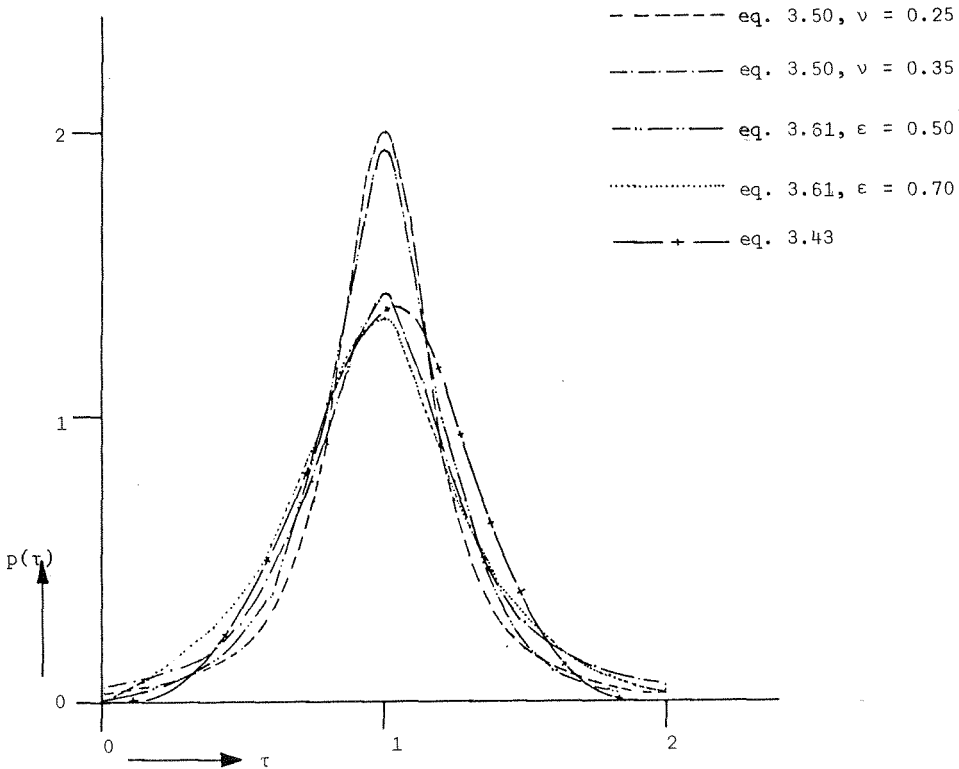

Figure 6 - *Marginal probability density functions of relative wave period*

The reason for choosing $\nu = \frac{1}{2}\varepsilon$ is that for a given spectrum these
parameters should satisfy the relation $\nu \simeq \frac{1}{2}\varepsilon$, as long as they are small
(ref. 35). It can be seen that with this choice (3.50) and (3.61) are
very similar, and that for $\varepsilon = 0.7$ and $\nu = 0.35$ both are similar to
the Rayleigh density for $\underline{T}^2$ (3.43), provided we consider the central
region of the distribution only. The functions shown behave quite
differently in the upper tail. Any judgement of the possible superiority
of one over the over depends on how much weight one wishes to give to these
different regions. Such evaluation should be carried out in the context
of the applications which are envisaged. Moreover, one should then look
beyond just the marginal densities.

For a gross comparison of the joint densities (3.48) and (3.56),
reference is made to figure 7, which shows a rough sketch of the general
pattern of lines of constant density.
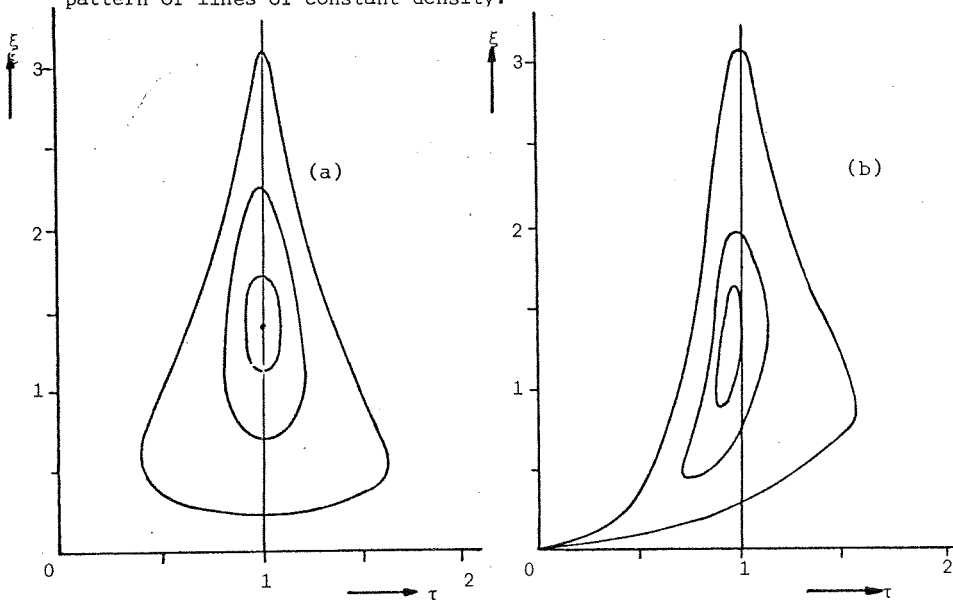
Figure 7 - Sketch of lines of constant joint probability density of amplitude
and period (a): eq. 3.48, Longuet-Higgins; (b): eq. 3.56, Cavanie
et al.

The two patterns are quite similar for the larger wave amplitudes,
showing an increasing concentration of wave periods around their (common)mean
with increasing amplitude. There is a distinct difference in the region
of lower amplitudes, however. Longuet-Higgins' solution is symmetrical
about the mean, while the solution given by Cavanie et al shows a decrease
of the conditional mean period with decreasing amplitude. The author

24

has often seen similar trends in Dutch wave data (see e.g. Svasek, ref. 46).
The data given by Cavanie et al (28240 waves) also show it quite clearly.
The data presented by Chakrabarti and Cooley (ref. 12) again give the same
picture. It is the author's impression that in this respect the solution
given by Cavanie et al is the most realistic, but the importance of such
differences as may exist between (3.48) and (3.56) is again open to
question. The fact that the difference is most pronounced for the smaller
wave amplitudes may suggest that it is not important, but this conclusion
would not necessarily be justified for problems involving particle velo-
cities and accelerations, wave steepness, and breaking, which are en-
hanced by decreasing the period. The last-mentioned two aspects will be
briefly considered below.

Distribution of wave steepness

The steepness ($\underline{S}$) of a wave with zero-upcrossing height $\underline{H}$ and
period $\underline{T}$ is here defined as

$$\underline{S} = \underline{H}/(g\underline{T}^2/2\pi), \tag{3.63}$$

and a corresponding normalized steepness ($\underline{s}$) as

$$\underline{s} = \underline{\xi}/\underline{\tau}^2. \tag{3.64}$$

This can also be written as

$$\underline{s} = \frac{g\sqrt{m_0}}{4\pi m_2} \underline{S} = 2 \frac{(gT_z^2/2\pi)}{H_s} \underline{S}, \tag{3.65}$$

using the significant wave height H_s, here equated to $4\sqrt{m_0}$, and the
mean zero-upcrossing period T_z, given by (3.6).

The distribution function of $\underline{s}$ can be calculated from the joint
probability density function of $(\underline{\xi},\underline{\tau})$ according to

$$P(s) \equiv Pr\{\underline{s} \le s\} = Pr\{\underline{\xi} < s\underline{\tau}^2\} = \int_0^\infty d\tau \int_0^{s\tau^2} p(\xi,\tau)d\xi. \tag{3.66}$$

The probability density of $\underline{s}$ is then given by

$$p(s) = \frac{dP(s)}{ds} = \int_0^\infty \tau^2 \left[p(\xi,\tau)\right]_{\xi=s\tau^2} d\tau. \tag{3.67}$$

It is expected that for $p(\xi,\tau)$ given by (3.48) or (3.56), these
integrals can only be evaluated numerically. However, substitution of

(3.41-3.43), proposed by Bretschneider, gives the explicit expression

$$P(s) = \frac{s^2}{2\Gamma^4(\frac{5}{4}) + s^2} \simeq \frac{s^2}{1.35 + s^2} .$$ (3.68)

Essentially the same result has been presented earlier by the author (ref. 3), as a special case of a more general expression based on a two-dimensional Rayleigh density for $(\underline{\xi}, \underline{\tau}^2)$, allowing for an arbitrary degree of correlation of $\underline{\xi}$ and $\underline{\tau}^2$.

In the case of a very narrow spectrum, such that the variability of the wave periods can be ignored, $\underline{S}$ is proportional to $\underline{H}$, and its distribution then is of the same type as that of $\underline{H}$. If the latter is of the classical Rayleigh-form, then it follows that

$$Pr\{\underline{s} > s\} = \exp(-\tfrac{1}{2}s^2).$$ (3.69)

Overvik and Houmb (ref. 40) define a steepness for each maximum $(\underline{h}_m)$ of $\underline{h}(t)$, by substituting

$$\underline{H} = 2|\underline{h}_m| \text{ and } \underline{T} = 2\pi |h/\ddot{h}|^{\frac{1}{2}}\big|_{\underline{h}=\underline{h}_m}$$ (3.70)

into (3.63), which gives

$$\underline{S} = -\frac{\pi}{g} \ddot{h}\big|_{\underline{h}=\underline{h}_m}$$ (3.71)

The distribution of $\underline{S}$ so defined is then calculated from the known (Gaussian) joint distribution of $\underline{h}(t)$, $\dot{\underline{h}}(t)$ and $\ddot{\underline{h}}(t)$, with the following result for the steepness, normalized as in (3.65):

$$Pr\{\underline{s} > s\} = \exp(-\frac{1 - \varepsilon^2}{2} s^2).$$ (3.72)

It follows that the steepness as defined by Overvik and Houmb is Rayleigh-distributed for all $\varepsilon < 1$. For a narrow spectrum, such that $\varepsilon \to 0$, eq. (3.72) approximates to (3.69).

According to Overvik and Houmb, the special case given by equation (3.69) should also apply to the steepness calculated from zero-crossing wave heights and periods for arbitrary ε, because in that case secondary maxima are neglected, which supposedly corresponds to the case $\varepsilon \to 0$. The present author does not see that this is a valid conclusion, at least on the grounds just stated. The arbitrary

26

neglect of the secondary maxima, which are present if $\varepsilon > 0$, does not suddenly cause them to be absent (as when $\varepsilon \to 0$). And if they are present then the distribution of $\underline{S}$ may well differ from (3.69).

In the evaluations of $P(s)$ referred to above, it is assumed that the steepness is unbounded. This is not quite correct because of wave breaking. An estimate of the probability of wave breaking (P_b) can be obtained by assuming an equivalency with periodic waves in deep water, which break if their steepness S exceeds a critical value (S_c) of approximately 0.13 (empirically) to 0.17 (theoretically). Denoting the corresponding normalized steepness with s_c, we have, using (3.68),

$$P_b = 1 - P(s_c) = \{1 + \tfrac{1}{2}\Gamma^{-4}(\tfrac{5}{4})s_c^2\}^{-1} \simeq \{1 + 0.74s_c^2\}^{-1}. \qquad (3.73)$$

For a developed wind sea, the ratio $H_s/(gT_z^2/2\pi)$ is typically about 0.05. Using $S_c \simeq 0.15$ then gives $s_c \simeq 6$ and $P_b \simeq 0.036$.

An equation equivalent to (3.73) has also been presented by Nath and Ramsey (ref. 37), who furthermore consider the distribution of the heights of the breaking waves, on the assumption that breaking waves of a given period T have a height equal to $S_c(gT^2/2\pi)$. Houmb and Overvik (ref. 26) have applied the same method, but they used Longuet-Higgins' joint probability density function for wave height and period.

The assumption just stated implies a clipping of the conditional distribution of $\underline{H}$, for given $\underline{T}$, which therefore becomes discontinuous at the breaking limit. Clipping of the height distribution has been used by Battjes (ref. 4) for calculations of set-up and longshore currents due to random waves on a beach, where breaking occurs due to depth limitations. Goda (ref. 18) has used an essentially similar analysis, but he assumed a range of possible breaking wave heights (for given period and depth), within which the probability of exceedence drops linearly from the highest value unaffected by breaking, to zero. The results obtained are quite good. But one should bear in mind that in the latter applications one is working with parameters of the distribution of all the wave heights, including the non-breaking ones (e.g., the overall mean square height). The results are therefore rather insensitive to possible errors in the estimates of probabilities in the region of breaking waves, as long as only a small fraction of the waves is breaking. Making estimates of probabilities for breaking waves exclusively, as is done in the refs. 28 and 20, is much more susceptible to significant errors.

27

3.3. Standard sea state spectra

3.3.1. Frequency spectra

In analyses of structures under dynamic loading, one is often con-
fronted with the problem of having to use a sea state spectrum, while
only a characteristic wave height and period are available. In such cases
it is necessary to make an assumption about the spectral form, the wave
height and period serving as scale parameters. Such assumptions are
often based on some standard form of spectra of pure wind-driven sea waves.
The Pierson-Moskowitz (PM) spectrum has been extensively used for this
purpose, but it has in this respect largely been replaced by the JONSWAP
(J) spectrum.

The PM-spectrum was introduced as a representation of spectra for
fully developed seas in deep water (ref. 42). Using frequencies in
cycles per unit time, it can be written as

$$E(f) = \alpha_{PM}(2\pi)^{-4}g^2f^{-5}\exp\{-\beta(f_0/f)^4\} \tag{3.74}$$

in which

$$\begin{aligned}
\alpha_{PM} &= 0.0081 \\
\beta &= 0.74 \\
f_0 &= g/(2\pi U_{19.5})
\end{aligned} \tag{3.75}$$

and $U_{19.5}$ is the wind speed at 19.5 m above sea level.
In the following, only this spectrum, and spectra which can be transformed
to exactly this expression, with the coefficients given by (3.75), will
be called the PM-spectrum. Note that it has only one free parameter, $U_{19.5}$,
which determines both the energy scale and the frequency scale
of the waves; the form of the spectrum is constant.

Shortly after its introduction, the PM-spectrum was already used
in recommendations for a standard form of sea state spectra. The ISSC
Committee I.1 for instance, in its report of 1964 (ref. 30), recommended
a spectrum similar to the PM-spectrum, but with two adjustable parameters,
H_v and T_v, the visually estimated wave height and period. On the
assumption that

$$H_v = 4\sqrt{m_0}, \quad T_v = m_0/m_1, \tag{3.76}$$

the following so-called ISSC-spectrum resulted:

$$E(f) = 0.11 \, H_v^2 T_v \, (T_v f)^{-5} \exp\{-0.44(T_v f)^{-4}\}. \tag{3.77}$$

This spectrum, and other spectra which can be written as

$$E(f) = Af^{-5} \exp\{-Bf^{-4}\}, \tag{3.78}$$

in which A and B are independent of f, will here be said to be of the PM-type.

It should be noted that already several years prior to the publication of the PM-spectrum, Bretschneider (ref. 8) had formulated a spectrum of the form (3.78). He uses significant wave height and period as scale parameters; these in turn can be obtained from the empirical growth curves of the SMB-type. Bretschneider's derivation of the spectrum is based on concepts which are not directly related to the concepts commonly used in definitions of variance density spectra (such as Fourier transforms). This makes a comparison of his results to those of others somewhat unfounded. However, in practice, this argument is given very little weight, or none at all. (The same can be said about the Neumann spectrum, which also came about in a way which had little to do, in a formal sense anyway, with common definitions of spectra. There is in fact a close link between the approaches taken by Neumann and Bretschneider.)

The JONSWAP (J) spectrum (ref. 22) applies to fetch-limited seas due to more or less stationary and homogeneous wind fields. This spectrum was formulated as a convenient means in the analysis of the JONSWAP data. It was not the purpose of that study to formulate a standard spectrum for general use. Nevertheless, it was very soon accepted as such.

The following five-parameter spectrum was found to give a uniformly good fit to nearly all of the spectra observed under "ideal" generation conditions (steady offshore wind, almost perpendicular to the shore, no swell) during JONSWAP:

$$E(f) = \alpha g^2 (2\pi)^{-4} f^{-5} \exp\{-\tfrac{5}{4}(f/f_m)^{-4}\}\gamma^{\exp\{-\tfrac{1}{2}(\frac{f - f_m}{\sigma f_m})^2\}} \tag{3.79a}$$

with

$$\sigma = \sigma_a \quad \text{for} \quad f \leq f_m \tag{3.79b}$$
$$\sigma = \sigma_b \quad \text{for} \quad f > f_m$$

It is written as the product of a PM-type spectrum, with two adjustable scale parameters α and f_m, in which f_m is the frequency at which this spectrum has its maximum, and a peak enhancement factor which equals γ for $f = f_m$, and unity for frequencies sufficiently far away from the peak.

The following fetch-dependence of α and f_m was proposed (ref. 22):

$$\alpha = 0.076 \ \tilde{x}^{-0.22} \quad , \ \tilde{f}_m = 3.5 \ \tilde{x}^{-0.33} \ , \tag{3.80}$$

in which

$$\tilde{x} = gx/U_{10}^2 \ , \ \tilde{f}_m = f_m U_{10}/g \tag{3.81}$$

x is the fetch, and U_{10} the wind speed at a height of 10 m above sea level.

The shape parameters γ, σ_a and σ_b displayed a large scatter (possibly due to small-scale perturbations of the mean wind field), but no significant trend with $\tilde{x}$. Their mean values are

$$\bar{\gamma} \simeq 3.3, \ \bar{\sigma}_a \simeq 0.07, \ \bar{\sigma}_b \simeq 0.09. \tag{3.82}$$

The spectrum (3.79), with the coefficients given by (3.80) through (3.82), is here called the J-spectrum. Spectra of the form (3.79) but with coefficients different from those given by (3.80) through (3.82) will be said to be of the J-type.

Subsequent to the publication of the J-spectrum, numerous additional fetch- or duration-limited sea state spectra were analyzed, including cases of highly non-stationary and inhomogeneous wind fields, from which the conclusion could be drawn that such spectra were generally of the J-type (see e.g. ref. 24 for a summary). It therefore appears to be justified to use a J-type standard spectrum for developing sea states.

A convenient parameterization for J-type spectra has been presented by the ISSC Committee I.1 in its 1976 report (ref. 31). For constant values of σ_a and σ_b, as given by (3.82), the first two moments of a J-type spectrum were computed numerically and presented as a function of γ, with the following results (due to J.A. Ewing):

γ	$=$	1	2	3	3.3	4	5	6
$\dfrac{m_0(\gamma)}{m_0(1)}$	$=$	1	1.24	1.46	1.52	1.66	1.86	2.04
$\dfrac{m_0(\gamma)/m_1(\gamma)}{m_0(1)/m_1(1)}$	$=$	1	0.95	0.93	0.92	0.91	0.90	0.89

With these numerical values, a J-type spectrum can be easily determined for given values of H_v, T_v and γ. Taking $\gamma = 3.3$, as a representative (?) mean value, and neglecting the effect of γ on the ratio m_0/m_1, the following result is obtained (ref. 31):

$$E(f) = 0.072 H_v^2 T_v (T_v f)^{-5} \exp\{-0.44(T_v f)^{-4}\} 3.3^{\exp\{-\frac{1}{2}(1.296 T_v f - 1)^2/\sigma^2\}}$$

$$(3.83)$$

Another parameterization has been worked out by Houmb and Overvik (ref. 26). They also assume $\sigma_a = 0.07$ and $\sigma_b = 0.09$, but they give not only α and f_m but also γ as function of $H_s(= 4\sqrt{m_0})$ and T_z ($= (m_0/m_2)^{\frac{1}{2}}$). In their formulation, γ decreases with decreasing mean steepness, approaching the value 1 for $S_s \equiv H_s/(gT_z^2/2\pi) \simeq 0.03$. This trend is reasonable, since one may expect the J-spectrum to approximate to the PM-spectrum for sufficiently large values of $\tilde{x}$, i.e. for sufficiently low values of S_s (which itself is a monotone non-increasing function of $\tilde{x}$). However, the value of 0.03 of S_s for which γ attains a value of about 1 is much lower than the value of S_s for the (original) PM-spectrum, which is $(\alpha_{PM}/\pi)^{\frac{1}{2}} \simeq 0.05$. According to Houmb and Overvik's results, $\gamma \simeq 4$ for $S_s \simeq 0.05$. The least which one can say therefore is that there are considerable discrepancies between these data and those used by Pierson and Moskowitz. In this connection it may be noted that in general little is known about the transition from a developing sea state into a fully developed one.

Houmb and Overvik have tabulated values of γ as high as about 7, for S_s up to 0.16. While a value of γ of about 7 has on occasion been measured, the steepness with which it is associated in ref.26 is unrealistically large.

The following brief comments are made with respect to the above.
1. There are applications in which the peakedness of the spectrum is important. Based on presently available data, it seems wise in such cases to distinguish between developing seas and fully developed seas, and to use J-type spectra for the former and PM-type spectra

31

for the latter.

2. The author has frequently heard statements to the effect that "The JONSWAP spectrum contains more energy than the Pierson-Moskowitz spectrum". Such statements are believed to be erroneous since they would imply that there are conditions in which both spectra would nominally be applicable. But this is not the case since the PM-spectrum is restricted to fully developed seas, and the J-spectrum to fetch-limited seas. Therefore, the statement cannot refer to a comparison of wave energies predicted from a given windfield. It can at most refer to a comparison of the two types of spectra fitted to known wave height and period values. But in that case the statement is meaningless. It would be trivial to say that the multiplication of a PM-type spectrum (or any spectrum, for that matter) with a factor exceeding 1 results in a spectrum with a larger total area than the originally assumed PM-type spectrum. However, it should also be obvious that hypotheses about the details of a process (such as the spectral distribution of energy) should at least be consistent with known overall-properties of that process (such as the total energy). Therefore, whether one fits a PM-type spectrum or a J-type spectrum to known values of a wave height (and period), the "predicted" values of the wave height (and period) should be the same in both cases. There seems to be little point in statements of the kind mentioned above.

3. The (maximum) peak enhancement factor γ is sometimes referred to as a ratio of the peak spectral density of the J-spectrum (or J-type spectra) to that of "the corresponding PM-spectrum". This again is considered to be erroneous, or at least meaningless. Comparing the peak spectral densities of two spectra which (one supposes) should represent the same sea state, but which do not even have the same total energy, is considered a meaningless exercise. Nevertheless, J-type spectra do have a higher peak spectral density than PM-type spectra for the same mean wave height and period. However, the ratio between them is not γ, but a factor $m_0(1)/m_0(\gamma)$ smaller (see p. 31). For $1 \leq \gamma \leq 6$, the resulting ratio varies between 1 and 2.94; for $\gamma = 3.3$ it is 2.17 (assuming $\sigma_a = 0.07$ and $\sigma_b = 0.09$).

4. The spectra measured during JONSWAP at different fetches, for a given wind speed, consistently displayed overshoot. This property has been preserved in the formulation of the J-spectrum, not only through the use of the peak enhancement factor, with $\gamma > 1$, but also because α decreases with increasing $\bar{x}$. (If only the latter condition would obtain, and $\gamma = 1$, there still would be overshoot. This is true e.g. for Bretschneider's spectrum ($\gamma = 1$), if combined with Bretschneider's

growth curves, which give a decreasing mean steepness (thus, a
decreasing α) with increasing $\tilde{x}$.)

5. The empirical dependence of the dimensionless variance $\tilde{m}_0 (= m_0 g^2/U_{10}^4)$
   on the dimensionless fetch $\tilde{x}$, as found in the JONSWAP measurements,
   indicates a smaller growth rate of the waves than in most other studies
   (see e.g. ref. 45).

It was noted above that the spectra of wind-driven waves are quite
similar, even in non-homogeneous and non-stationary wind fields (see
ref. 24 for examples). This fact, which has been explained theoretically
by Hasselmann on the grounds of non-linear interactions (see ref. 22
for details and additional references), is presently being incorporated
in wave prediction models. This development reverses a long trend of
increasing complexity of these models, from the simple (H_s, T_s)-growth
curves in constant wind fields (Sverdrup-Munk) to numerical two-
dimensional spectral models using an energy balance equation for each
spectral component, with various non-linear couplings included. In
contrast to the latter category, the more recent models by Sanders
(ref. 45) and Hasselmann et al (ref. 24) are based on a standard
spectrum shape at every growth stage of the waves, as indicated by
the parameters gH_s/U^2 or $f_m U/g$, so that only a few dependent parameters
need be predicted. These models are in this respect very similar to
the earlier, less sophisticated adaptations of Sverdrup and Munk's
method by Wilson, who generalised it to variable wind fields, and
by Bretschneider, who added a standard spectral form.

3.3.2. Directional spectra

A discussion on standard sea state spectra would not be complete
without some reference to the fact that the sea waves form a random,
moving surface. So far only temporal variations at fixed points have
been considered. This should be extended with the inclusion of spatial
properties.

Considering the sea surface elevation as a stationary random
process in time and space (horizontal position vector), one can
define a three-dimensional spectral density function by a straight-
forward generalization of the one-dimensional case. The three independent
variables in this general spectrum are the frequency (f) and the two
components (k_1, k_2) of a (horizontal) wave vector $\vec{k}$ with respect to a
chosen reference, in which case we have $E(f, k_1, k_2)$. Equivalently,
one can work with the frequency f, the magnitude k of $\vec{k}$, and the
orientation of $\vec{k}$ with respect to the chosen reference: $E(f, k, \theta)$. The
integral of this spectral density function over all values of $\vec{k}$ yields

the one-dimensional (frequency) spectrum E(f).

If linearity can be assumed, then there is a unique relation between k and f, so that the dimensionality of the spectral density function is reduced to two, in which case we have $E(k_1, k_2)$ or $E(k, \theta)$, or $E(f, \theta)$. In the latter two cases a conditional spectral density as a function of θ, at fixed values of k or f, can be defined:

$$D(\theta; f) = \frac{E(f, \theta)}{E(f)} . \tag{3.84}$$

It follows that

$$\int_{-\pi}^{\pi} D(\theta; f) d\theta = 1 . \tag{3.85}$$

Longuet-Higgins has derived many results concerning the statistical properties of a random, moving surface considered as a Gaussian process with an arbitrary two-dimensional spectrum. The reader is referred to ref.34 for details.

While the two-dimensional spectrum of ocean waves in general can assume a great variety of shapes, it seems plausible that it would have some standard form in the case of pure wind-driven waves. However, good measurements of $E(f, \theta)$ are difficult to obtain, particularly as regards the directional density. The most widely used methods of estimating this function have a rather poor directional resolving power. In such cases one usually assumes a plausible analytical expression for $D(\theta; f)$, usually unimodal, and then estimates the associated parameters. Some such results obtained from an array of three mechanically linked pitch-and-roll buoys arranged in a cloverleaf pattern, have been reported by Ewing (ref. 17) and later also by Mitsuyasu (ref. 36). Both give values of the parameters of the following function:

$$D_1(\theta; f) = A(s)\{\cos (\frac{\theta - \theta_0}{2})\}^{2s} , \quad 0 \leq \theta \leq 2\pi, \tag{3.86}$$

in which A(s) is a normalizing coefficient such that eq. (3.85) is fulfilled. The azimuth θ_0 is a mean direction, and s is a shape parameter determining the width of the distribution. Both can vary with f, although the mean directions at different frequencies can be expected to coincide with each other and with the mean wind direction in a stationary, homogeneous wind field. Mitsuyasu has fitted the following functions to the observed variation of s with f:

34

$$s = 11.5 \; \tilde{f}_m^{-7.5} \tilde{f}^5 \qquad \text{for } \tilde{f} \leq \tilde{f}_m$$
$$ = 11.5 \; \tilde{f}^{-2.5} \qquad \text{for } \tilde{f} \geq \tilde{f}_m, \tag{3.87}$$

in which $\tilde{f}$ is a normalized frequency:

$$\tilde{f} = 2\pi f \; U/g \; (= U/c) \tag{3.88}$$

and $\tilde{f}_m$ is the normalized frequency of the maximum density of the frequency spectrm. Mitsuyasu's data clearly indicate a maximum of s (i.e. minimum directional spreading) for $f = f_m$. Furthermore, the maximum value of s, which equals $11.5 \; \tilde{f}_m^{-2.5}$, increases with increasing growth stage (dimensionless fetch) of the waves.

Combining the behavior of $D(\theta;f)$ with that of $E(f)$, as discussed above, the following picture emerges. In a young sea, the wave energy is distributed rather narrowly over the frequencies (JONSWAP) but rather broadly over the directions of propagation (small s), while the reverse is true in a developed sea (Pierson-Moskowitz; large s); throughout this development the components with maximum energy density have the narrowest angular spreading.

4. Long-term wave statistics

4.1. Generalities

The main purpose in long-term wave statistics is to characterize the long-term variability of the short-term sea states, both with regard to service conditions and to extreme conditions.

In the short-term description, a particular sea state is considered, characterized by various probabilistic and/or spectral distribution functions and their parameters, such as

$$(H_s, \; T_z, \; \nu, \; \ldots\ldots, \; \theta_o, \; s, \; \ldots\ldots)$$

or

$$(m_0, \; m_1, \; m_2, \; \ldots\ldots, \; \theta_o, \; s, \; \ldots\ldots).$$

Let us denote these for generality be

$$(a_1, \; a_2, \; a_3, \; \ldots\ldots, \; a_n),$$

which, for brevity, is represented as an n-dimensional vector $\vec{a}$.

In the short-term view, a sea state occupies finite (non-zero) intervals in time and space, but in the long term view one can reduce these to point values, and define a value of $\vec{a}$ for a continuous range of location $\vec{x}$ and time t.

If one considers the growth, propagation and decay of wind waves due to a given windfield, then $\vec{a}$ can be considered as a slowly varying function of $(\vec{x},t)$, governed by deterministic laws based on the mechanics of the air-sea interactions. (Numerical spectral wave prediction models are based on this approach.)

In the long-term description of ocean waves, the wind fields them- selves are not given in deterministic but in probabilistic terms, and the sea state parameters can then be considered as a random (vector) process in $\vec{x}$ and t, written as $\underline{\vec{a}}(\vec{x},t)$. It is this random process which is the basic object of study in long-term wave statistics.

The terminology, methods and results of general random-process theory have been widely accepted and systematically applied in the short-term description of sea states, but not nearly as much in the long-term description. There would seem to be a potential for further developments in this direction, despite the fact that the long-term random process $\underline{\vec{a}}(\vec{x},t)$ is more difficult to handle than the short-term process $\underline{h}(\vec{x},t)$ in several respects. This is due e.g. to the scarcity of basic data and the impossibility of controlled experimentation. Furthermore, whereas there is at least some basis for a deductive approach to the analysis of the sea state in the short term, mainly because of the approximate applicability of the central limit theorem, such is much less the case for the long-term situation.

The location vector $\vec{x}$ and the time t have so far in this chapter been treated on an equal basis. In practice, one deals with $\underline{\vec{a}}$ as a process in time, for given, discrete values of $\vec{x}$. We shall here omit the $\vec{x}$-dependence in the notation and write simply $\underline{\vec{a}}(t)$.

Because of seasonal effects, the process $\underline{\vec{a}}(t)$ is in general non- stationary. This complication is circumvented by restricting the time intervals on which $\underline{\vec{a}}(t)$ is defined to appropriately small parts of each calenderyear, e.g. the four seasons, the twelve individual months, etc. Each year then gives one realization of each of the processes $\underline{\vec{a}}(t)$ so defined.

In the analysis of $\underline{\vec{a}}(t)$ as a random process, one can consider statistics of its instantaneous values (at arbitrary instants) as well

as the joint statistics of its values at several instants separated by
certain lags. The former may be sufficient for certain applications in
which the sequential behavior of sea states is not important (example:
conventional practice of evaluating fatigue, using the Palmgren-Miner
rule), but otherwise the latter approach is called for (example: knowledge
of the duration of calms is needed in operations planning). Some aspects
of both levels of approach will be considered in the following, but no
comprehensive review will be given. However, we shall first make a few
remarks about the data base.

There are three principal sources of basic data:
(a) visual observations
(b) instrumental records
(c) hindcast sea states in historical storms.

Visual observations generally are available over a longer time span
than instrumental records, but they suffer from the drawback of a
poorly defined calibration. On the other hand, routine instrumental
records by themselves contain no directional information. Hindcast
data generally can cover a fairly long time span, although only for
storm conditions. If all the options are available then it may be best
to use visually observed data for estimating long-term service conditions,
to use hindcast data for extreme conditions, and to use the available few
years' data from instrumental records for a calibration of the other two
data sources.

4.2. Statistics of instantaneous values

4.2.1. Distribution of sea state parameters

The previous remarks were of a rather general nature. We shall now
be somewhat more specific, and to that end suppose that in the available
data three parameters are given for each sea state: a characteristic wave
height H_c (such as $4(m_0)^{\frac{1}{2}}$, or a visually estimated height), a characteris-
tic wave period T_c, and a characteristic direction θ_c. In the long-term
view, these have a value (are defined) at each instant. The statistics
of these for arbitrary instants are wholly described by the joint pro-
bability density function $p(H_c, T_c, \theta_c)$.

Note that random sampling here consists of picking an instant at
random and observing the corresponding values of $(\underline{H}_c, \underline{T}_c, \underline{\theta}_c)$. In principle
we can select such instants from continous time. The notion of "return

period" is therefore not applicable here - at least not in a meaningful way. By arbitrarily introducing a discrete time base, e.g. by considering observations made once per three hours, one can indeed convert fractions of the total number of observations into durations (i.e., one can <u>define</u> return periods), but in that case these will reflect the subjectively chosen time step, rather than an objective characteristic of the wave climate. For instance, a statement that in a certain locality "a significant wave height of 10 m has a return period of 15 years" is meaningless if not supplemented with information about the (average) time interval between successive definitions of H_s. If that interval is chosen to coincide with the interval between successive observations of H_s in the past, say 3 hours, then the "return period" of 15 years referred to represents the average interval between successive events $\{\underline{H}_s > 10 \text{ m}\}$, if observations are made once every three hours. But if the observations are made once per hour, then the return period of the event $\{\underline{H}_s > 10 \text{ m}\}$ would reduce to 5 years. This clearly illustrates that the return period as used in this context is not an intrinsic property of the observed wave climate.

Various analytical distribution functions can be fitted to the observed data. For this purpose, and also for graphical or tabular presentation of data, it is convenient to treat the joint probability density as the product of marginal and conditional densities. The Weibull type is often used for the marginal distributions of $\underline{H}_c$ and $\underline{T}_c$. The goodness-of-fit is frequently checked visually, although of course more quantitative measures can be defined, evaluated and used as a criterion in choosing among various possible types (see e.g. ref. 41).

Having found one or more distribution functions which are deemed to give an acceptable fit to the data, by the criteria chosen, then these are also used for extrapolation. This procedure rests on the belief that there is no basic change in the factors causing the growth or limitation of the variables considered (except for a change of scale). Weak though this basis may seem, it would be hard to justify any other procedure in the absence of indications for such basic changes (such as depth-limitations on the wave heights).

4.2.2. Distribution of response peaks

So far the long-term distribution of sea state parameters at arbitrary instants has been considered. This can be used to calculate the long-term distribution of some response parameter, such as the rms-stress in a structural member. More frequently the long-term

distribution of the peak values of the response is of interest, e.g.
for evaluating fatigue. Although peak values are not function values at
arbitrary instants, and therefore, strictly speaking, do not belong in the
section on "Statistics of instantaneous values", they are nevertheless
included here because the aspect of sea state sequences is ignored in the
considerations.

Consider the maxima $\underline{r}_m$ of some response $\underline{r}(t)$ of a given structure
in a wave field. By way of illustration, we shall assume that in a given
sea state $\underline{r}_m$ is Rayleigh-distributed, with mean square value equal to $2\sigma_r^2$,
and that the average number of maxima of $\underline{r}(t)$ per unit time is λ_r.
Both σ_r and λ_r are functions of the sea state parameters (for a given
structure). In the case considered above, we have only (H_c, T_c, θ_c) as
sea state parameters. Knowing σ_r and λ_r as functions of these, and $p(H_c, T_c, \theta_c)$,
the long-term distribution of $\underline{r}_m$ can be evaluated.

Under the assumptions stated, the conditional distribution of $\underline{r}_m$,
for given (H_c, T_c, θ_c) is given by

$$\Pr\{\underline{r}_m > r \mid H_c, T_c, \theta_c\} = \exp\left\{-\frac{r^2}{2\sigma_r^2(H_c, T_c, \theta_c)}\right\} \qquad (4.1)$$

The fraction of time during which $\underline{H}_c$, $\underline{T}_c$ and $\underline{\theta}_c$ simultaneously are in
the ranges $(H_c, H_c + dH_c)$, $(T_c, T_c + dT_c)$ and $(\theta_c, \theta_c + d\theta_c)$ respectively,
is given by $p(H_c, T_c, \theta_c)dH_c dT_c d\theta_c$, as follows from the definition of
$p(H_c, T_c, \theta_c)$. The expected number of maxima of $\underline{r}(t)$ per unit time under
these conditions is $\lambda_r(H_c, T_c, \theta_c)$, of which a fraction given by (4.1)
exceeds r. It follows that the expected number of events $\{\underline{r}_m > r\}$ per
unit time, regardless of the values of (H_c, T_c, θ_c), equals

$$\iiint \Pr\{\underline{r}_m > r \mid H_c, T_c, \theta_c\}\lambda_r(H_c, T_c, \theta_c)p(H_c, T_c, \theta_c)dH_c dT_c d\theta_c \ , \qquad (4.2)$$

in which the integration is carried out over all possible values of (H_c, T_c, θ_c).
The fraction of all the maxima of $\underline{r}(t)$ exceeding r is then obtained by
dividing (4.2) by the expected number of maxima of $\underline{r}(t)$ per unit time,
which is given by

$$\iiint \lambda_r(H_c, T_c, \theta_c)p(H_c, T_c, \theta_c)dH_c dT_c d\theta_c \ . \qquad (4.3)$$

The result equals the marginal (long-term) probability that $\underline{r}_m$ shall
exceed the level r. If $\underline{r}$ is a stress then this result can be used in
evaluating fatigue according to the Palmgren-Miner rule.

It may be noted that the reciprocal of (4.3) equals the average
duration between successive events $\{\underline{r}_m > r\}$. As such it represents the

return period of that event. The value of a return period is unambiguous
in this case since the occurrence of a maximum is a discrete event, in
contrast to what was noted above with respect to the significant wave
height (or other parameters which are defined for continuous time).

Although the value of the return period of the event $\{\underline{r}_m > r\}$ is
unambiguous, one should be careful in its interpretations, and not lose
sight of the fact that it is based on the fraction of all the maxima of
$\underline{r}(t)$ exceeding r, no matter in which sequence they occur. High values
of $\underline{r}_m$ tend to occur only in isolated stormy periods, in each of which
several may exceed r. Thus, the return period of the event $\{\underline{r}_m > r\}$ is
in general by no means equal to the average time interval between
successive storms in which the event $\{\underline{r}_m > r\}$ occurs (at least once),
but shorter than that. The latter is in general a more meaningful
quantity in the context of a design process. Its evaluation requires
knowledge of the sequential behavior of sea state parameters, which
will be considered in the following paragraph.

4.3. Sea state sequences

4.3.1. Definitions

Apart from its intrinsic interest, the problem of sea state
sequences has practical relevance for operations planning (duration
of calms and storms) and design (frequency and duration of storms;
probability of encounter of rare values in the service life of a structure).
Only these aspects will be considered here.

Within the context of wave statistics, it is natural to define
"calms" and "storms" at some location as events in which consecutive
values of the intensity of the wave action at that location, as mea-
sured by the characteristic height $\underline{H}_c$, are below or above a certain
treshold.

To be more specific, we consider $\underline{H}_c$ as a random function of
continuous time, t (the modifications required in case of a discrete
time base are obvious), and we choose a treshold value H. The time
interval between an upcrossing and the next downcrossing of the level
H by the process $\underline{H}_c(t)$ is said to correspond to a storm, and that between
a downcrossing and the next upcrossing to a calm. The lengths of
these intervals are called durations of storms ($\underline{D}_s$) and of calms ($\underline{D}_c$),
respectively.

In the applications referred to above, knowledge of the probability
distributions of $\underline{D}_s$ and of $\underline{D}_c$ is required, for various values of H. These

are considered in the next section, 4.3.2. Following that, we shall deal with encounter probabilities of extreme sea states (section 4.3.3.) and of extreme response values (section 4.3.4.).

4.3.2. Durations of sea states

Some data on sea state durations have been published (see e.g. refs. 14, 25, 29, 44 and a series of reports by Draper in the format recommended by him in ref. 15, which format includes data on persistence). In analyses of the data, and for subsequent generalizations and deductions, one needs plausible hypotheses about the nature of the processes considered.

For increasing values of the treshold level H, the occurrence of storms becomes increasingly rare, and memory-effects may become weaker. This is some justification for introducing the hypothesis that storm occurrences constitute a Poisson process. In such Poisson-model, the number of events in an interval of a given duration is Poisson-distributed, and the time interval between successive events is exponentially distributed.

Russel and Schueller (ref. 44) have tested the hypothesis that hurricane occurrences in a certain part of the Gulf of Mexico would constitute a Poisson process, and found it valid. At about the same time, Houmb (ref. 25) applied the Poisson-model to visual North Sea wave data. In fact, he considers the sequence of upcrossings of the level H by $\underline{H}_c(t)$ as one Poisson process, and the sequence of downcrossings as another. From this he concluded that both $\underline{D}_c$ and $\underline{D}_s$ should be exponentially distributed. Houmb also made an empirical check of some of his hypotheses and results. These were found to be correct within the 5% level of significance, using the chi-square test.

In the author's opinion, considering the occurrences of storms as a Poisson process implies that such storms are reduced to point-events in time, or in other words, that $\underline{D}_s$ is neglected compared to $\underline{D}_c$, so that $\underline{D}_c$ is approximately exponentially distributed. In this view, the assumed applicability of the Poisson-model would not permit any deduction concerning the distribution of the duration of storms.

In a subsequent paper (ref. 29), Houmb and Vik return to the problem of the duration of sea state. They abandon the Poisson model. Observed durations of storms and of calms, mainly from instrumental wave data, were found to be Weibull distributed, with exponents generally in the range 0.5 to 0.8, dependent on the threshold level considered. In addition, Houmb and Vik use a relationship between the average duration of storms (or of calms), their frequency of occurrence, and the probability distribution of the instantaneous value of $\underline{H}_c(t)$.

The frequency of occurrence of storms in turn is expressed in terms of the joint distribution of $\underline{H}_c(t)$ and its derivative by means of Rice's formula for the average threshold crossing rate. Since the work by Houmb and Vik has been reported at the present seminar, no further details are given here.

4.3.3. Encounter probability of extreme sea state

In the design of offshore structures one has to consider extreme sea states, i.e. sea states of such intensity that there is only a small probability of that intensity being exceeded in the anticipated service life of the structure. Such service life is generally far longer than the time span covered by the available wave data. One is forced therefore to make a substantial extrapolation of the probability distributions estimated from the observed frequencies of occurrence.

To define "storms", a fairly high threshold level (H) is chosen for the characteristic wave height $\underline{H}_c(t)$, so that the day-to-day occurrences are filtered out. On the other hand, the threshold should be low enough that a sufficient number of events in the available data is above it, since these events are the basis for extrapolation. The choice is to some extent one of convenience, since the extrapolation is based only on an upper range of the observed values of $\underline{H}_c(t)$, in which case the threshold value chosen a priori does not affect the end results.

Having chosen a threshold value H, the expected number of storms per unit time in which $\underline{H}_c(t) > H$, written as $\mu(H)$, can be estimated from the data. This information is usually supplemented with the assumptions that the occurrences of storms are independent events, such that they form a Poisson process, and that the intensities in the storms which do occur are mutually independent.

In the further elaboration of the data, various approaches are possible. One can consider the maximum characteristic height per storm, written as max $\underline{H}_c$, given that $\underline{H}_c(t) > H$. Its probability distribution, written as

$$Q(H_o;H) \equiv \Pr\{\max \underline{H}_c > H_o | \underline{H}_c(t) > H\}, \qquad (4.4)$$

can be estimated in the range of $(H_o;H)$ covered by the data. If a distribution function is found which is deemed to represent these data well enough then that function is used for extrapolation beyond the measured range.

42

The product $\mu(H)Q(H_o;H)$ equals the expected number of storms per unit time, above the threshold H, in which the maximum value of the characteristic height $\underline{H}_c$ exceeds H_o. Although it may be convenient in practice to estimate this quantity from the two factors μ and Q, estimated separately, it is the product which is the more fundamental quantity in the subsequent developments, and for this reason it is given a separate symbol (n). Furthermore, as noted above, the choice of the threshold level H does not affect this quantity, although it does affect μ and Q taken separately. We can write therefore

$$n(H_o) = \mu(H)Q(H_o;H). \qquad (4.5)$$

The expected number of storms in a given duration (lifetime) L, in which the maximum $\underline{H}_c$ exceeds an extreme value H_o, is equal to $n(H_o)L$. On the basis of the Poisson model and the independence assumption, the probability that no sea state shall occur in the duration L with a characteristic height $\underline{H}_c$ exceeding H_o is then estimated as $\exp\{-n(H_o)L\}$.

An alternative interpretation of the quantity just stated is that it represents the distribution function of the largest characteristic wave height in the duration L:

$$\Pr\{(\max \underline{H}_c \text{ in duration L}) < H_o\} \simeq \exp\{-n(H_o)L\}. \qquad (4.6)$$

The value of H_o for which

$$n(H_o)L = 1, \qquad (4.7)$$

written as $\hat{H}_c(L)$, can (loosely) be called the characteristic wave height with return period L, in the sense that storms in which the maximum $\underline{H}_c$ exceeds $\hat{H}_c(L)$ occur on the average once in a duration L. Since the distribution $Q(H_o;H)$, to which $n(H_o)$ is proportional, is of the exponential type as defined by Gumbel (ref. 20), $\hat{H}_c(L)$ is also approximately equal to the mode of the distribution (4.6), i.e. to the most probable value of the largest characteristic height in a duration L. On the basis of (4.6), this modal value itself has a probability of $1 - e^{-1} \simeq 0.63$ of being exceeded at least once in a duration L.

The approach sketched above is somewhat similar to the one used by Thom (ref. 48), who considers the annual maximum of $\underline{H}_c(t)$ as the basic random variable. He has fitted the Fisher-Tippett type I distribution of extreme values (the double exponential type) to series of the annual maximum significant wave heights, estimated visually from Ocean Station Vessels.

This method is sound in principle, but it can only be used if records are
available for at least 10 years or so, since only one data point per year
is retained in the analysis.

4.3.4. Encounter probability of extreme response peak

In the approaches mentioned above, one is dealing with extreme values
of the characteristic (e.g., significant) wave height $\underline{H}_c(t)$. Another approach
is to consider extreme values of the individual wave heights. A number of
different methods of this kind have been discussed by Nolte (ref. 38).
Reference is also made to Borgman (ref. 6).

In the following we shall not deal with individual wave heights, but
with individual peaks of some response to the waves. This generalization is
included here because it is needed in applications, in which the ultimate
interest is not in the waves themselves but in their effects.

The problem to be considered is the estimation, on the basis of infor-
mation on past storms, of the probability that at least one response peak
$(\underline{r}_m)$ shall exceed an extreme value r, in a time interval of duration L.

Since the exceedence level r is given to be extreme, it is for all
practical purposes certain that r will only be exceeded during storms. This
impolies that the required probability is virtually equal to the probability
of occurrence, in a time interval of duration L, of at least one storm in
which at least one value of $\underline{r}_m$ exceeds r. The latter probability is fairly
easily estimated, at least insofar as it is justified to treat storms as
independent events.

Let $\underline{N} = \underline{N}(r;L)$ denote the number of storms in the duration L in which
at least one response peak exceeds r. The required probability can then be
written as $1 - \text{Pr}\{\underline{N} = 0\}$.

Let μ again denote the average frequency of occurrence of storms; μL
then respresents the expected number of storms in the duration L. Furthermore,
let $Q(r)$ represent the probability that in an arbitrary storm at least one
response peak exceeds r:

$$Q(r) \equiv 1 - \text{Pr}\{\max \underline{r}_m \leq r | \text{arbitrary storm}\}. \tag{4.8}$$

Considering encounter probabilities of extreme values in a lifetime
of a structure, we have $\mu L \gg 1$ and $Q(r) \ll 1$. Treating the storms as inde-
pendent events, $\underline{N}$ is then very nearly Poisson-distributed, in which case the
required probability is given by

$1 - \Pr\{\underline{N} = 0\} = 1 - \exp \{- E(\underline{N})\},$

in which

$$E\{\underline{N}\} = E\{\underline{N}(r;L)\} = \mu L \, Q(r) \, . \qquad (4.10)$$

The parameters μ and $Q(r)$ must be estimated from the data. Regarding $Q(r)$, we shall first consider a given, single storm.

A single storm is supposed to be described by the time history of the short-term sea state parameters such as $(H_c, T_c, \theta_c, \ldots)$. We consider response maxima $\underline{r}_m$, which for a given sea state are assumed to be Rayleigh-distributed, with mean square value equal to $2\sigma_r^2$, and with mean frequency λ_r, both of which vary with the sea state parameters $(H_c, T_c, \theta_c, \ldots)$. Considering a storm lasting from $t = t_1$ to $t = t_2$, the expected number of maxima $\underline{r}_m$ which exceed r is

$$m(r) = \int_{t_1}^{t_2} \lambda_r(t) \exp \{- \tfrac{1}{2}r^2/\sigma_r^2(t)\}dt, \qquad (4.11)$$

where it should be understood that λ_r and σ_r actually vary with t implicitly, through their dependence on $(H_c, T_c, \theta_c, \ldots)$. The probability that no $\underline{r}_m$-value shall exceed r during the storm is then estimated as

$$R(r) \equiv \Pr\{\max \underline{r}_m \le r \,|\, \text{given storm}\} \simeq \exp \{- m(r)\} \qquad (4.12)$$

(see also eq. 3.36).

The probability (4.12) is conditional in that it is given that a storm occurs with a given time history of $(H_c, T_c, \theta_c, \ldots)$. In the long-term view, the storm may or may not occur, and its parameters are random variables, with an associated multidimensional probability density function. Moreover, the variable pattern of their variation with time during a storm should be taken into account. This greatly complicates the transition from the conditional probability (4.12) to the non-conditional probability (4.8).

A first step in getting around this difficulty is to recognize that the sea state parameters have an effect on m only through λ_r and σ_r, and their variation with time. (Needless to say, this is a reduction in complexity only if the number of sea state parameters was more than two.) Furthermore, for large r, only a relatively small time interval around the peak of the response intensity contributes significantly to m. (The actual values of t_1 and t_2 in (4.11) are then immaterial, provided t_1 and t_2 are sufficiently far away from the time of the maximum response intensity.) This implies that m is mainly determined by the storm's maximum σ_r-value, and by the number of $\underline{r}_m$-values around the storm's peak which contribute effectively to m. Borgman

(refs. 6, 7) has shown how an effective number of peaks (n) and an effective intensity (σ) can be estimated from given time histories of $\lambda_r(t)$ and $\sigma_r(t)$, such that

$$m(r) \equiv \int_{t_1}^{t_2} \lambda_r(t) \exp \{- \tfrac{1}{2}r^2/\sigma_r^2(t)\}dt \simeq n \exp \{- \tfrac{1}{2}r^2/\sigma^2\} \qquad (4.13)$$

for all not-too-small values of r. (Borgman actually deals with individual wave heights instead of response peaks, but that does not affect his method.) In this approximation, the response's time history during the storm is described by just two constants, at least insofar as it affects m(r) and R(r). We can write therefore

$$m = m(r;n,\sigma), \qquad (4.14)$$

and interpret R as a probability which is conditional on the occurrence of specific values (n,σ) of the random variables $(\underline{n},\underline{\sigma})$:

$$R = R(r;n,\sigma) = Pr\{\max \underline{r}_m \leq r | \underline{n} = n, \underline{\sigma} = \sigma\}. \qquad (4.15)$$

For each storm, one pair of values (n,σ) can be calculated. The joint probability density $p(n,\sigma)$ of $(\underline{n},\underline{\sigma})$ can therefore be estimated from past storms. Compounding this with the conditional probability $R(r;n,\sigma)$, the probability that at least one response peak shall exceed r in a storm picked at random from the population of storms becomes

$$Q(r) = 1 - \iint R(r;n,\sigma)p(n,\sigma)dnd\sigma$$

$$= 1 - \iint e^{-m(r;n,\sigma)}p(n,\sigma)dnd\sigma. \qquad (4.16)$$

Together with (4.10) and (4.9), this is a solution to the problem which was posed.

In most practical cases, the variability of $\underline{n}$ has much less effect on $Q(r)$ than the variability of $\underline{\sigma}$. A good approximation can then be obtained by assigning some mean value $(\bar{n})$ to $\underline{n}$, and to work with the probability density $p(\sigma)$ of $\underline{\sigma}$ only. We then have

$$m = m(r;\bar{n},\sigma), \quad R = R(r;\bar{n},\sigma) \qquad (4.17)$$

and

$$Q = Q(r;\bar{n}) = 1 - \int R(r;\bar{n},\sigma)p(\sigma)\,d\sigma$$

$$= 1 - \int e^{-m(r;\bar{n},\sigma)}p(\sigma)d\sigma. \qquad (4.18)$$

Antoher simplifying approximation, which can be applied whether we use (4.16) or (4.18), consists of equating $\underline{\sigma}$, the effective value of $\underline{\sigma}_r$, to the maximum $\underline{\sigma}_r$-value occurring during a storm. The use of the latter obviates the need of completely following Borgman's procedure for a more precise estimate of $\underline{\sigma}$.

A method of estimating $Q(r)$ which would seem to be simpler still is to treat $\underline{m}(r)$ itself as the basic random variable. For a given value of the level r, one value of m can be calculated per storm, using (4.11). The probability density of $\underline{m}$ can therefore be estimated from past storms, with r as a parameter; it is written as $p(m;r)$. In this approach, R is conditional on the occurrence of a specific value of $\underline{m}(r)$:

$$R = R(m) = \Pr\{\max \underline{r}_m \leq r \mid \underline{m}(r) = m\} = e^{-m}. \qquad (4.19)$$

Note that the dependence on r is absent in this conditional probability. It re-enters the problem through the probability of the event $\{m < \underline{m}(r) \leq m + dm\}$, which is used in the calculation of the non-conditional probability $Q(r)$:

$$Q(r) = 1 - \int R(m)p(m;r)dm$$

$$= 1 - \int e^{-m}p(m;r)dm. \qquad (4.20)$$

An advantage of this approach is that in the calculations one deals throughout with a single random variable, while still taking full account of the effects of the joint variability of the storm parameters.

It should be remembered that the methods sketched above (eqs.4.16, 4.18 and 4.20) differ only in the estimation of $Q(r)$, i.e. the probability that in a storm picked at random at least one response peak shall exceed r. The subsequent calculations of $E(\underline{N})$ and $\Pr\{\underline{N} = 0\}$ are the same (eqs. 4.10 and (4.9).

Since $\Pr\{\underline{N}(r;L) = 0\}$ is the probability that not a single $\underline{r}_m$-value exceeds r in the entire interval of duration L, it can also be interpreted as the probability that the largest $\underline{r}$-value in a duration L shall not ex-

ceed the level r. (Reference is made to a similar statement which was made above concerning the distribution of the largest value of $\underline{H}_c$.) Likewise, the value of r for which $E\{\underline{N}(r,L)\} = 1$, written as $\hat{r}(L)$, can be called the response value with a return period L, in the sense that storms, in which at least one response peak exceeds $\hat{r}(L)$, occur on the average once in a duration L. The value $\hat{r}$ is approximately equal to the most probable value of the largest $\underline{r}_m$-value in a duration L, and the probability that in a duration L at least one storm occurs in which at least one response peak exceeds $\hat{r}$, is $1 - e^{-1} \simeq 0.63$.

Throughout the developments in this section on sea state sequences, storms have been treated as independent events, both with respect to the time of their occurrence and with respect to their parameters. While this may not be correct in a strict sense, it has so far been generally accepted as a basis for analysis and prediction. At any rate, there is at present insufficient knowledge about possible dependencies between storms to incorporate that in a model. Moreover, if in fact some dependence is present then the independence assumption is conservative, in the sense that it then overpredicts the encounter probabilities of extreme values.

References

1. Arhan, M., Cavanie, A., Ezraty, R., May 1976, Etude theorique et experimen-
   tale de la relation hauteur-periode des vagues de tempête, Ref. I.F.P.
   24 191.
2. Barber, N.F., 1950, Ocean waves and swell, Lecture published by the
   Institute of Civil Engrs., London.
3. Battjes, J.A., Febr. 1972, Run-up distributions of waves breaking on
   slopes, J. of the Waterways, Harbors and Coastal Engineering Division,
   Proceedings ASCE, vol. 97, No. WW 1, p. 91-114.
4. Battjes, J.A., 1974, Computation of set-up, longshore currents, run-up
   and overtopping due to wind-generated waves, Communications on Hydraulics,
   Department of Civil Engineering, Delft University of Technology,
   Report no. 74-2.
5. Bonneau, E., May-June 1971, Statistique des maximums absolus d'un
   processus aléatoire stationnaire, Gaussien et centré, Rech. Aérosp.
   No. 1971-3, p. 169-171.
6. Borgman, L., 1970, Maximum wave height probabilities for a random number
   of random intensity storms, Proceedings 12th Int. Coastal Eng. Conf.,
   Washington, D.C., Vol. I, p. 53-64.
7. Borgman, L.E., May 1973, Probabilities for highest wave in hurricane,
   J. of Waterways, Harbors and Coastal Engineering Division, Proceedings
   of the ASCE, vol. 99, No. WW2, p. 185-207.
8. Bretschneider, C.L., 1959, Wave variability and wave spectra for wind
   generated gravity waves, U.S. Army Corps of Engineers, B.E.B., Tech.
   Mem. 118.
9. Cartwright, D.E., 1958, On estimating the mean energy of sea waves from
   the highest waves in a record, Proc. Roy. Soc. London, A, 247, p. 22-48.
10. Cartwright, D.E., Longuet-Higgins, M.S., 1956, The statistical distribution
    of the maxima of a random function, Proc. Roy. Soc. London, A, 237,
    p. 212-232.
11. Cavanié, A, Arhan, M. and Ezraty, R., 1976, A statistical relationship
    between individual heights and periods of storm waves, Proceedings
    BOSS'76, Vol. II, 13.5 p. 354-360.
12. Chakrabarti, S.K. and Cooley, R.P., 1977, Statistical distribution of
    periods and heights of ocean waves, Journal of Geoph. Res., 82, 9,
    p. 1363-1368.
13. Cramer, H., 1966, On the intersections between the trajectories of a
    normal stationary stochastic process and a high level, Arkiv. Mat., 6.
14. Dorrestein, R., 1967, Wind and wave data of Netherlands lightvessels
    since 1949, Mededelingen en verhandelingen KNMI, 90.

15. Draper, L., 1966, The analysis and presentation of wave data - a plea for uniformity, Proceedings 10th Int. Coastal Eng. Conf., Tokyo, Vol. I, p. 1-11.

16. Earle, M.D., Ebbesmeyer, C.C., Evans, D.J., Aug. 1974, Height-period joint probabilities in Hurricane Camille, Journal of the Waterways, Harbors and Coastal Engineering Division, Proceedings of the ASCE, vol. 100, no. WW3, p. 257-264.

17. Ewing, J.A., 1969, Some measurements of the directional wave spectrum, J. Marine Res., 27, p. 163-171.

18. Goda, Y., 1975, Irregular wave deformation in the surfzone, Coastal Engineering in Japan, Vol. 18, p. 13-26.

19. Goda, Y., 1976, On wave groups, Proceedings BOSS'76, Vol. I. p. 115-128.

20. Gumbel, E.J., 1958, Statistics of extremes, Columbia University Press, New York, N.Y.

21. Haring, R.E. et al, 1976, Extreme wave parameters based on continental shelf storm wave records, Preprint, 15th Int. Coastal Eng. Conf., Honolulu, Hawaii.

22. Hasselmann, K. et al, 1973, Measurements of wind-wave growth and swell decay during the joint North Sea wave project(JONSWAP), Deutsches Hydrographisches Zeitschrift, Hamburg, Reihe A (8°) Nr. 12.

23. Hasselmann, K., Munk, W. and MacDonald, G., 1963, Bispectra of ocean waves, Time series analysis, M. Rosenblatt (ed.).

24. Hasselmann, K., Ross, D.B., Müller, P. and Sell, W., March 1976, A parametric wave prediction model, J. of Physical Oceanography, vol. 6, no. 2.

25. Houmb, O.G., 1971, On the duration of storms in the North Sea, 1st Conf. on Port and Ocean Eng. under Arctic Conditions, Trondheim, Vol. 1, p. 423-439.

26. Houmb, O.G. and Overvik, T., 1976, Parameterization of wave spectra and long term joint distribution of wave height and period, Proceedings BOSS '76, Vol. I. 2.4. p. 144-169.

27. Houmb, O.G. and Overvik, T., 1977, On the statistical properties of 115 wave records from the Norwegian Continental Shelf, Rep. Div. of Port and Ocean Eng., Univ. of Trondheim.

28. Houmb, O.G. and Rye, H., 1973, Analysis of wave data from the Norwegian Continental Shelf, Proceedings of the second International Conference on Port and Ocean Engineering under Arctic Conditions, p. 780-788.

29. Houmb, O.G. and Vik, I., 1977, On the duration of sea state, Rep. Div. of Port and Ocean Eng., Univ. of Trondheim. (Also presented as a short contribution to the Seminar on Safety of Structures under Dynamic Loading, Trondheim, 1977.)

30. ISSC, 1964, Report Committee 1, Proceedings 2nd International Ship
    Structures Congress, Delft, Netherlands, 20-24 July 1964, Vol. I.

31. ISSC, 1976, Report Committee 1, 6th International Ship Structures
    Congress, Boston, 1976.

32. Jahns, H.O. and Wheeler, J.D., 1972, Long term wave probabilities based
    on hindcasting of severe storms, Proceedings Fourth Offshore Technology
    Conference, 1972, Paper number OTC 1590.

33. Longuet-Higgins, M.S., 1952, On the statistical distribution of the
    heights of sea waves, J. Mar. Res. XI, 3, p. 245-266.

34. Longuet-Higgins, M.S., 1957, The statistical analysis of a random, moving
    surface, Phil. Trans. Roy. Soc. London, Ser. A, 249, p. 321-387.

35. Longuet-Higgins, M.S., 1975, On the joint distribution of the period and
    amplitude of sea waves, J. of Geophysical Research, vol. 80, no. 18,
    June 20, 1975, p. 2688-2694.

36. Mitsuyasu, H. et al, 1975, Observations of the directional spectrum
    of ocean waves using a cloverleaf buoy, J. Phys. Ocean., 5, p. 750-760.

37. Nath, J.H. and Ramsey, T.L., 1974, Probability distributions of breaking
    wave heights, Proc. of the International Symposium on Ocean Wave
    Measurements and Analysis, ASCF,New Orleans, Sept. 9-11, 1974, p. 379-395.

38. Nolte, K.G., 1973, Statistical methods for determining extreme sea states,
    Proc. 2nd Conf. on Port and Ocean Eng. under Arctic Conditions,
    Reykjavik, p. 705-742.

39. Ochi, M.K., 1973, On prediction of extreme values, J. of Ship Res.,
    vol. 17, no. 1.

40. Overvik, T. and Houmb, O.G., 1977, A note on the distribution of wave
    steepness, Rep. Div. of Port and Ocean Eng., Univ. of Trondheim.

41. Petrauskas, C. and Aagaard, P.M., 1970, Extrapolation of Historical Storm
    Data for Estimating Design Wave Heights, Offshore Techn. Conf., Houston.

42. Pierson, W.J.,Jr. and Moskowitz, L., 1964, A proposed spectral form for
    fully developed wind seas based on the similarity theory of S.A.
    Kitaigordskii, J. of Geophysical research, vol. 69, no. 24, Dec. 1964,
    p. 5181-5190.

43. Rice, S.O., 1944, Mathematical Analysis of Random Noise, Bell. System
    Tech. Journ., 23, 1944, p. 282-332 and 24, 1945, p. 46-156.

44. Russell,L. and Schueller, G., 1971, Probabilistic Models for Texas Gulf
    Coast Hurricane Occurrences, Offshore Techn. Conf., Houston.

45. Sanders, J.W., 1976, A growth-stage scaling model for the wind-driven
    sea, Deutsches Hydrographisches Zeitschrift, Band 29, Heft 4, p. 136-161.

46. Svasek, J.N., 1969, Statistical evaluation of wave conditions in a deltaic area,
    Symposium "Research on wave action", Delft Hydraulics Laboratory, 1969.

47. Taylor, G.I., 1921, Diffusion by continuous movements, Proc. London
    Math. Soc., Ser. 2, 20, 1921, p. 196-211.

48. Thom, H.C.S., 1971, Asymptotic Extreme-value Distributions of Wave Heights in the Open Ocean, J. Mar. Res., 29, 1, p. 19-27.

49. Titov, L.F. 1971, Wind-driven waves, Israel Program for Scientific Translations, Jerusalem.

50. Tucker, M.J., 1963, Analysis of records of sea waves, Proc. Instn. of Civ. Engrs, 26, p. 304-316.

51. Watson, G.S., 1954, Extreme values in samples from m-dependent stationary stochastic processes, The Annals of Mathematical Statistics, vol. 25, p. 798-800.

52. Wooding, R.A., May 1955, An approximate joint probability distribution for wave amplitude and frequency in random noise, New Zealand J. of Science and Technology, vol. 36, sec. B, no. 6, p. 537-544.

53. Wu, H.J., 1973, A discussion of the wave records obtained at weather station Papa, Pacific Ocean, Proceedings POAC 1973, vol. I, p. 49-60.